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PERFORMANCE ANALYSIS OF VARIABLE SAMPLE SIZE CONTROL CHART WITH MEASUREMENT ERROR

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ABSTRACT

Measurement error frequently distorts outcomes in real-world applications, affecting process results. This paper explores how measurement error affects the detection capabilities of a new adaptive control chart with variable sample size in identifying out-of-control conditions. The analysis is based on a model incorporating linear covariates, focusing specifically on the variable sample size with the EWMA chart's ability to detect shifts in the mean. Our findings demonstrate that measurement error considerably influences the chart's performance concerning mean detection. Additionally, we show that the proposed variable sample size strategy outperforms traditional methods under these conditions.

KEYWORDS

Measurement error; control chart; adaptive control chart; variable sample size; EWMA.

1. INTRODUCTION

Control charts are widely recognized tools in modern industries, with Shewhart (1924) control charts being among the most prominent. Despite their popularity, they are unable to detect small shifts in a process quickly enough. The control limits (*CL*) for monitoring the process mean in this chart are set at $\mu \pm 3\sigma$. While effective, this method does not perform well with small shifts. To enhance the capability of control charts, adaptive control charts have been introduced. These charts are considered adaptive when at least one design parameter changes based on process data Costa and De Magalhaes (2007). A significant portion of research on adaptive control charts has focused on improving their sensitivity to small shifts in process parameters through methods like variable sampling intervals (*VSI*) and variable sample size (*VSS*). There is often a problem with control charts regarding the variability of measurement errors (*ME*). This problem arises when the variable of interest X, cannot be measured accurately. Using imprecise measurement tools affects the control charts ability to detect situations where the process is out-of-control (*OC*). Additionally, the variable of interest may be associated with the *ME* system used through a covariate. Mittag and Stemann (1998) studied how *ME* impact the $\bar{X}$ and S control charts. They looked at a model where Y represents the measured value due to random errors in the actual value X. Linna and Woodall (2001) extended previous studies by examining models that incorporate additional factors, known as covariates, which influence measurements. They investigated how these covariate-based models affect the performance of various control charts. Later, Linna, Woodall and Busby (2001) specifically analyzed the effect of including covariates in a multivariate Shewhart chart for monitoring the mean. Earlier in this field of research, Prabhu, Runger and Keats (1993) and Costa (1994) pioneered the application of the Variable Sample Size (VSS) technique to control charts.

Later, De Magalhaes et al. (2009) conducted a statistical analysis of a two-state hierarchy with adaptive parameters for $\bar{X}$ charts. Their study evaluated and compared seven different adaptive control chart methods, with a particular focus on *VSS*. Several optimization techniques exist for determining the optimal sample sizes in *VSS* control charts. For instance, Wu and Luo (2002) formulated an algorithm to identify sample sizes for *NP*-control charts. The concept of Exponentially Weighted Moving Average (*EWMA*) control charts was first proposed by Roberts (1959) and has since become a staple in statistical process control (*SPC*) for detecting small shifts in process parameters. Amjad et al. (2020) discussed the *ME* significantly impacts monitoring mean and variance shifts in industrial production. Quality control techniques, particularly *Max-EMWA* control charts, incorporate covariate models and multiple measurements to mitigate these effects, as evidenced by Monte Carlo simulations and real-life data examples. Afshan and Muhammad Noor-ul-Amin et al. (2021) studied *ME* impact joint monitoring schemes for mean and coefficient of variation (*CV*) in control charts, highlighting reduced detection capability. Multiple measurements are suggested to mitigate these effects, based on simulations and real data findings. Wang et al. (2023) analyzed the effects of measurement error on the Bayesian EWMA control chart by integrating various ranked set sampling designs and loss functions. Their study used performance metrics, such as Average Run Length (ARL) and Standard Deviation of Run Length (SDRL), evaluated through Monte Carlo simulations and real data. The results showed that median Ranked Set Sampling (RSS) outperformed other sampling methods under measurement error, enhancing the control chart's performance in accurately detecting shifts. This research demonstrates the advantages of RSS in quality control applications affected by measurement errors.

In industrial and quality control processes, precise detection of shifts in process parameters is crucial for maintaining product quality and minimizing production costs. Traditional control charts, while widely used, often struggle to detect small shifts effectively, especially in the presence of ME. The inclusion of additional factors, such as covariates and multiple measurements, has shown promise in mitigating the impact of ME on control chart performance. However, there remains a need for an adaptive approach that can dynamically adjust to varying process conditions, particularly when errors and variability are high. This study aims to develop an enhanced VSS EWMA control chart that incorporates covariates and multiple measurements. By exploring the effects of measurement errors in a controlled environment and proposing adaptive strategies, this research seeks to advance the efficiency and robustness of control charts, offering a practical solution for real-time quality monitoring in complex industrial processes. The article is structured as follows: Section 2 introduces an Adaptive Control Chart utilizing *VSS*. Section 3 discusses *VSS* in the context of a model incorporating covariates. Section 4 presents methods for evaluating *VSS* performance and examines the impact of the *ME* model on *VSS*. Lastly, Section 5 compares the performance of these charts, followed by the conclusion in the final section.

2. EWMA BASED ON VARIABLE SAMPLE SIZE

The *EWMA* control charts were initially introduced by Roberts in 1959. Suppose a quality characteristic is normally distributed with a mean (μ) and a standard deviation (σ). The *EWMA* statistic for the *ith* sample is defined as follows:

$$Z_i = \lambda \bar{X}_i + (1-\lambda) Z_i - 1 \,;\; 0 < \lambda \le 1 \qquad (1)$$

To create an *EWMA* control chart with *VSS EWMA*, we utilize specific statistics and *CL*, along with two warning Limits. These limits are the Upper Warning Limit (*UWL*) and the Lower Warning Limit (*LWL*), which are defined as follows:

$$UWL = \mu + L_1 \sigma \sqrt{\frac{\lambda}{n_i(2-\lambda)}(1-(1-\lambda)^{2i}} \qquad (2)$$

$$LWL = \mu - L_1 \sigma \sqrt{\frac{\lambda}{n_i(2-\lambda)}(1-(1-\lambda)^{2i}} \qquad (3)$$

where λ is smoothing parameter or exponential weight constant. For $i = 0, Z_i = Z_0$ is starting value and is often taken equal to the process target value.

The region between the *LWL* and the *UWL* is designated as the safety zone. The regions situated between the Lower control limit (*LCL)* and *LWL*, as well as between *UWL* and the Upper control limit (*UCL)*, are referred to as warning zones. For the *ith* sample, a smaller sample size (n_1) is used if the previous sample statistic ($Z_i - 1$) lies within the safety zone, whereas a larger sample size (n_2) is employed if $(Z_i - 1)$ falls within either of the warning zones.

$$\begin{aligned} LWL < Z_{i-1} < UWL &\rightarrow n_i = n_1 \\ UWL < Z_{i-1} < UCL &\rightarrow n_i = n_2 \\ LCL < Z_{i-1} < LWL &\rightarrow n_i = n_1 \end{aligned} \qquad (4)$$

The *EWMA* statistic, when considering variable sample sizes, remains consistent with the *EWMA* statistic as described in Equation (1), with the initial value $Z_0 = \mu$. The *CL* for *VSS EWMA* are adjusted as follows:

$$\begin{aligned} UCL_i &= \mu + L_2 \sigma \sqrt{\frac{\lambda}{n_i(2-\lambda)}(1-(1-\lambda)^{2i}} \\ CL &= \mu \\ LCL_i &= \mu - L_2 \sigma \sqrt{\frac{\lambda}{n_i(2-\lambda)}(1-(1-\lambda)^{2i}} \end{aligned} \qquad (5)$$

For $i = 0$, let $Z_0 = \mu_0$ and the initial *CL*, UCL_0 and LCL_0, can be determined by using the following method:

$$UCL_0 = \mu + L_2\sigma\sqrt{\frac{\lambda}{n_1(2-\lambda)}}$$
$$CL = \mu \qquad (6)$$
$$LCL_0 = \mu - L_2\sigma\sqrt{\frac{\lambda}{n_1(2-\lambda)}}$$

In the traditional *VSS* control charts, $Z_0 = \mu_0$ serves as the initial point, with the centerline μ located in the safe zone. Consequently, a n_1 is utilized for the initial set of samples. This is why n_1 is employed as the starting sample size in (6). Furthermore, several articles in the *VSS* control chart literature, including Flaig (1991), also advocate for selecting a n_1. Subsequent n_i for $(i = 2,3, ...)$ are then determined using (4).

3. PROPOSED ADAPTİVE EWMA CONTROL CHART UNDER MEASUREMENT ERROR

In this section, we present a new *VSS* strategy aimed at enhancing the effectiveness of the *EWMA* control chart by using *ME*. While this approach can be applied to various control charts, this article specifically focuses on its application in the context of *ME*.

Consider a process that follows to normal distributions. We can state that the process is in-control (*IC*) if $X_i \sim N(0, \sigma^2)$. Conversely, the process is deemed *OC* if $X_i \sim N(\delta, \sigma^2)$. This discussion primarily centers on changes in the μ, so we assume $\delta \neq 0$. According to the description, n_1 are assigned when the current *Z* value is closer to the centerline, whereas n_2 are assigned when the *Z* value is closer to the *UCL* or *LCL*.

Let n_1 denote the minimum sample size and n_2 denote the maximum sample size. The n_1 occurs when the current *Z* value equals centerline, indicating the smallest necessary sample. Conversely, the n_2 occurs when the current *Z* value reaches *UCL* or *LCL*, signifying a n_2 is needed. Thus, the function defining the relationship between n_i and *Z* value can be understood as varying between n_1 and n_2, depending on how far the current *Z* value deviates from the centerline and approaches the *CL*.

The n_i function can be defined as follows:

$$n_i = f\left(|Z_{i-1}|\right) \qquad (7)$$

The *UCL* and *LCL* of this chart can be presented as follows:

$$Z_i = \lambda\left(\frac{\sum_{i=1}^{n_i} x_i}{n_i}\right) + (1-\lambda)2_{i-1}\ ;\ 0 < \lambda \le 1 \qquad (8)$$

Now, the values of *CL* can be determined as follows;

$$UCL_0 = \mu + L_3\sigma\sqrt{\frac{\lambda}{n_1(2-\lambda)}}$$
$$CL = \mu \quad (9)$$
$$LCL_0 = \mu - L_3\sigma\sqrt{\frac{\lambda}{n_1(2-\lambda)}}$$

For every sample, it is necessary to compute several key metrics: n_i, UCL_i, LCL_i, and Z_i.

3.1 Covariate Model

Consider a scenario where the true value of characteristic X, when the process is in control (IC), is normally distributed with a specific variance (σ^2). However, we cannot directly observe this true value. Instead, we observe a related value Y, which is linked to X by the formula $Y = A + BX + \varepsilon$. Here, A and B are constants, and ε represents a random error term that is independent of X and is normally distributed with a $\mu = 0$ and σ_m^2. Assuming that all the model parameters are provided.

The relationship between X and Y clearly indicates that Y is normally distributed with a mean of $A + B\mu$ and a variance of $B^2\sigma^2 + \sigma_m^2$. For each sampling occasion, we collect n observations of Y, compute their average $\overline{Y_i}$, and then use this to calculate the *CL* are;

$$UCL_0 = \mu + L_3\sqrt{\frac{\lambda\left(B^2\sigma^2 + \sigma_m^2\right)}{n_1(2-\lambda)}} \quad (10)$$

$$LCL_0 = \mu - L_3\sqrt{\frac{\lambda\left(B^2\sigma^2 + \sigma_m^2\right)}{n_1(2-\lambda)}} \quad (11)$$

We can utilize their limiting values, as demonstrated by Lucas and Saccucci (1990).

$$\sqrt{\frac{\lambda\left(B^2\sigma^2 + \sigma_m^2\right)}{n_1(2-\lambda)}}$$

In this context, the asymptotic standard deviation of Z_t.

3.2 Multiple Measurements

According to Linna and Woodall (2001), one effective way to minimize the impact of *ME* is to increase the number of measurements taken per sampled unit. By conducting multiple measurements and calculating their average, the precision of the measurement improves significantly. This approach also reduces the variability caused by *ME*, particularly as the number of measurements increases, especially if an infinite number were taken in theory, the variability due to *ME* would approach zero. However, it's important to balance this with the additional costs and time required for conducting multiple observations. It's crucial to note that when *ME* are absent, taking multiple measurements does not enhance the effectiveness of control charting methods; instead, it merely adds to the cost associated with extra measurements.

When enough measurements are taken, it becomes feasible to consider that our process operates effectively without significant *ME*. However, it's crucial to weigh the costs in terms of both time and resources associated with conducting additional measurements. These factors cannot be underestimated and must be carefully evaluated in the context of our specific application. It's important to note that the σ_m^2 due to *ME* should be sufficiently large, and the impact of additional observations on these factors should be minimal, to justify the practical value of conducting extra measurements.

To calculate the *VSS EWMA* statistic, initially, we collect k measurements for each of n instances of Y during each sampling period. We then compute the average of these observed values, denoted as $\overline{Y}$. Now, we proceed to compute the *VSS EWMA* statistic using its defined formula.

$$Z_i = \lambda\left(\frac{\sum_{i=1}^{n_i} x_i}{n_i}\right) + (1-\lambda)2_{i-1}\ ;\ 0 < \lambda \le 1 \tag{12}$$

Let Y_i represent the average of the observations gathered at time i, where $(i = 1,2,3, \dots)$.The parameter λ is a smoothing factor that falls within the range of 0 to 1, and Z_i denotes the initial value.

According to Linna and Woodall (2001), demonstrating that the σ_m^2 of the overall mean is;

$$\frac{B^2\sigma^2}{n} + \frac{\sigma_m^2}{nk}$$

Therefore, the *CL* are

$$UCL_0 = \mu + L_3\sqrt{\frac{\lambda}{(2-\lambda)}\left(\frac{B^2\sigma^2}{n_1} + \frac{\sigma_m^2}{n_1 k}\right)} \tag{13}$$

$$LCL_0 = \mu - L_3\sqrt{\frac{\lambda}{(2-\lambda)}\left(\frac{B^2\sigma^2}{n_1} + \frac{\sigma_m^2}{n_1 k}\right)} \tag{14}$$

4. PERFORMANCE EVALUATION

In a control chart, we aim to achieve two primary goals. Firstly, the process is stabilizes, we aim for our chart to indicate an *OC* condition (false alarm) at a specified frequency. Statistically, we desire the probability of the mean exceeding the *CL* to correspond to our intended rate when the process is *IC*. Secondly, when the process is unstable, we want the chart to detect this change as quickly as possible. In statistical terms, we seek to minimize the probability of the mean remaining within *CL* when the process is *OC*. Several methods have been suggested for assessing the effectiveness of a chart about the mentioned objectives. The most commonly recognized metric is *ARL*, which relies on the distribution of *RL*. The *RL* signifies the number of observations required for a control chart to indicate a deviation or a single observation from the *RL* distribution. In the realm of statistical quality control, the mean of the *RL* distribution represents the *ARL*. This metric represents the average number of observations needed for a control chart to indicate a signal. When it comes to evaluating performance measures within the *ME*, the primary method is commonly used: Monte Carlo simulation. In this study, we exclusively utilize the Monte Carlo simulation uses repeated random sampling to estimate statistical properties. The *ARL* measures the average number of samples taken before a control chart signals a process shift. However, it typically requires discretization of the process's continuity into multiple steps to enhance accuracy.

Table 1

***ARL (SDRL)* for the *VSS* with a Covariate Model across Different Values of $\frac{\sigma_m^2}{\sigma^2}$**

Shift	No Error	0.1	0.2	0.3	0.5	1
0	500.80 (500.01)	503.37 (495.518)	506.32 (505.70)	499.62 (489.34)	500.02 (493.37)	500.84 (489.60)
0.1	135.86 (129.63)	147.29 (143.35)	155.80 (150.56)	164.87 (161.36)	182.81 (176.83)	223.65 (217.16)
0.2	35.27 (29.88)	38.93 (33.85)	42.59 (37.13)	46.76 (42.12)	55.11 (49.39)	73.06 (67.48)
0.4	8.90 (4.93)	9.66 (5.49)	10.27 (6.048)	11.21 (6.95)	12.87 (8.30)	17.05 (12.40)
0.6	4.918 (1.900)	5.24 (2.15)	5.52 (2.33)	5.824 (2.571)	6.49 (2.92)	8.12 (4.34)
0.75	3.76 (1.231)	3.97 (1.309)	4.15 (1.430)	4.384 (1.582)	4.78 (1.850)	5.78 (2.55)
1	2.824 (0.73)	2.958 (0.812)	3.05 (0.846)	3.187 (0.90)	3.406 (1.037)	4.012 (1.37)
1.25	2.32 (0.517)	2.419 (0.565)	2.49 (0.601)	2.576 (0.643)	2.761 (0.715)	3.176 (90.911)
1.5	2.06 (0.324)	2.122 (0.3748)	2.182 (0.424)	2.241 (0.464)	2.357 (0.532)	2.658 (0.676)
2	1.822 (0.387)	1.881 (0.334)	1.911 (0.307)	1.945 (0.287)	1.992 (0.279)	2.136 (0.386)
3	1.093 (0.291)	1.161 (0.368)	1.228 (0.419)	1.314 (0.464)	1.453 (0.497)	1.741 (0.438)

In Table 1 shows the *ARL* for the *VSS* with a covariate model across different values of $\frac{\sigma_m^2}{\sigma^2}$ where $B = 1$. The *IC ARL* value remains consistent across all combinations to ensure an equitable basis for comparison. The impact on the *OC ARL* rises as the ratio $\frac{\sigma_m^2}{\sigma^2}$ increases. This outcome corresponds with the conclusions drawn in the study by Linna and Woodall (2001). In Table 2, we present the *ARL* for the *VSS* with a covariate model across different values of *B*. The parameters used are consistent with those in Table 1, under the condition where $\frac{\sigma_m^2}{\sigma^2} = 1$. It's evident that as *B* values increase, their impact on *ARL* decreases. This finding is consistent with the observations made by Linna & Woodall (2001). In addition, both Tables 1 and 2 demonstrate that as the size of the shift increases, the impact of *ME* on *ARL* diminishes. Notably, variable a shows no significant effect on *ARL* performance in this particular study.

Table 2

ARL (SDRL)* for the *VSS* with a Covariate Model across Different Values of *B

Shift	No Error	1	2	3	5
0	500.06 (496.90)	500.16 (500.04)	500.33 (496.50)	500.35 (524.20)	500.07 (495.02)
0.1	135.97 (130.87)	225.89 (221.39)	335.97 (335.73)	410.24 (405.01)	461.08 (452.56)
0.2	35.30 (29.842)	73.13 (67.91)	165.72 (159.99)	249.19 (241.13)	367.90 (362.40)
0.4	8.92 (4.94)	16.86 (12.08)	44.80 (39.59)	89.24 (84.84)	195.70 (191.07)
0.6	4.83 (1.866)	8.073 (4.217)	19.137 (13.979)	40.37 (34.99)	102.199 (95.57)
0.75	3.781 (1.24)	5.80 (2.550)	12.27 (7.911)	24.503 (19.66)	67.123 (60.93)
1	2.811 (0.743)	4.002 (1.33)	7.434 (3.751)	13.543 (8.944)	36.302 (30.69)
1.25	2.317 (0.512)	3.153 (0.896)	5.372 (2.21)	9.034 (5.024)	22.58 (17.56)
1.5	2.072 (0.328)	2.667 (0.674)	4.226 (1.487)	6.842 (3.274)	15.801 (11.088)
2	1.821 (0.385)	2.136 (0.384)	3.139 (0.892)	4.562 (1.670)	9.185 (5.138)
3	1.092 (0.290)	1.749 (0.433)	2.215 (0.441)	2.952 (0.805)	5.051 (2.016)

In Table 3, you can find the *ARL* results for the covariate model, which includes multiple measurements across various ratios $\frac{\sigma_m^2}{\sigma^2}$ of a specified variable. This data pertains specifically to cases where the number of measurements k is less than **5** and the B exceeds 1. It's evident that when five measurements per unit are feasible, for $\frac{\sigma_m^2}{\sigma^2}$ ratios below 0.3, we can infer that the process operates practically free from *ME*. However, for ratios exceeding 0.3, the impact is significantly diminished compared to the case with k equal to 1, as shown in Table 1, even when $\frac{\sigma_m^2}{\sigma^2} = 1$.

Table 3
***ARL (SDRL)* for the *VSS* with Multiple Measurements**
using $k = 5$, $B = 1$ Across Different Values of $\frac{\sigma_m^2}{\sigma^2}$

Shift	No Error	0.1	0.2	0.3	0.5	1
0	500.66 (500.23)	500.12 (500.56)	500.59 (500.52)	500.69 (491.18)	500.17 (490.41)	500.27 (485.28)
0.1	136.915 (131.57)	138.18 (133.15)	141.98 (138.75)	146.66 (138.60)	147.93 (143.64)	159.76 (155.19)
0.2	34.92 (29.69)	35.83 (30.38)	37.08 (31.22)	37.61 (32.12)	39.18 (33.71)	42.79 (36.95)
0.4	8.940 (4.89)	9.095 (5.032)	9.278 (5.239)	9.338 (5.297)	9.705 (5.605)	10.415 (6.203)
0.6	4.873 (1.901)	5.006 (1.949)	4.997 (1.938)	5.096 (2.045)	5.185 (2.087)	5.539 (2.323)
0.75	3.767 (1.220)	3.807 (1.249)	3.837 (1.257)	3.872 (1.287)	3.954 (1.335)	4.164 (1.455)
1	2.825 (0.747)	2.835 (0.755)	2.865 (0.773)	2.888 (0.7687)	2.942 (0.796)	3.066 (0.845)
1.25	2.323 (0.5128)	2.350 (0.533)	2.361 (0.535)	2.377 (0.545)	2.416 (0.567)	2.502 (0.595)
1.5	2.071 (0.333)	2.082 (0.341)	2.090 (0.339)	2.1048 (0.365)	2.127 (0.3793)	2.176 (0.411)
2	1.823 (0.385)	1.8427 (0.369)	1.8485 (0.363)	1.8523 (0.3617)	1.8714 (0.349)	1.913 (0.3082)
3	1.0947 (0.2928)	1.1089 (0.3115)	1.1187 (0.3234)	1.129 (0.335)	1.1598 (0.3664)	1.2307 (0.4213)

In Table 4, the outcomes are displayed for various measurements corresponding to different values of *B*. It is evident that as *B* increases, the impact on the *ARL* decreases. This finding aligns with the conclusions drawn from the data presented in Table 2.

Table 4

ARL (SDRL)* for the *VSS* with a Multiple Measurements $k = 5$, $\frac{\sigma_m^2}{\sigma^2} = 1$ across Different Values of *B

Shift	No Error	1	2	3	5
0	500.84 (489.17)	500.07 (494.85)	500.55 (487.756)	500.52 (496.68)	500.23 (496.79)
0.1	27.79 (22.66)	156.67 (152.86)	320.12 (312.06)	395.07 (383.56)	456.91 (447.22)
0.2	7.409 (3.713)	42.49 (36.65)	143.95 (139.87)	239.90 (238.11)	364.98 (359.89)
0.4	3.133 (0.896)	10.38 (6.127)	37.34 (31.679)	84.534 (78.76)	191.03 (185.27)
0.6	2.203 (0.439)	5.537 (2.339)	16.173 (11.52)	35.534 (30.265)	101.15 (95.410)
0.75	1.979 (0.279)	4.169 (1.447)	10.28 (5.993)	22.411 (17.23)	66.219 (59.73)
1	1.650 (0.477)	3.062 (0.8497)	6.547 (3.054)	12.484 (7.889)	35.57 (30.63)
1.25	1.195 (0.396)	2.512 (0.609)	4.831 (1.875)	8.392 (4.477)	21.95 (16.65)
1.5	1.0187 (0.1354)	2.183 (0.4228)	3.867 (1.290)	6.417 (2.983)	15.277 (10.634)
2	1 (0)	1.909 (0.311)	2.882 (0.775)	4.339 (1.553)	8.976 (4.951)
3	1 (0)	1.226 (0.418)	2.0977 (0.352)	2.851 (0.753)	4.915 (1.913)

Table 5
ARL (SDRL)* for the *VSS* with Multiple Measurements across Different Values of *k

Shift	No Error	5	10	20	50
0	500.19 (495.39)	500.87 (510.03)	500.17 (494.38)	500.37 (496.20)	500.29 (499.63)
0.1	135.35 (128.76)	148.35 (143.39)	142.16 (136.524)	137.55 (130.87)	138.03 (134.23)
0.2	35.904 (30.59)	39.13 (33.48)	36.922 (31.823)	36.44 (30.906)	35.312 (29.803)
0.4	8.911 (4.858)	9.607 (5.454)	9.311 (5.242)	9.056 (5.0254)	9.051 (5.093)
0.6	4.864 (1.894)	5.200 (2.093)	5.0306 (2.0393)	4.956 (1.917)	4.914 (1.937)
0.75	3.755 (1.2028)	3.976 (1.337)	3.842 (1.259)	3.789 (1.232)	3.785 (1.2277)
1	2.8058 (0.734)	2.931 (0.798)	2.878 (0.7808)	2.8432 (0.7548)	2.839 (0.7605)
1.25	2.322 (0.514)	2.4078 (0.5629)	2.356 (0.533)	2.346 (0.5293)	2.3358 (0.5163)
1.5	2.0688 (0.3332)	2.122 (0.3762)	2.0916 (0.344)	2.0809 (0.3387)	2.0739 (0.3305)
2	1.8178 (0.3901)	1.875 (0.3413)	1.8538 (0.3600)	1.8367 (0.3742)	1.8247 (0.384)
3	1.0917 (0.2886)	1.156 (0.362)	1.1284 (0.3345)	1.1145 (0.3184)	1.1033 (0.3043)

Furthermore, Table 5 presents findings for the *VSS* with a multiple measurements across various k values. Notably, k increases, the impact of *ME* diminishes. However, it's crucial to consider the trade-off between the increased cost and time associated with additional measurements and the tolerable level of *ME*. It's important to note that the results presented reflect the worst-case scenario, as we selected $B = 1$ and $\frac{\sigma_m^2}{\sigma^2} = 1$, representing the most affected combination. Thus, it's reasonable to expect even better outcomes in other scenarios.

In all the computations we used the Markov Chain method. Moreover, the values of the constants are $L = 3.234$. Note also that in all instances the *CL* utilized are those associated with the limiting values. Some of the results from the Tables are presented in Figures 1-3.

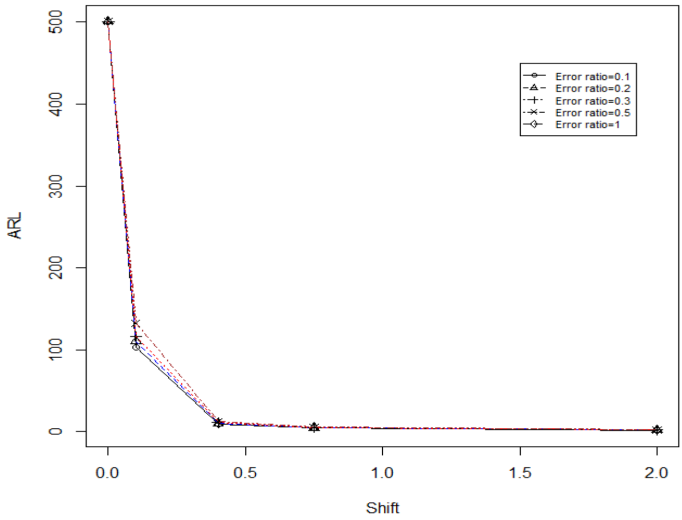

Figure 1 : *ARLs* at Different Values of Error Ratio with $\lambda = 0.1$

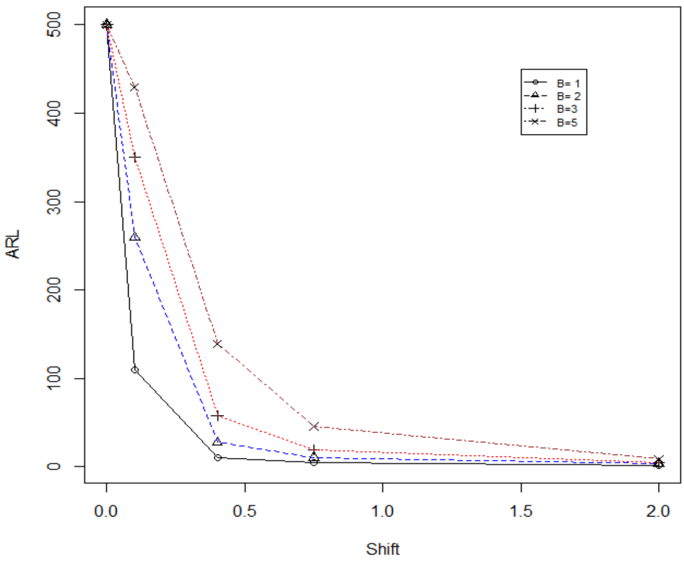

Figure 2 : *ARLs* at Different Values of *B* with $\lambda = 0.1$

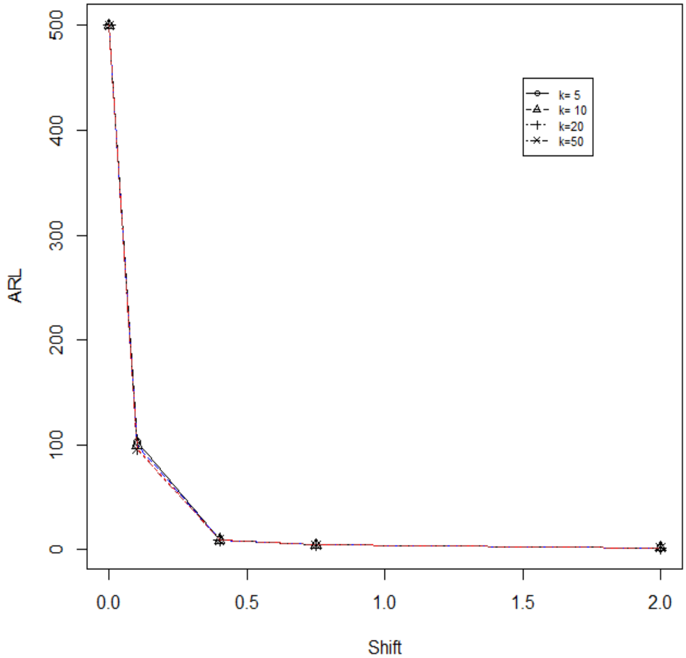

Figure 3 : *ARL* at Different Values of *K* with $\lambda = 0.1$

5. APPLICATION

In this section, we utilize a real dataset to illustrate the application of the proposed control chart with *ME*. The dataset pertains to a hard-bake process used in conjunction with photolithography in semiconductor manufacturing. Our objective is to establish statistical control over the flow width of the resist using both existing and proposed control charts. The dataset, sourced from Montgomery (2007), includes 45 samples, each consisting of measurements from 5 wafers, with measurements taken at 1-hour intervals. Flow width measurements are in microns. The first 25 samples are treated as the Phase I dataset, assuming the process is IC. The subsequent 20 samples (from sample 26 to 45) constitute the Phase II dataset, where each observation is incremented by 0.1 microns to simulate an OC process. We apply competing control charts with an IC ARL of 500 and a sample size of 5, using parameters $\lambda = 0.2$, $A = 0$, $B = 1$. Figures 4, 5, and 6 illustrate the performance of the proposed control chart under different error conditions (0, 0.1, and 0.25). These figures plot the control statistics against the sample number, showcasing the CL and indicating when the process is considered OC.

Figure 4: The proposed control chart with error = 0. The chart remains stable for the first 25 samples, indicating an IC process. From sample 26 onwards, the chart signals an OC process, reflecting the introduced shift.

Figure 5: Proposed control chart with error = 0.1. Similar to Figure 4, the chart remains stable for the first 25 samples and indicates an OC process starting from sample 26, but with a slight delay compared to the error-free scenario.

Figure 6: Proposed control chart with error = 0.25. The chart shows stability for the first 25 samples and signals an OC process starting from sample 26. The higher error rate results in a more pronounced deviation from CL.

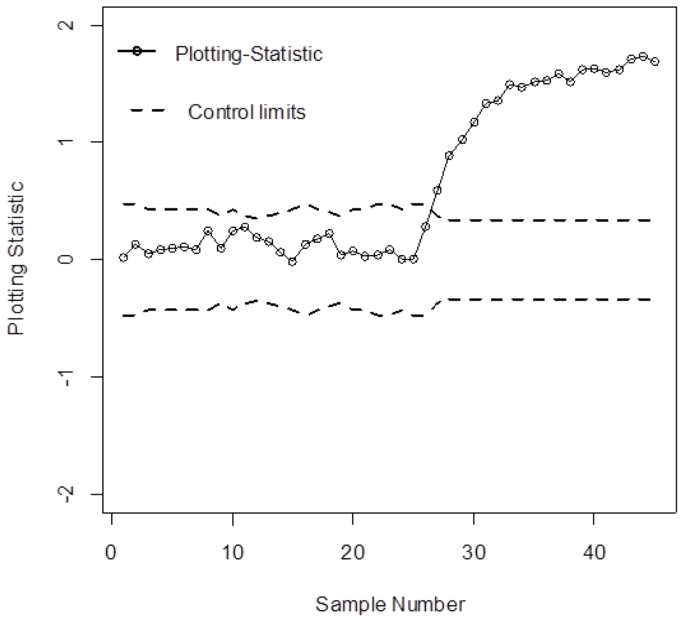

Figure 4: Proposed Control Chart with Error Ratio = 0

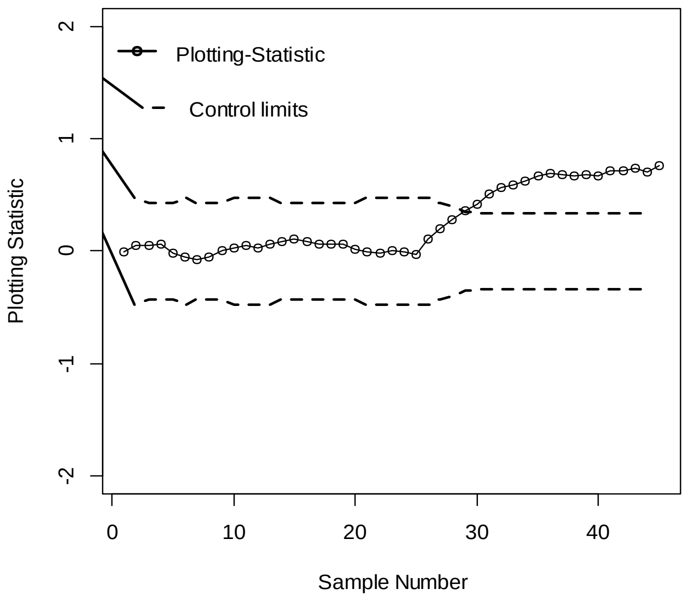

Figure 5: Proposed Control Chart with Error Ratio = 0.1

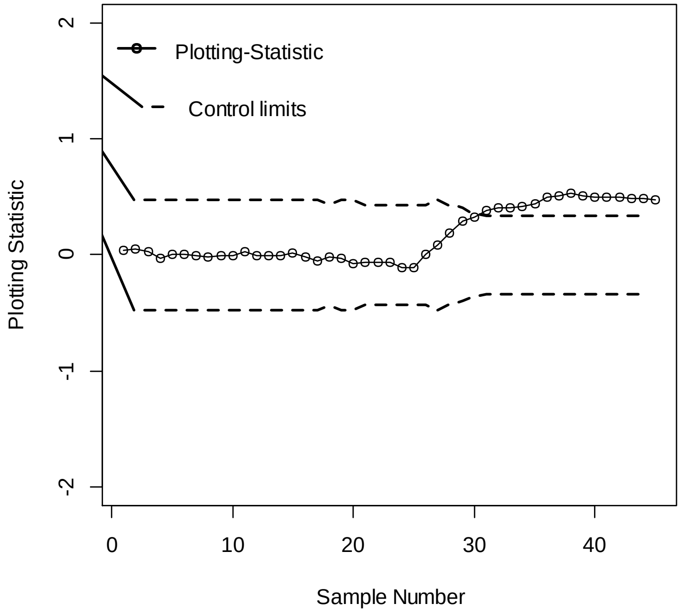

Figure 6: Proposed Control Chart with Error Ratio = 0.25

6. CONCLUSIONS

In this article, we introduced a new *VSS* approach for *ME* using *EWMA* statistics. We assessed the effectiveness of the of the *VSS EWMA* control chart for monitoring the mean, taking into account *ME* and incorporating a model that includes covariates. Our findings indicated that *ME* can adversely affect the *ARL* of the control chart. One potential solution to mitigate this issue is to take multiple measurements; however, this approach incurs additional costs and time. An economic analysis tailored to the specific context may help determine the feasibility of this solution. Nonetheless, the additional time may not be a significant concern, as modern industrial processes often involve automated measurement systems. To enhance the robustness of control charts in the presence of measurement error, further studies should explore hybrid models that integrate additional adaptive features, such as dynamically adjusting sample intervals alongside sample sizes. Additionally, economic considerations should be factored into the selection of control chart parameters to balance the trade-off between increased sampling costs and improved detection accuracy.

Data Availability

The datasets used and/or analyzed during the current study are available from the corresponding author upon reasonable request.

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