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INFERENCE ON THE ACCELERATED COMPETING FAILURE MODEL FROM THE RAYLEIGH DISTRIBUTION UNDER TYPE-I CENSORED DATA

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ABSTRACT

In a different area of life testing, designing experiments needs higher stress level than normal stress one. Also, the time to failure of experimental units is resulted by one of a fetal risk factors, only. In this paper, we consider the simple step-stress model with competing risks under Type-I censoring. The cumulative exposure model is assumed when the lifetime of test units follows Rayleigh distribution. Under this setup, we obtain the maximum likelihood estimates and the Bayes Estimators of the unknown parameters using Markov chain Monte Carlo (MCMC) method under various loss functions. Furthermore, to demonstrate the proposed methods, dataset is analyzed. Also, the confidence intervals are derived by using the asymptotic distributions of the maximum likelihood estimates. For comparison, we obtain the highest posterior density (HPD) credible intervals based on different prior distributions. Their performance is assessed through Monte Carlo simulations.

KEYWORDS

Simple step-stress accelerated life testing; Type-I censoring; Competing risks; Rayleigh distribution; Cumulative exposure model; Maximum likelihood estimation; Bayesian estimation; linear exponential and Squared error loss functions.

Abbreviations and Notation that used in this paper

ALT	accelerated life test
BEs	Bayes estimates
CDF	cumulative distribution function
CEM	cumulative exposure model
CIs	confidence intervals
HPD	highest posterior density
MCMC	markov chain monte carlo method
MH	metropolis–hastings
MLE	maximum likelihood estimate (or estimator)
MSE	mean squared error
PDF	probability density function
SEL	squard error loss.
τ	the total ideal test time.

τ'	the stress time changed.
T_{ij}	the lifetime of the jth failure cause.
δ_j	the indicator variable denote to cause of failure of the j-th unit.
s_i	i-th stress level for $i = 1,2$
$F_i(t)$	cumulative exposure model for i-th level stress
$f_{T,\delta}(t,j)$	the joint pdf of (T,δ)

1. INTRODUCTION

In life testing and reliability studies, the experimenter may not always obtain complete information on failure times for all experimental units. Thus, censoring is common to be performed. A censoring scheme (CS), which can balance between total time spent for the experiment, number of units used in the experiment, and the efficiency of statistical inference based on the results of the experiment. Among different censoring schemes, the conventional Type-I right censoring corresponds to the situation when the experiment gets terminated at a pre-fixed time point.

The life testing experiments, which units failure with more than one causes of failure, to measure effect of one cause of failure respected to the other causes which defined by competing risks problem. For analyzing a competing risks model, each complete observation must be in a bi-variate format composed of the failure time and the corresponding cause of failure.The causes of failure can be assumed independent or dependent. In most situations, the analysis of competing risks data assumes s-independent causes of failure. Prentice et al. [17] summarized the two approaches of modeling the competing risks data: the cause-specific hazard functions and the latent failure times for each risk factor. Berkson and Elveback [6], Cox and David [7] and Crowder [8] have all investigated the competing risks models with each risk factor having some specific parametric lifetime distributions.

With today’s high technology, some life tests result in none or very few failures, by the end of the test. In such cases, an approach is to do life test at higher-than-usual stress conditions, in order to obtain failures quickly. This can be achieved by using accelerated life test (ALT). ALT is achieved by subjecting units and components to test conditions such that failure occurs sooner. Thus, prediction of the long-term reliability can be made within a short period of time. Results from the ALT are used to extrapolate the unit characteristics at any future time and given at normal operating conditions. so, the ALTs are widely used in reliability analysis. When the experiment running, firstly at normal conditions is called partially step-stress partially.This problem is discussed with different authors, for Burr-XII distribution under type-I censoring Abd-Elfattah et al. [1], for Burr-XII distribution under type-I and adaptive type-II progressively hybrid censored Nassar et al. [15], for Exponentiated Gamma with unified hybrid censored data Alrashidi et al. [4] and Yao and Gui [23] evaluated the parameters and the accelerating factor based on constant stress for partially accelerating life tests when the potential failure times have an exponentiated Rayleigh distribution. The problem of the competing risks model is discussed under accelerate life test model in Ganguly and Kundu [9]. Almarashia et al. [3] discussed the problem of partially step-stress ALTs (accelerated life tests) form Rayleigh competing risks model. Type-II censored scheme Shi et al. [19] considered a constant-stress accelerated life test (CSALT) with competing risks for failure from exponential distribution under progressive Type II hybrid censoring. The maximum likelihood estimator and Bayes estimator of the parameter were derived. Xu and Tang [22] and Wu et al. [21] considered different inferential issues regarding the constant-stress accelerated competing failure models when the lifetime of different risk factors follows Weibull distributions. Based on Nelson's cumulative exposure (CE) model, Balakrishnan and Han [5], Han and Balakrishnan [11] developed the exact inference for a simple step-stress model with competing risks for failure from the exponential distribution under type-II and type-I censoring scheme, respectively. Liu and Shi [14] considered a simple step-stress model with progressively censored competing risks data from Weibull distribution.Abd-Elfattah et al. [2] considered the simple step-stress model with competing risks for failure from Weibull distribution under progressive Type-II censoring. Also, Han and Kundu [12] introduced a step-stress model with competing risks for failure from the generalized exponential distribution under type-I censoring. Nelson [16] introduced the step-stress ALTs that allows test conditions to change during testing. Among step stress experiments, the cumulative exposure model (CEM) is one of the most useful and used models. A simple step stress model starts with initial low stress and if it does not fail in a predetermined time point, τ, the stress level is increased. Simple step stress models contain only one stress change point. The CEM defined by Nelson [16] for simple step-stress testing with stresses and is given as

$$F_0(t) = \begin{cases} F_1(t) & \text{if } t \leq \tau \\ F_2(t-\tau+\tau') & \text{if } t \geq \tau \end{cases} \tag{1}$$

where τ' (the equivalent start time) is the solution of $F_1(\tau) = F_2(\tau')$

The Rayleigh distribution had played an important role in modeling the lifetime of random phenomena. It arises in many areas of applications, including reliability, life testing and survival analysis. It is frequently used to model wave heights in oceanography, and in communication theory to describe hourly median and instantaneous peak power of received radio signals. It has been used to model the frequency of different wind speeds over a year and a wind turbine sites. The distance from one individual to its nearest neighbor when the spatial pattern is generated by Poisson distribution follows a Rayleigh distribution. Gong et al. [10] discussed the statistical inference of the parameters, reliability function, and hazard function of the generalized Rayleigh distribution under progressive first-failure censoring samples. In communication theory, Rayleigh distribution is used to model scattered signals that reach a receiver by multiple paths. Depending on the density of scatter, the signal will display different fading characteristics. Rayleigh distribution is used to model dense scatter

The probability density function (p.d.f.) of Rayleigh distribution is given by

$$f(t) = \frac{t}{\theta^2}\, e^{-\frac{t^2}{2\theta^2}}, 0 < t < \infty, \theta > 0 \tag{2}$$

Cumulative distribution function (c.d.f) is given by

$$F(t) = 1 - e^{-\frac{t^2}{2\theta^2}} \tag{3}$$

where is θ the scale parameter. The corresponding survival function is

$$S(t) = e^{-\frac{t^2}{2\theta^2}}$$

The hazard function of t, denoted as $h(t) = f(t)/S(t)$ is obtained as

$$h(t) = \frac{t}{\theta^2}$$

This article is organized as follows, In Section 2, introduces the model formulation. The classical maximum likelihood estimation (MLEs) and The asymptotic confidence intervals (CIs) of the unknown parameters discussed In Section 3. In Section 4, the Bayes estimates (BEs) of model parameters using (MCMC) method are obtained. In Section 5, the performance of these confidence/credible intervals is evaluated in terms of probability coverages via Monte Carlo simulations. In Section 6, we present a real data analysis to prove the efficiency of the Rayleigh distribution in this article, and some concluding remarks are finally made in Section 7.

2. MODEL DESCRIPTION AND TEST ASSUMPTIONS

Let n independent units are put on a life test, and the ideal prefixed test time τ is considered. All n units are initially put on lower stress s_1 and run until time τ Then the stress is changed to the high level s_2 The following assumptions are provided for a LBE distributed lifetime units

1. The failure of a product occurs only due to one of the 2 independent competing failure causes with lifetimes T_1 and T_2. Then the lifetime of the product is $T = min(T_1, T_2)$.
2. Test procedure is done at stresses s_1 and s_2 $(s_1 < s_2)$ levels.
3. Scale parameter θ_{ij} is the log-linear function of stresses as
$$log(\theta_{ij}) = \beta_j + \beta_{0j}\,\varphi(s_i), i, j = 1,2 \tag{4}$$
where β_j, β_{0j} are unknown parameters, $\varphi(s_i)$ is a given decreasing function of stress level s. We adopt the Arrhenius model in this article, so $\varphi(s_i) = 1/s_i$.
4. For any stress level s_i, the lifetime of the jth failure cause T_{ij} $(i, j = 1,2)$ are independently and identically distributed variables from the Rayleigh distribution with scale parameter θ_{ij}
5. In this test, the cumulative exposure model which is defined by Nelson [14] for the simple step-stress testing with stresses s_1 and s_1 is used.

Based on the given assumptions above, Rayleigh cumulative exposure (RCE) model is given as follows. Firstly, the equivalent start time τ' for the RCE model which is the solution of $F_1(\tau) = F2(\tau')$ is equal to

$$\tau' = \left(\frac{\theta_{2j}}{\theta_{1j}}\right)\tau$$

Then, by replacing τ' in (1), the CDF of the lifetime T_{ij} under the simple step-stress ALT are given as

$$G_j(t) = \begin{cases} F_1(t) = 1 - e^{-\frac{t^2}{2\theta_{1j}^2}} & \text{if } 0 < t < \tau \\ F_2(t-\tau+\tau') = 1 - e^{-\frac{1}{2\theta_{2j}^2}\left[t-\tau\left(1-\frac{\theta_{2j}}{\theta_{1j}}\right)\right]^2} & \text{if} \end{cases} \quad (5)$$

and the corresponding probability density function (PDF) of T_j is given by

$$g_j(t) = \begin{cases} f_1(t) = \frac{1}{\theta_{1j}^2}\, t\, e^{-\frac{t^2}{2\theta_{1j}^2}} & \text{if } 0 < t < \tau \\ f_2(t-\tau+\tau') = \frac{1}{\theta_{2j}^2}\left[t-\tau\left(1-\frac{\theta_{2j}}{\theta_{1j}}\right)\right] e^{-\frac{1}{2\theta_{2j}^2}\left[t-\tau\left(1-\frac{\theta_{2j}}{\theta_{1j}}\right)\right]^2} & \text{if } \tau \le t < \infty \end{cases} \quad (6)$$

for $j = 1,2$. Since we will observe only the smaller of T_1 and T_2, let $T = min(T_1, T_2)$ denote the overall failure time of a test unit. Then, its CDF and PDF are readily obtained to be

$$F_T(t) = 1 - (1 - G_1(t))(1 - G_2(t))$$

$$F_T(t) = \begin{cases} 1 - e^{-\frac{1}{2}\left(\frac{1}{\theta_{11}{}^2}+\frac{1}{\theta_{12}{}^2}\right)t^2} & \text{if } 0 < t < \tau \\ 1 - e^{-\frac{1}{2}\left\{\frac{1}{\theta_{21}^2}\left\{t-\tau\left(1-\frac{\theta_{21}}{\theta_{11}}\right)\right\}^2+\frac{1}{\theta_{22}^2}\left\{t-\tau\left(1-\frac{\theta_{22}}{\theta_{12}}\right)\right\}^2\right\}} & \text{if } \tau \le t < \infty \end{cases} \quad (7)$$

$$f_T(t) = \begin{cases} \left(\frac{1}{\theta_{11}{}^2}+\frac{1}{\theta_{12}{}^2}\right) t\, e^{-\frac{1}{2}\left(\frac{1}{\theta_{11}{}^2}+\frac{1}{\theta_{12}{}^2}\right)t^2} & \text{if } 0 < t < \tau \\ \left\{\frac{1}{\theta_{21}^2}\left\{t-\tau\left(1-\frac{\theta_{21}}{\theta_{11}}\right)\right\}+\frac{1}{\theta_{22}^2}\left\{t-\tau\left(1-\frac{\theta_{22}}{\theta_{12}}\right)\right\}\right\} & \\ \times\, e^{-\frac{1}{2}\left\{\frac{1}{\theta_{21}^2}\left\{t-\tau\left(1-\frac{\theta_{21}}{\theta_{11}}\right)\right\}^2+\frac{1}{\theta_{22}^2}\left\{t-\tau\left(1-\frac{\theta_{22}}{\theta_{12}}\right)\right\}^2\right\}} & \text{if } \tau \le t < \infty \end{cases} \quad (8)$$

Let δ be the indicator of the failure cause, then we derive the joint PDF of (T, δ) as $f_{T,\delta}(t, j) = g_j(t)(1 - G_{j'}(t))$

$$f_{T,\delta}(t) = \begin{cases} \frac{1}{\theta_{1j}{}^2}\, t\, e^{-\frac{1}{2}\left(\frac{1}{\theta_{11}{}^2}+\frac{1}{\theta_{12}{}^2}\right)t^2} & \text{if } 0 < t < \tau \\ \frac{1}{\theta_{2j}^2}\left\{t-\tau\left(1-\frac{\theta_{2j}}{\theta_{1j}}\right)\right\} & \\ \times\, e^{-\frac{1}{2}\left\{\frac{1}{\theta_{21}^2}\left\{t-\tau\left(1-\frac{\theta_{21}}{\theta_{11}}\right)\right\}^2+\frac{1}{\theta_{22}^2}\left\{t-\tau\left(1-\frac{\theta_{22}}{\theta_{12}}\right)\right\}^2\right\}} & \text{if } \tau \le t < \infty \end{cases} \quad (9)$$

for $j, j' = 1,2$ and $j \ne j'$.

Suppose there are D_1 failures before the stress changing time τ. If we denote d_{1j} and $d_{2j}(j = 1,2)$ as the number of failures due to failure cause j under stress level s_1 and s_2, respectively, then $D_1 = d_{11} + d_{12}$. Since each failure time is accompanied by the corresponding cause of failure.let $\delta = (\delta_1, \delta_2, \dots, \delta_m)$ be the observed sequence of the cause of failure corresponding to the observed failure time $t = (t_1, t_2, \dots, t_d)$.

3. MAXIMUM LIKELIHOOD ESTIMATION

In this section, we discussed the point and asymptotic confidence intervals with MLE under the assumption of the cumulative exposure model, we formulate the likelihood function of $\Theta = (\theta_{1j}, \theta_{2j})$ based on the type-I censored data as

$$L(\Theta) \propto \prod_{i,j=1}^{2} \theta_{ij}^{-2d_{ij}} \sum_{i=1}^{d_{1.}} t\, e^{-\frac{1}{2}\left(\frac{1}{\theta_{11}{}^2}+\frac{1}{\theta_{12}{}^2}\right)\sum_{i=1}^{d_{1.}} t^2} \sum_{i=d_{1.}+1}^{d_{..}} \left\{t-\tau\left(1-\frac{\theta_{2j}}{\theta_{1j}}\right)\right\}$$
$$e^{-\frac{1}{2}\left\{\frac{1}{\theta_{21}{}^2}\sum_{i=d_{1.}+1}^{d_{..}}\left\{t-\tau\left(1-\frac{\theta_{21}}{\theta_{11}}\right)\right\}^2+\frac{1}{\theta_{22}{}^2}\sum_{i=d_{1.}+1}^{d_{..}}\left\{t-\tau\left(1-\frac{\theta_{22}}{\theta_{12}}\right)\right\}^2\right\}} \tag{10}$$
$$e^{-\frac{1}{2}(d-d_{..})\left\{\frac{1}{\theta_{21}{}^2}\left\{\tau_\xi-\tau\left(1-\frac{\theta_{21}}{\theta_{11}}\right)\right\}^2+\frac{1}{\theta_{22}{}^2}\left\{\tau_\xi-\tau\left(1-\frac{\theta_{22}}{\theta_{12}}\right)\right\}^2\right\}}$$

for $0 < t_{1:d} < \ldots\ldots < t_{d_{1.}:d} < \tau < t_{d_{1.}+1:d} < \ldots . < t_{d_{..}:d} < \tau_\xi$,

where $d_{..} = d_{1.} + d_{2.} = (d_{11} + d_{12}) + (d_{21} + d_{22})$

Then, the log Likelihood function of Θ is obtained as

$$\iota(\Theta) \propto -\sum_{i,j=1}^{2} 2d_{ij}\log\theta_{ij} + \sum_{i=1}^{d_{1.}} \log t - \frac{1}{2}\left(\frac{1}{\theta_{11}^2}+\frac{1}{\theta_{12}^2}\right)\sum_{i=1}^{d_{1.}} t^2$$
$$+\sum_{i=d_{1.}+1}^{d_{..}} log\left\{t-\tau\left(1-\frac{\theta_{2j}}{\theta_{1j}}\right)\right\}$$
$$-\frac{1}{2\theta_{21}^2}\sum_{i=d_{1.}+1}^{d_{..}}\left\{t-\tau\left(1-\frac{\theta_{21}}{\theta_{11}}\right)\right\}^2-\frac{1}{2\theta_{22}^2}\sum_{i=d_{1.}+1}^{d_{..}}\left\{t-\tau\left(1-\frac{\theta_{22}}{\theta_{12}}\right)\right\}^2$$
$$-(d-d_{..})\frac{1}{2\theta_{21}^2}\left\{\tau_\xi-\tau\left(1-\frac{\theta_{21}}{\theta_{11}}\right)\right\}^2-(d-d_{..})\frac{1}{2\theta_{22}^2}\left\{\tau_\xi-\tau\left(1-\frac{\theta_{22}}{\theta_{12}}\right)\right\}^2 \tag{11}$$

3.1 Point Estimation

The MLEs of the parameters θ_{1j} and θ_{2j} can be obtained by setting the first partial derivative of logL about θ_{1j} and θ_{2j} to zero, namely

$$\frac{\partial\iota(\theta)}{\partial\theta_{1j}} = \frac{-2d_{1j}}{\theta_{1j}} + \frac{1}{\theta_{1j}^3}\sum_{i=1}^{d_{1.}} t^2 - \frac{\theta_{2j}}{\theta_{1j}^2}\tau\omega_1{}^{-1} + \frac{1}{\theta_{1j}^2\theta_{2j}}\tau\omega_1 + (d-d_{..})\frac{1}{\theta_{1j}^2\theta_{2j}}\tau\omega_2 \tag{12}$$

$$\frac{\partial\iota(\theta)}{\partial\theta_{2j}} = \frac{-2d_{2j}}{\theta_{2j}} + \frac{1}{\theta_{1j}}\tau\omega_1{}^{-1} - \frac{1}{\theta_{2j}^2}\omega_1\left\{\frac{1}{\theta_{1j}}\tau - \frac{1}{\theta_{2j}}\omega_1\right\}$$
$$-(d-d_{..})\frac{1}{\theta_{2j}^2}\omega_2\left\{\frac{1}{\theta_{1j}}\tau - \frac{1}{\theta_{2j}}\omega_2\right\} \tag{13}$$

where,

$$\omega_1 = \sum_{i=d_{1.}+1}^{d_{..}}\left\{t - \tau\left(1 - \frac{\theta_{2j}}{\theta_{1j}}\right)\right\}$$

and

$$\omega_2 = \left\{\tau_\xi - \tau\left(1 - \frac{\theta_{22}}{\theta_{12}}\right)\right\}.$$

Now, we have a system of three nonlinear equations in three unknowns θ_{1j} and θ_{2j}. It is obvious that a closed form solution is quite difficult to obtain. It is obvious that a closed form solution is quite difficult to obtain.

3.2 Asymptotic Confidence Intervals

The asymptotic variance and covariance matrix of maximum likelihood estimates are given by the elements of the inverse of the Fisher information matrix as follows

$$I_{ij}(\underline{\theta}) \cong E\left\{-\frac{\partial^2 logL}{\partial\theta_i\theta_j}\right\}$$

Unfortunately, the exact mathematical expressions for the previous expectation are complicated to obtain. Therefore, the Fisher information matrix is given by

$$I_{ij}(\underline{\theta}) \cong \left\{-\frac{\partial^2 logL}{\partial\theta_i\theta_j}\right\}$$

which is obtained by approximating the expectation on operation E and replacing θ_{1j} and θ_{2j} with $\hat{\theta}_{1j}$ and $\hat{\theta}_{2j}$, respectively. The asymptotic variance and covariance matrix F of the maximum likelihood estimates can be written as follows:

$$I^{-1}(\theta_{1j},\theta_{2j}) = \begin{bmatrix} \frac{\partial^2\iota(\Theta)}{\partial\theta_{1j}^2} & \frac{\partial^2\iota(\Theta)}{\partial\theta_{1j}\partial\theta_{2j}} \\ \frac{\partial^2\iota(\Theta)}{\partial\theta_{1j}\partial\theta_{2j}} & \frac{\partial^2\iota(\Theta)}{\partial\theta_{2j}^2} \end{bmatrix}^{-1} \tag{14}$$

where,

$$\frac{\partial^2\iota(\Theta)}{\partial\theta_{1j}^2} = \frac{2d_{1j}}{\theta_{1j}^2} - \frac{3}{\theta_{1j}^4}\sum_{i=1}^{d_{1.}} t^2 - \frac{\theta_{2j}}{\theta_{1j}^3}\tau\omega_1^{-1}\left\{\frac{\theta_{2j}}{\theta_{1j}}\tau\omega_1^{-1} - 2\right\}$$
$$-\frac{\tau}{\theta_{1j}^3}\left\{\frac{\tau}{\theta_{1j}} + \frac{2}{\theta_{2j}}\omega_1\right\} + (d-d_{..})\frac{\tau}{\theta_{1j}^3}\left\{\frac{\tau}{\theta_{1j}} + \frac{2}{\theta_{2j}}\omega_2\right\}, \tag{15}$$

$$\frac{\partial^2 \iota(\Theta)}{\partial\theta_{1j}\partial\theta_{2j}} = \frac{\tau\omega_1^{-1}}{\theta_{1j}^2}\left\{\frac{\theta_{2j}}{\theta_{1j}}\tau\omega_1^{-1} - 1\right\} + \frac{\tau}{\theta_{1j}^2\theta_{2j}}\left\{\frac{\tau}{\theta_{1j}} - \frac{1}{\theta_{2j}}\omega_1\right\}$$
$$+(d - d_{..})\frac{\tau}{\theta_{1j}^2\theta_{2j}}\left\{\frac{\tau}{\theta_{1j}} - \frac{1}{\theta_{2j}}\omega_2\right\} \quad (16)$$

and

$$\frac{\partial^2 \iota(\Theta)}{\partial\theta_{2j}^2} = \frac{2d_{ij}}{\theta_{2j}^2} - \frac{1}{\theta_{1j}^2}\tau^2\omega_1^{-2} - \frac{1}{\theta_{1j}\theta_{2j}^2}\tau\left\{\frac{1}{\theta_{1j}}\tau - \frac{2}{\theta_{2j}}\omega_1\right\}$$
$$+\frac{1}{\theta_{2j}^3}\omega_1\left\{\frac{2}{\theta_{1j}}\tau - \frac{3}{\theta_{2j}}\omega_1\right\} - (d - d_{..})\frac{1}{\theta_{1j}\theta_{2j}^2}\tau\left\{\frac{1}{\theta_{1j}}\tau - \frac{2}{\theta_{2j}}\omega_2\right\}$$
$$+(d - d_{..})\frac{1}{\theta_{2j}^3}\omega_2\left\{\frac{2}{\theta_{1j}}\tau - \frac{3}{\theta_{2j}}\omega_2\right\} \quad (17)$$

Upon inverting this matrix and denoting $\hat{V}_{ij} = \hat{I}_{ij}^{-1}$ Therefore, the approximate $100(1-r)\%$ confidence intervals for the parameters θ_{1j} and θ_{2j} are expressed, respectively:

$$\left[max\left(0, \hat{\theta}_{1j} - Z_{\gamma/2}\sqrt{\hat{V}_j}(1,1), \hat{\theta}_{1j} + Z_{\gamma/2}\sqrt{\hat{V}_j}(1,1)\right)\right]$$

and

$$\left[max\left(0, \hat{\theta}_{2j} - Z_{\gamma/2}\sqrt{\hat{V}_j}(2,2), \hat{\theta}_{2j} + Z_{\gamma/2}\sqrt{\hat{V}_j}(2,2)\right)\right]$$

where $j = 1{,}2$ and $z_{\gamma/2}$ is the $(1 - z_{\gamma/2})th$ quantile of a standard normal distribution.

4. BAYES ESTIMATION

In this section, we calculate Bayes estimators of unknown parameters. Bayesian estimation approach has received a lot of attention for analysing failure time data.

4.1 Posterior Distribution

In Bayesian approach, parameters are considered as a random variables. The formulation of posterior distribution depend on likelihood function and prior distribution. So, prior infromation of Lomax parameters are considered as non informative prior for accelerate factor. The prior distributions of parameters $(\theta_{1j}, \theta_{2j})$ are formulated as

$$\eta(\theta_{1j}) \propto \frac{1}{\theta_{1j}}, \theta_{1j} > 0, j = 1{,}2 \quad (18)$$

and

$$\eta(\theta_{2j}) \propto \frac{1}{\theta_{2j}}, \theta_{2j} > 0, j = 1{,}2 \quad (19)$$

respectively.Then, the joint prior density of $(\theta_{1j}, \theta_{2j})$ is given by

$$\eta(\theta_{1j},\theta_{2j}) \propto \prod_{i,j=1}^{2} \theta_{ij}^{-1} \tag{20}$$

Hence, under model assumption proposed in Section 2 and prior distribution given by (20) the posterior distribution is defined by

$$\eta^{*}(\theta_{1j},\theta_{2j}) \propto L((\theta_{1j},\theta_{2j})\, \eta(\theta_{1j},\theta_{2j})$$

$$\propto \prod_{i,j=1}^{2} \theta_{ij}^{-2d_{ij}-1} \sum_{i=1}^{d_{1.}} t\, e^{-\frac{1}{2}(\frac{1}{\theta_{11}{}^{2}}+\frac{1}{\theta_{12}{}^{2}})\sum_{i=1}^{d_{1.}} t^{2}} \sum_{i=d_{1.+1}}^{d_{..}} \left\{t-\tau\left(1-\frac{\theta_{2j}}{\theta_{1j}}\right)\right\}$$
$$e^{-\frac{1}{2}\left\{\frac{1}{\theta_{21}{}^{2}}\sum_{i=d_{1.+1}}^{d_{..}}\left\{t-\tau\left(1-\frac{\theta_{21}}{\theta_{11}}\right)\right\}^{2}+\frac{1}{\theta_{22}{}^{2}}\sum_{i=d_{1.+1}}^{d_{..}}\left\{t-\tau\left(1-\frac{\theta_{22}}{\theta_{12}}\right)\right\}^{2}\right\}}$$
$$e^{-\frac{1}{2}(d-d_{..})\left\{\frac{1}{\theta_{21}{}^{2}}\left\{\tau\xi-\tau\left(1-\frac{\theta_{21}}{\theta_{11}}\right)\right\}^{2}+\frac{1}{\theta_{22}{}^{2}}\left\{\tau\xi-\tau\left(1-\frac{\theta_{22}}{\theta_{12}}\right)\right\}^{2}\right\}} \tag{21}$$

Multiply the joint prior density (20) with likelihood equation in (10) of $(\theta_{1j},\theta_{2j})$, the joint posterior density is constructed as follows

$$\eta^{*}(\theta_{1j},\theta_{2j}|t) = \varphi L((\theta_{1j},\theta_{2j})\, \eta(\theta_{1j},\theta_{2j}) \tag{22}$$

where,

$$\varphi = 1/\int_{0}^{\infty}\int_{0}^{\infty} L((\theta_{1j},\theta_{2j})\, \eta(\theta_{1j},\theta_{2j}) d\theta_{1j} d\theta_{2j}$$

is normalized constant.

The square error (SE) loss function and linear exponential (LINEX) loss function are considered to obtain BEs of the model parameters $(\theta_{1j},\theta_{2j})$ under type-I censoring.

Under Squared Error Loss Function (SE)

The Bayes estimates of the unknown parameters $\theta = (\theta_{1j},\theta_{2j}), j = 1,2$ under (SE) denoted by $\tilde{\theta}_{(BSE)}$; can be calculated through the following equations as follows

$$\tilde{\theta}_{(BSE)} = E(\theta|\underline{t}) = \int_{0}^{\infty} \theta\pi^{*}(\theta|\underline{t})\mathrm{d}\theta$$

Under Asymmetric LINEX Loss Function

Under the assumption that the minimal loss occurs at $\theta = (\alpha_{j},\beta_{1j},\beta_{2j}), j = 1,2$, the LINEX loss function can be expressed as

$$L_{BL}(\Delta) \propto e^{c\Delta} - c\Delta - 1, c \neq 1 \tag{23}$$

where $\Delta = (\hat{\theta} - \theta)$ is an estimate of θ.

The posterior expectation of the LINEX loss function is

$$E(L_{BL}(\hat{\theta} - \theta)) \propto e^{c\hat{\theta}}\, E[e^{c\hat{\theta}}] - c(\hat{\theta} - E[\theta]) - 1 \tag{24}$$

The Bayes estimator of θ, denoted by $\hat{\theta}$ under LINEX loss function, is the value of $\hat{\theta}$ which minimizes (23). It is

$$\hat{\theta}_{BL} = -\frac{1}{c} log \left\{E\left[e^{c\hat{\theta}}\right]\right\} \tag{25}$$

It may be noted here that the posterior distribution of $(\theta_{1j}, \theta_{2j})$ in equation (22) takes a ratio form that involves an integration in the denominator and cannot be reduced to a closed form. Hence, the evaluation of the posterior expectation for obtaining the Bayes estimator of $(\theta_{1j}, \theta_{2j})$ will be tedious.

The solution of equation (22) can be discussed with different methods such as numerical integrations or by Lindly approximations. One of most important method employed in this case called MCMC which employed in this section to present Bayes estimators under squared error loss (SEL) function and LINEX loss function. In the following, we discuss the problem of generate samples from the posterior distribution.

4.2 MCMC Approximation

One of the important MCMC approach called Metropolis-Hastings (M-H) algorithm. This is a special case of the Markov chain Monte Carlo (MCMC) approach, whose use has become widespread in the general statistical literature. The Metropolis-Hastings sampling need to conditional posterior PDF's of $(\theta_{1j}, \theta_{2j})$, hence posterior distribution given by (12) is reduced to conditional distributions described respectively as follows

$$\eta^*(\theta_{1j}|\theta_{2j}) \propto \prod_{j=1}^{2} \theta_{1j}^{-2d_{1j}-1} \sum_{i=1}^{d_{1.}} t\, e^{-\frac{1}{2}\left(\frac{1}{\theta_{11}{}^2}+\frac{1}{\theta_{12}{}^2}\right)\sum_{i=1}^{d_{1.}} t^2} \sum_{i=d_{1.}+1}^{d_{..}} \left\{t - \tau\left(1-\frac{\theta_{2j}}{\theta_{1j}}\right)\right\}$$
$$e^{-\frac{1}{2}\left\{\frac{1}{\theta_{2j}^2}\sum_{i=d_{1.}+1}^{d_{..}} \left\{t-\tau\left(1-\frac{\theta_{2j}}{\theta_{1j}}\right)\right\}^2\right\}} e^{-\frac{1}{2}(d-d_{..})\left\{\frac{1}{\theta_{2j}^2}\left\{\tau_\xi-\tau\left(1-\frac{\theta_{2j}}{\theta_{1j}}\right)\right\}^2\right\}}$$

$$\eta^*(\theta_{2j}|\theta_{1j}) \propto \prod_{j=1}^{2} \theta_{2j}^{-2d_{2j}-1} \sum_{i=d_{1.}+1}^{d_{..}} \left\{t - \tau\left(1-\frac{\theta_{2j}}{\theta_{1j}}\right)\right\} e^{-\frac{1}{2}\left\{\frac{1}{\theta_{2j}^2}\sum_{i=d_{1.}+1}^{d_{..}} \left\{t-\tau\left(1-\frac{\theta_{2j}}{\theta_{1j}}\right)\right\}^2\right\}}$$
$$e^{-\frac{1}{2}(d-d_{..})\left\{\frac{1}{\theta_{2j}^2}\left\{\tau_\xi-\tau\left(1-\frac{\theta_{2j}}{\theta_{1j}}\right)\right\}^2\right\}}$$

The conditional posterior distributions of $(\theta_{1j}, \theta_{2j})$ in previous equations cannot be reduced analytically to well known distribution. The Metropolis-Hastings algorithm is used to generate random samples from these distributions, see Upadhyay and Gupta [20]. For more information concerning the application of M-H, readers may refer to Robert et al. [18]. The following algorithm is proposed to compute Bayes estimators of $U = U(\theta_{1j}, \theta_{1j})$ under SE and LINEX loss functions.

Algorithm (1)

1. Start with $\theta_{1j}^{(0)} = \hat{\theta}_{1jMLE}, \theta_{2j}^{(0)} = \hat{\theta}_{2jMLE}$
2. Set $i = 1$.

3. Generate $\theta_{1j}^{(i)}$ and $\theta_{2j}^{(i)}$ from equations (20)-(22) respectively
4. Set $i = i + 1$.
5. Repeat steps (3)–(4) K times.
6. The approximate means of U and e^{-cU} are given respectively by

$$E(U) = \frac{1}{K-M} \sum_{i=M+1}^{K} U\left(\theta_{1j}^{(i)}, \theta_{2j}^{(i)}\right) \tag{26}$$

$$E(e^{-cU}) = \frac{1}{K-M} \sum_{i=M+1}^{K} exp\left\{-cU\left(\theta_{1j}^{(i)}, \theta_{2j}^{(i)}\right)\right\} \tag{27}$$

where M is the burn-in period. Then, from equations (26) and (27), the Bayes estimators of U under balanced SE and balanced LINEX loss functions are given respectively by:

$$\widehat{U}_{SE} = \Omega \widehat{U}_{ML} + (1-\Omega)E(U) \tag{28}$$

$$\widehat{U}_{LINK} = -\frac{1}{C} In\left[\Omega e^{-c\widehat{U}_{ML}} + (1-\Omega)E(e^{-cU})\right] \tag{29}$$

It is clear that the balanced loss functions are more general, which include the MLE and both symmetric and asymmetric BEs as special cases.

5. REAL DATA ANALYSIS

In this section, we analyze a real-life data set from Lawless [13] for illustration purposes. The data comes from an experiment in which new models of a small electrical appliance were being tested. The appliances were operated repeatedly by an automatic testing machine. There are 18 different possible causes of failure for the appliance. We will focus on failure mode 9. Therefore, we denote $\delta_i = 1$ if the failure occurs in mode 9 and $\delta_i = 2$ if the failure occurs in any other mode. []

Data Set:

(11,2) (35,2) (49,2) (170,2) (329,2) (381,2) (708,2) (958,2) (1062,2)
(1167,1) (1594,2) (1925,1) (1990,1) (2223,1) (2327,2) (2400,1) (2451,2)
(2471,1) (2551,1) (2568,1) (2694,1) (2702,2) (2761,2) (2831,2) (3034,1)
(3059,2) (3112,1) (3214,1) (3478,1) (3504,1) (4329,1) (6976,1) (7846,1)

We will assume $\tau = 25$ and time truncation $T = 30$. Hence, we obtained Type-I censored data (1 for censored and 0 for non-censored) with level of stress (indictor by 1 and 2) represented in the following Table 1.

Table 1
Type-I Censored Data after Truncation and Level of Stress

Observation Index	Time	Event	Stress level	Censored
1	11.67	1	1	1
2	19.25	1	1	1
3	19.90	1	1	1
4	22.23	1	1	1
5	24.00	1	1	1
6	24.71	1	1	1
7	25.51	1	0	1
8	25.68	1	0	1
9	26.94	1	0	1
10	30.34	1	0	0
11	31.12	1	0	0
12	32.14	1	0	0
13	34.78	1	0	0
14	35.04	1	0	0
15	43.29	1	0	0
16	69.76	1	0	0
17	78.46	1	0	0
18	0.11	2	1	1
19	0.35	2	1	1
20	0.49	2	1	1
21	1.70	2	1	1
22	3.29	2	1	1
23	3.81	2	1	1
24	7.08	2	1	1
25	9.58	2	1	1
26	10.62	2	1	1
27	15.94	2	1	1
28	23.27	2	1	1
29	24.51	2	1	1
30	27.02	2	0	1
31	27.61	2	0	1
32	28.31	2	0	1
33	30.59	2	0	0

We divided all data by 100 for illustrative of convergence. To determine whether the data makes a good fit for the Rayleigh distribution, we made a Kolmogorov Smirnov goodness-of-fit test for the type-I censored data. The MLE of the parameter $\theta = 20.5160$ and K-S statistic is 0.1647 with p-value is 0.2983. The results of p-values for each stress level $s_i, i = 1{,}2$ shows us that the distributions provide an excellent fit to the given data because all p-values exceed 0.05.

Also, the following figures; empirical cdf and histogram, shows that the data fit the given distribution.

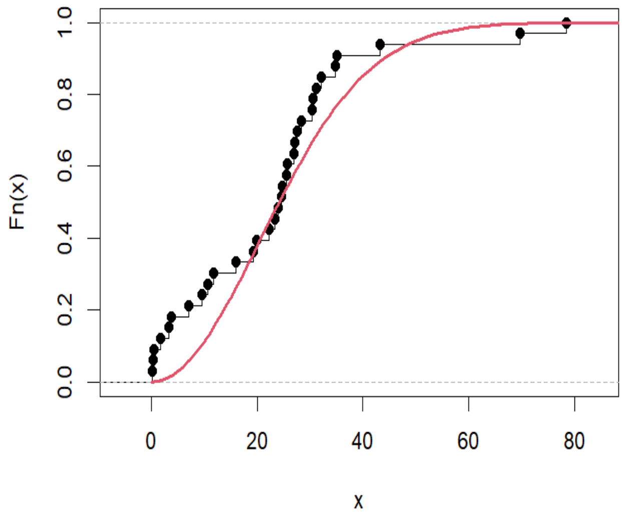

Figure 1: Plots of the Empirica CDF of Rayleigh Distribution from Dataset

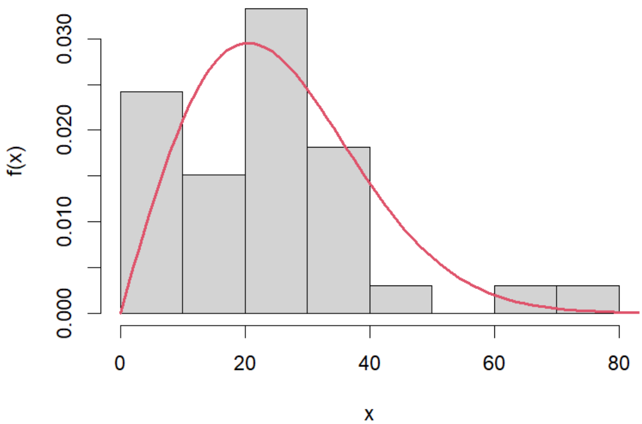

Figure 2: Histogram Density of Rayleigh Distribution from Dataset

Also,we compute the parameters of MLEs of $\theta_{11}, \theta_{12}, \theta_{21}$, and θ_{22}. Bayes estimates is computed utilizing the MH algorithm with the informative prior. Note that the non-informative prior are assumed where $a_{11} = b_{11} = a_{12} = b_{12} = a_{21} = b_{21} = a_{22} = b_{22} = 0$. Eventually, 2000 burn-in samples are terminated from the entire 10000 samples generated by the posterior density, and adopted technique to produce Bayes estimates under SE and LINEX (with LN-1: $c = 0.5$ and LN-2: $c = -0.5$) loss function.The MLEs and BEs of $\theta_{11}, \theta_{12}, \theta_{21}$ and θ_{22} are introduced in Table 2.

Table 2
MLE's and BEs based on Type-I Censoring Scheme for the given Real Data Set

Parm	MLE	BE: Non-IF		
		SE	LN-1	LN-2
$\boldsymbol{\theta_{11}}$	17.04133	16.92059	16.91672	16.92442
$\boldsymbol{\theta_{12}}$	17.08704	16.94032	16.92745	16.95365
$\boldsymbol{\theta_{21}}$	22.27253	23.09254	23.01561	23.18192
$\boldsymbol{\theta_{22}}$	22.17453	22.39877	22.38491	22.41293

6. SIMULATION STUDY

We will use a simulation study to estimate the unknown parameters of the Rayliegh distribution under Type-I censoring based on the SS-ALT model in order to assess the effectiveness of the proposed approach. For different sample combinations, the MLE and Bayesian estimation methods' respective MSE, Average and coverage probability (CP) are evaluated Sample combinations:

- The sample size n has been taken as $30.60.90$ and 120
- Different values of parameters as actual values have been chosen as: $\theta_{11} = 0.50$, $\theta_{12} = 0.51, \theta_{21} = 0.52$ and $\theta_{22} = 0.53$
- Time (Time censoring) has been taken as $T = 1.2, 1.5$ and 1.75
- Stress constant has been chosen as $\tau = 0.8$
- We replicate the number of replications 1000 times
- To obtain the MLEs of the model parameters, the Newton-Raphson method is employed to simultaneously solve the nonlinear equations.
- To obtain the Bayesian of the model parameters, the Metropolis-Hastings (M-H) algorithm is employed.
- Number of MCMC: 10,000
- Bayesian hyper-parameters:
    1. Informative (IF) prior case:
    $$a_{11} = 2.5, b_{11} = 5.5, a_{12} = 2.5, b_{12} = 5.5 a_{21} = 2.5, b_{21} = 5.5, a_{22} = 2.5, b_{22} = 5.5,$$
    2. Non-informative (Non-IF) prior case:
    $$a_{11} = b_{11} = a_{12} = b_{12} = a_{21} = b_{21} = a_{22} = b_{22}$$
- Linex constant values:

$$LN - 1: C = 0.5,\ LN - 1: C = -0.5$$

All the average estimates (Avg) and Mean Square Error (MSE) for methods are reported in Table 3 to Table 5 for different combinations of samples size n and different time censoring T. Further, the first column donates the (Avg.) and in the second column, related (MSEs).

For confidence intervals, we have asymptotic confidence interval for MLEs and HPD for Bayesian estimates based on MCMC which reported in Table 6 to Table 8 for different combinations of samples size n and different time censoring T. Further, the first column represents lower bound confidence interval, the second column represents upper bound of CI, the third column represent the average interval lengths (AILs) and in the last column, related coverage probabilities (CPs) in prencetage (%)

Table 3
Point Estimates (Avg. and MPoint Estimates (Avg. and MSE) of MLE and BE for Simple Step Stress Rayleigh Model under Censoring Type-I given $T = 1.2$

n	Parm	Estimate	MLE	BE: Non-IF			BE: IF		
				SE	LN-1	LN-2	SE	LN-1	LN-2
30	θ_{11}	Avg.	0.61729	0.61822	0.61818	0.61826	0.61805	0.61801	0.61809
		MSE	0.01880	0.01965	0.01964	0.01966	0.01943	0.01942	0.01944
	θ_{12}	Avg.	0.44328	0.44495	0.44493	0.44498	0.44409	0.44406	0.44411
		MSE	0.01155	0.01190	0.01190	0.01190	0.01179	0.01180	0.01179
	θ_{21}	Avg.	0.90575	0.90564	0.90555	0.90573	0.90472	0.90462	0.90482
		MSE	0.18684	0.18643	0.18634	0.18653	0.18551	0.18541	0.18561
	θ_{22}	Avg.	0.42171	0.42510	0.42508	0.42513	0.42612	0.42610	0.42615
		MSE	0.01676	0.01695	0.01695	0.01695	0.01688	0.01688	0.01688
60	θ_{11}	Avg.	0.62202	0.62250	0.62246	0.62253	0.62231	0.62227	0.62234
		MSE	0.01895	0.01974	0.01973	0.01975	0.01950	0.01949	0.01951
	θ_{12}	Avg.	0.44446	0.44496	0.44494	0.44498	0.44491	0.44489	0.44493
		MSE	0.01016	0.01038	0.01038	0.01038	0.01026	0.01026	0.01026
	θ_{21}	Avg.	0.86686	0.86880	0.86871	0.86888	0.86885	0.86877	0.86894
		MSE	0.15314	0.15476	0.15468	0.15483	0.15349	0.15342	0.15357
	θ_{22}	Avg.	0.42016	0.42413	0.42411	0.42416	0.42493	0.42490	0.42496
		MSE	0.01644	0.01650	0.01651	0.01650	0.01633	0.01633	0.01632
90	θ_{11}	Avg.	0.61775	0.61868	0.61865	0.61872	0.61825	0.61822	0.61829
		MSE	0.01741	0.01828	0.01827	0.01829	0.01827	0.01826	0.01828
	θ_{12}	Avg.	0.44271	0.44337	0.44335	0.44339	0.44295	0.44293	0.44297
		MSE	0.01023	0.01048	0.01048	0.01048	0.01048	0.01049	0.01048
	θ_{21}	Avg.	0.84638	0.84925	0.84917	0.84934	0.84902	0.84894	0.84911
		MSE	0.13620	0.13902	0.13895	0.13910	0.13773	0.13765	0.13780
	θ_{22}	Avg.	0.41895	0.42348	0.42345	0.42350	0.42270	0.42267	0.42272
		MSE	0.01627	0.01647	0.01648	0.01647	0.01633	0.01633	0.01633
120	θ_{11}	Avg.	0.61877	0.61870	0.61867	0.61874	0.61902	0.61899	0.61906
		MSE	0.01757	0.01814	0.01813	0.01815	0.01828	0.01827	0.01829
	θ_{12}	Avg.	0.44170	0.44313	0.44311	0.44314	0.44276	0.44274	0.44278
		MSE	0.01022	0.01054	0.01054	0.01053	0.01049	0.01049	0.01048
	θ_{21}	Avg.	0.83872	0.84103	0.84095	0.84112	0.84093	0.84085	0.84101
		MSE	0.12981	0.13092	0.13085	0.13099	0.13049	0.13042	0.13056
	θ_{22}	Avg.	0.41839	0.42257	0.42254	0.42260	0.42255	0.42252	0.42258
		MSE	0.01617	0.01620	0.01620	0.01620	0.01612	0.01612	0.01612

Table 4
Point Estimates (Avg. and MSE) of MLE and BE for Simple Step Stress Rayleigh Model under Censoring Type-I given $T = 1.5$

n	Parm	Estimate	MLE	BE: Non-IF			BE: IF		
				SE	LN-1	LN-2	SE	LN-1	LN-2
30	θ_{11}	Avg.	0.61352	0.61346	0.61342	0.61350	0.61349	0.61345	0.61353
		MSE	0.01743	0.01793	0.01792	0.01794	0.01792	0.01791	0.01793
	θ_{12}	Avg.	0.44053	0.44239	0.44237	0.44242	0.44205	0.44203	0.44208
		MSE	0.01227	0.01262	0.01262	0.01262	0.01255	0.01255	0.01255
	θ_{21}	Avg.	0.90977	0.91618	0.91607	0.91629	0.91366	0.91355	0.91377
		MSE	0.19134	0.19634	0.19622	0.19646	0.19329	0.19318	0.19340
	θ_{22}	Avg.	0.45064	0.45893	0.45889	0.45896	0.45869	0.45866	0.45873
		MSE	0.01281	0.01325	0.01325	0.01325	0.01317	0.01317	0.01317
60	θ_{11}	Avg.	0.61531	0.61497	0.61493	0.61501	0.61624	0.61619	0.61628
		MSE	0.01738	0.01788	0.01787	0.01789	0.01831	0.01830	0.01832
	θ_{12}	Avg.	0.43860	0.44060	0.44057	0.44062	0.44115	0.44113	0.44117
		MSE	0.01148	0.01168	0.01168	0.01168	0.01180	0.01180	0.01180
	θ_{21}	Avg.	0.87205	0.88278	0.88266	0.88289	0.88185	0.88174	0.88195
		MSE	0.15846	0.16546	0.16536	0.16557	0.16415	0.16405	0.16425
	θ_{22}	Avg.	0.44612	0.45564	0.45560	0.45568	0.45601	0.45597	0.45606
		MSE	0.01238	0.01307	0.01306	0.01307	0.01305	0.01305	0.01305
90	θ_{11}	Avg.	0.61766	0.61909	0.61905	0.61913	0.61797	0.61793	0.61801
		MSE	0.01732	0.01848	0.01847	0.01849	0.01808	0.01807	0.01809
	θ_{12}	Avg.	0.44196	0.44508	0.44506	0.44511	0.44517	0.44515	0.44519
		MSE	0.01049	0.01084	0.01084	0.01084	0.01059	0.01060	0.01059
	θ_{21}	Avg.	0.84970	0.86515	0.86502	0.86527	0.86646	0.86633	0.86659
		MSE	0.14084	0.15004	0.14994	0.15015	0.15041	0.15030	0.15052
	θ_{22}	Avg.	0.44953	0.46344	0.46338	0.46350	0.46309	0.46303	0.46314
		MSE	0.01110	0.01209	0.01209	0.01209	0.01216	0.01216	0.01217
120	θ_{11}	Avg.	0.61981	0.62052	0.62048	0.62056	0.62078	0.62074	0.62082
		MSE	0.01789	0.01878	0.01877	0.01879	0.01902	0.01901	0.01903
	θ_{12}	Avg.	0.44248	0.44543	0.44541	0.44545	0.44587	0.44585	0.44589
		MSE	0.01033	0.01061	0.01061	0.01061	0.01061	0.01061	0.01061
	θ_{21}	Avg.	0.85649	0.87416	0.87402	0.87431	0.87549	0.87535	0.87564
		MSE	0.14832	0.15889	0.15876	0.15901	0.15915	0.15903	0.15927
	θ_{22}	Avg.	0.44783	0.46323	0.46316	0.46330	0.46330	0.46323	0.46336
		MSE	0.01144	0.01269	0.01268	0.01269	0.01267	0.01267	0.01268

Table 5
Point Estimates (Avg. and MSE) of MLE and BE for Simple Stress Strength Rayleigh Model under Censoring Type-I given $T = 1.75$

n	Parm	Estimate	MLE	BE: Non-IF			BE: IF		
				SE	LN-1	LN-2	SE	LN-1	LN-2
30	θ_{11}	Avg.	0.61355	0.61483	0.61478	0.61487	0.61224	0.61219	0.61228
		MSE	0.01756	0.01870	0.01869	0.01871	0.01782	0.01781	0.01783
	θ_{12}	Avg.	0.44059	0.44238	0.44235	0.44240	0.44287	0.44284	0.44290
		MSE	0.01272	0.01298	0.01298	0.01298	0.01307	0.01307	0.01307
	θ_{21}	Avg.	0.92743	0.93780	0.93767	0.93793	0.93424	0.93412	0.93435
		MSE	0.21399	0.22235	0.22221	0.22249	0.21634	0.21621	0.21647
	θ_{22}	Avg.	0.47277	0.48187	0.48184	0.48191	0.48263	0.48259	0.48267
		MSE	0.01054	0.01098	0.01098	0.01098	0.01103	0.01103	0.01103
60	θ_{11}	Avg.	0.61532	0.61526	0.61521	0.61530	0.61572	0.61568	0.61577
		MSE	0.01714	0.01778	0.01777	0.01779	0.01795	0.01794	0.01797
	θ_{12}	Avg.	0.43948	0.44220	0.44218	0.44223	0.44324	0.44321	0.44326
		MSE	0.01089	0.01123	0.01123	0.01122	0.01110	0.01110	0.01109
	θ_{21}	Avg.	0.87356	0.89258	0.89246	0.89271	0.89167	0.89154	0.89180
		MSE	0.16295	0.17634	0.17622	0.17645	0.17269	0.17257	0.17281
	θ_{22}	Avg.	0.46886	0.48316	0.48311	0.48321	0.48324	0.48319	0.48329
		MSE	0.00933	0.01039	0.01038	0.01039	0.01033	0.01032	0.01033
90	θ_{11}	Avg.	0.61978	0.61944	0.61940	0.61948	0.61995	0.61991	0.61999
		MSE	0.01791	0.01839	0.01838	0.01840	0.01856	0.01855	0.01857
	θ_{12}	Avg.	0.44134	0.44451	0.44449	0.44454	0.44517	0.44514	0.44519
		MSE	0.01064	0.01099	0.01099	0.01099	0.01087	0.01087	0.01086
	θ_{21}	Avg.	0.86424	0.89169	0.89152	0.89185	0.89144	0.89128	0.89160
		MSE	0.15711	0.17382	0.17368	0.17396	0.17378	0.17365	0.17392
	θ_{22}	Avg.	0.47069	0.49037	0.49029	0.49044	0.48959	0.48952	0.48966
		MSE	0.00878	0.01070	0.01069	0.01071	0.01048	0.01047	0.01048
120	θ_{11}	Avg.	0.61840	0.61874	0.61870	0.61878	0.61794	0.61790	0.61798
		MSE	0.01727	0.01802	0.01801	0.01803	0.01798	0.01797	0.01799
	θ_{12}	Avg.	0.44261	0.44686	0.44684	0.44688	0.44703	0.44700	0.44705
		MSE	0.00999	0.01038	0.01038	0.01038	0.01035	0.01035	0.01035
	θ_{21}	Avg.	0.83418	0.86923	0.86904	0.86943	0.87148	0.87128	0.87167
		MSE	0.12603	0.14804	0.14788	0.14820	0.14983	0.14967	0.14999
	θ_{22}	Avg.	0.47176	0.49600	0.49590	0.49610	0.49691	0.49680	0.49701
		MSE	0.00820	0.01098	0.01097	0.01100	0.01126	0.01124	0.01128

Table 6
Interval Estimates (AIL and CP in %) of MLE and BE for Simple Stress Strength Rayleigh Model under Censoring Type-I given $T = 1.2$

n	Parm	Asy-CI		HPD: Non-IF		HPD: IF	
		AIL	CP	AIL	CP	AIL	CP
30	θ_{11}	0.27839	96.1	0.28599	96.3	0.28222	96.3
	θ_{12}	0.33046	98.4	0.34090	98.1	0.33800	98.3
	θ_{21}	0.76493	94.2	0.62378	95.4	0.64311	95.2
	θ_{22}	0.27812	98.9	0.30649	99.0	0.30531	98.6
60	θ_{11}	0.24986	96.2	0.26445	95.5	0.24832	96.7
	θ_{12}	0.30033	99.1	0.29594	98.3	0.28457	98.2
	θ_{21}	0.71053	93.3	0.60941	95.7	0.60313	96.3
	θ_{22}	0.25943	99.9	0.31045	99.0	0.31063	98.3
90	θ_{11}	0.23344	96.8	0.24322	96.9	0.24676	96.4
	θ_{12}	0.29627	99.3	0.29606	97.9	0.28995	99.0
	θ_{21}	0.67568	92.5	0.57697	95.3	0.55468	95.1
	θ_{22}	0.24601	99.9	0.31374	98.4	0.30005	98.7
120	θ_{11}	0.23070	96.3	0.23260	97.4	0.24209	96.8
	θ_{12}	0.29242	99.1	0.29101	98.6	0.28669	98.6
	θ_{21}	0.65895	92.4	0.55279	95.5	0.55771	95.4
	θ_{22}	0.23891	99.6	0.28834	98.4	0.29515	98.1

Table 7
Interval Estimates (AIL and CP in %) of MLE and BE for Simple Stress Strength Rayleigh Model under Censoring Type-I given $T = 1.5$

n	Parm	Asy-CI		HPD: Non-IF		HPD: IF	
		AIL	CP	AIL	CP	AIL	CP
30	θ_{11}	0.26442	96.1	0.27012	95.8	0.27377	95.8
	θ_{12}	0.33827	98.5	0.36680	97.8	0.34531	98.6
	θ_{21}	0.77869	94.9	0.64609	95.4	0.63600	95.6
	θ_{22}	0.31641	99.0	0.35097	99.0	0.34854	99.2
60	θ_{11}	0.25057	97.0	0.24704	96.5	0.25845	96.4
	θ_{12}	0.31339	99.6	0.31089	98.4	0.32229	97.5
	θ_{21}	0.72871	92.7	0.61957	95.1	0.61939	95.4
	θ_{22}	0.28680	99.5	0.32868	99.2	0.32638	98.8
90	θ_{11}	0.23126	96.2	0.24733	96.1	0.24459	96.1
	θ_{12}	0.30019	99.0	0.30907	98.8	0.29942	98.6
	θ_{21}	0.70311	92.6	0.60954	95.3	0.59711	96.2
	θ_{22}	0.26656	99.1	0.33692	99.7	0.32044	98.8
120	θ_{11}	0.23318	96.8	0.24564	97.1	0.24342	96.9
	θ_{12}	0.29792	99.4	0.29547	99.5	0.29662	98.3
	θ_{21}	0.73473	92.1	0.60238	95.3	0.61090	95.4
	θ_{22}	0.26852	99.6	0.34720	99.2	0.33303	99.8

Table 8
Interval Estimates (AIL and CP in %) of MLE and BE for Simple Step Stress Rayleigh Model under Censoring Type-I given $T = 1.75$

n	Parm	Asy-CI		HPD: Non-IF		HPD: IF	
		AIL	CP	AIL	CP	AIL	CP
30	θ_{11}	0.26805	96.9	0.27614	97.1	0.27145	96.1
	θ_{12}	0.34861	98.9	0.36376	98.7	0.37772	98.0
	θ_{21}	0.85921	93.5	0.72401	95.9	0.68547	95.3
	θ_{22}	0.33418	99.2	0.34970	99.1	0.35845	97.9
60	θ_{11}	0.24301	96.7	0.25399	96.4	0.25015	97.4
	θ_{12}	0.30159	98.4	0.30884	99.2	0.31324	98.1
	θ_{21}	0.76398	92.6	0.65952	95.1	0.62183	95.2
	θ_{22}	0.29314	98.4	0.32988	99.0	0.31229	98.9
90	θ_{11}	0.23394	96.9	0.23364	97.2	0.24233	97.5
	θ_{12}	0.30178	99.1	0.30730	98.7	0.29314	98.9
	θ_{21}	0.77061	92.1	0.64326	95.5	0.63705	95.2
	θ_{22}	0.28450	98.7	0.32968	99.4	0.33397	99.1
120	θ_{11}	0.22375	96.3	0.23124	97.2	0.24122	97.1
	θ_{12}	0.28951	99.1	0.28320	98.4	0.27295	98.9
	θ_{21}	0.64823	92.0	0.55202	96.0	0.54440	95.2
	θ_{22}	0.27184	99.1	0.33112	98.8	0.33676	99.0

To show the MCMC convergence, using samples, trace plots of the posterior distributions of $\theta_{11}, \theta_{12}, \theta_{21}$ and θ_{22} are plotted in Figures (3-6). It displays 10,000 outputs for the unknown parameters $\theta_{11}, \theta_{12}, \theta_{21}$ and θ_{22} with their sample mean (represented by soled lines (—)) and two bounds BCIs (represented by dashed lines (- - -)). It indicates that the MCMC sampling procedure has converged well. It also shows that the burn-in sample has an appropriate size to erase the effect of the initial values. Also,the marginal posterior density estimates of $\theta_{11}, \theta_{12}, \theta_{21}$ and θ_{22} with their histograms and sample means (vertical dashed lines (:)), based on MCMC samples of size 10,000 are represented in Figures (3-6).Thus, the results of the proposed methods give a good explanation to our model.

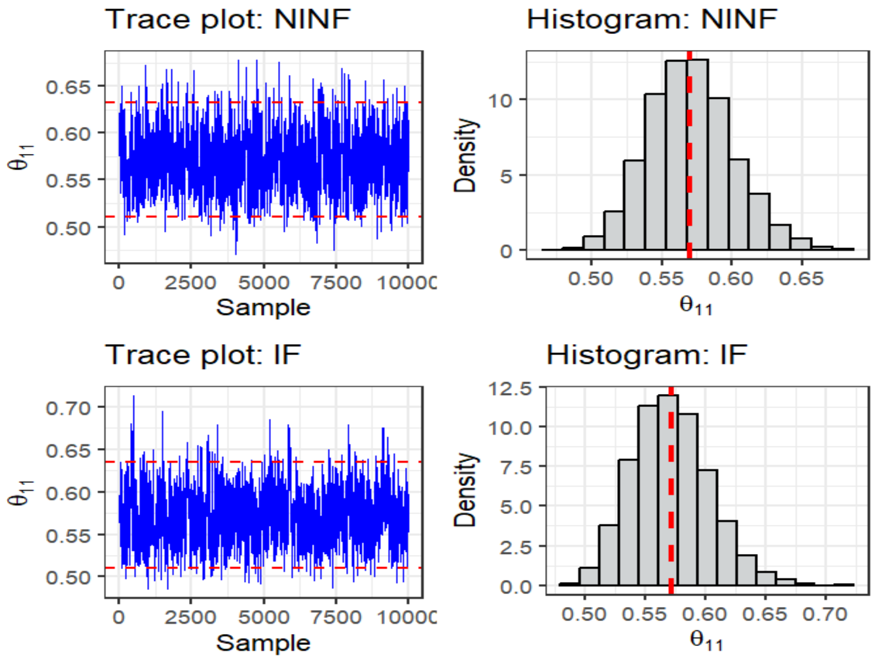

Figure 3: Convergence of MCMC Estimates for θ_{11}

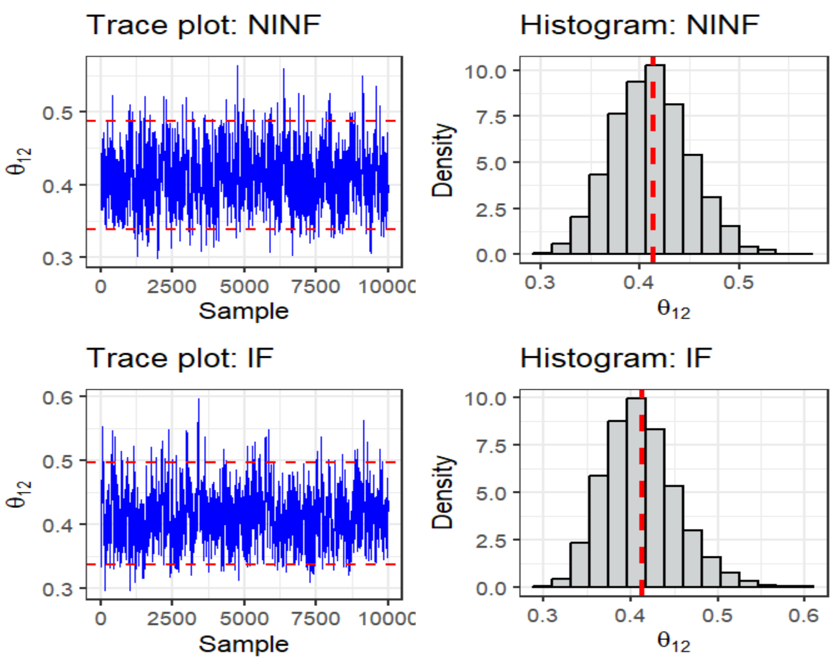

Figure 4: Convergence of MCMC Estimates for θ_{12}

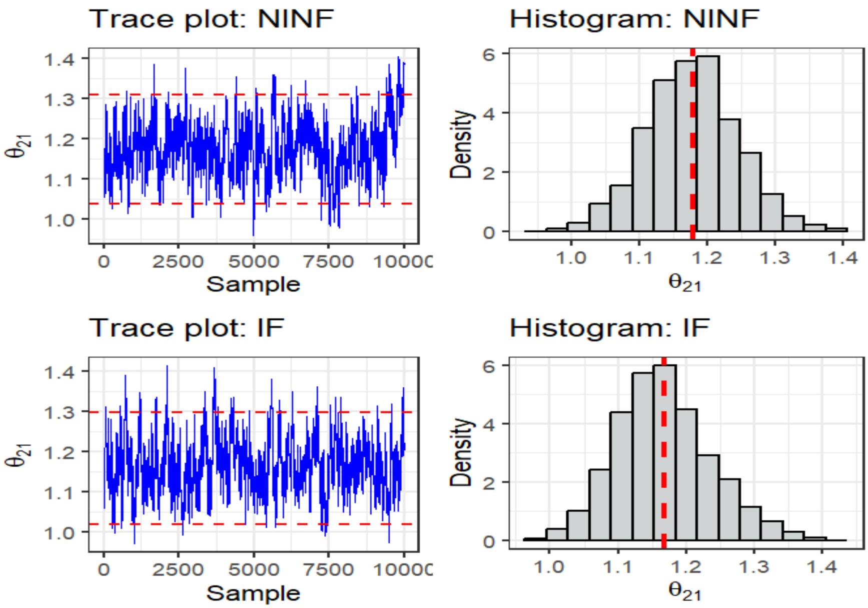

Figure 5: Convergence of MCMC Estimates for θ_{21}

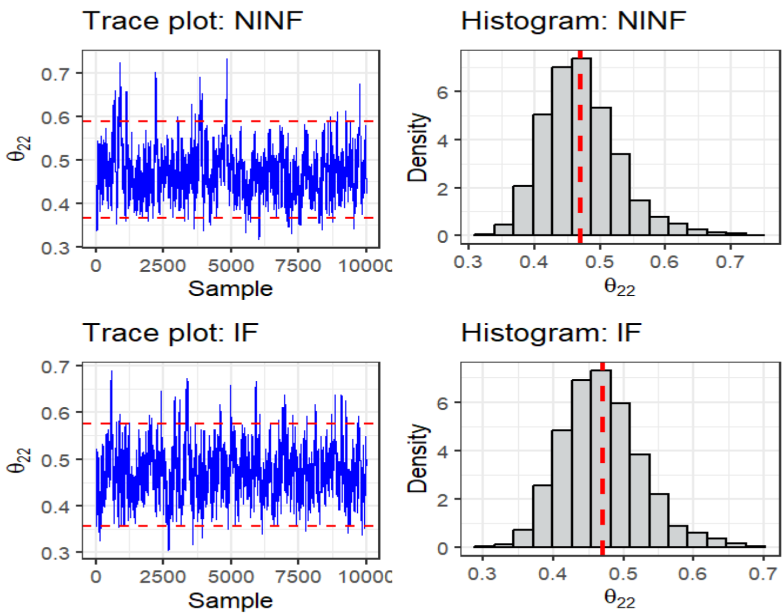

Figure 6: Convergence of MCMC Estimates for θ_{22}

7. CONCLUSION

In this paper, we studied a simple step-stress ALT model for the Rayleigh distribution under type-I censored data in presence of competing causes of risks. We have derived the MLEs and asymptotic confidence interval estimates for the unknown parameters. Also, we computed BEs and the corresponding HPD interval estimates under informative priors based on two different types of loss functions LINEX and squared error loss functions. We have then performed a simulation study to assess the performance of all these procedures and an explanatory instance has been offered to demonstrate all the methods of inference developed. The calculations have been made based on different sample sizes and different Time truncation (Time censoring) $T = 1.2, 1.5$ and 1.75. Also, real data is analyzed. To determine whether the data makes a good fit for the Rayleigh distribution, we made a Kolmogorov Smirnov goodness-of-fit test for the type-I censored data.

From the results in Tables (1-6), we have observed the following:

1. For the real data sets, the Rayleigh distribution gives a good fit for the real data.
2. The Figures (1-2); empirical cdf and histogram, shows that the data fit the given distribution.
3. The MSEs of MLEs of the considered parameters decrease as the sample size increases, except for a few cases. This may be due to fluctuations in data.
4. The BEs of the considered parameters decrease as the sample size increases, except for a few cases. This may be due to fluctuations in data.
5. The length of approximate and credible CIs decreases as the sample size increases
6. The CP% of approximate CIs increases as the sample size increases and the CP% of the highest posterior density (HPD) credible intervals increases as the sample size increases.

Finally, the simulation results demonstrate that both the MLEs and Bayesian estimates become better in terms of MSEs as sample size increases. For future research, we will study a simple step-stress ALT model for the Rayleigh distribution under censored data in presence of dependent competing causes of risks and We will compare the results in the two cases.

REFERENCES

[1] Abd-Elfattah, A., EL-Sherpieny, E. and Nassr, S. (2009). The bayesian estimation in step partially accelerated life tests for the burr type xii parameters using type i censoring. *The Egyptian Statistical Journal*, 53(2), 125-137.

[2] Abdelfattah, A.M., El-Sherpieny, E.-S.A. and Khalil, O.F. (2023). Analysis simple step stress modelunder competing extension weibull failure distribution based on progressive type-ii censoring. *Mathematical Modelling of Engineering Problems*, 10(1), 93-108.

[3] Almarashia, A.M., Ali, A., Abd-Elmougod, G. and Abdel-Khalek, S. (2019). Statistical analysis of Rayleigh competing risks models based on partially step stress type-ii censoring samples. *Journal of Nonlinear Science and Applications*, 12, 230-238.

[4] Alrashidi, A., Rabie, A., Mahmoud, A.A., Nasr, S.G., Mustafa, M.S., Al Mutairi, A., Hussam, E. and Hossain, M.M. (2024). Exponentiated gamma constant-stress partially accelerated life tests with unified hybrid censored data: Statistical inferences. *Alexandria Engineering Journal*, 88, 268-275.

[5] Balakrishnan, N. and Han, D. (2008). Exact inference for a simple step-stress model with competing risks for failure from exponential distribution under type-ii censoring. *Journal of Statistical Planning and Inference*, 138(12), 4172-4186.

[6] Berkson, J. and Elveback, L. (1960). Competing exponential risks, with particular reference to the study of smoking and lung cancer. *Journal of the American Statistical Association*, 55(291), 415-428.

[7] Cox, D.R. (1959). The analysis of exponentially distributed life-times with two types of failure. *Journal of the Royal Statistical Society Series B: Statistical Methodology*, 21(2), 411-421.

[8] Crowder, M.J. (2001). *Classical competing risks*. Chapman and Hall/CRC.

[9] Ganguly, A. and Kundu, D. (2016). Analysis of simple step-stress model in presence of competing risks. *Journal of Statistical Computation and Simulation*, 86(10), 1989-2006.

[10] Gong, Q., Chen, R., Ren, H. and Zhang, F. (2024). Estimation of the reliability function of the generalized rayleigh distribution under progressive first-failure censoring model. *Axioms*, 13(9), 580.

[11] Han, D. and Balakrishnan, N. (2010). Inference for a simple step-stress model with competing risks for failure from the exponential distribution under time constraint. *Computational Statistics & Data Analysis*, 54(9), 2066-2081.

[12] Han, D. and Kundu, D. (2014). Inference for a step-stress model with competing risks for failure from the generalized exponential distribution under type-i censoring. *IEEE Transactions on Reliability*, 64(1), 31-43.

[13] Lawless, J.F. (2011). *Statistical models and methods for lifetime data*. John Wiley & Sons.

[14] Liu, F. and Shi, Y. (2017). Inference for a simple step-stress model with progressively censored competing risks data from weibull distribution. *Communications in Statistics-Theory and Methods*, 46(14), 7238-7255.

[15] Nassar, M., Nassr, S. and Dey, S. (2017). Analysis of burr type-xii distribution under step stress partially accelerated life tests with type-i and adaptive type-ii progressively hybrid censoring schemes. *Annals of Data Science*, 4, 227-248.

[16] Nelson, W. (1990). Accelerated life testing: statistical models, data analysis and test plans. *Accelerated life testing: statistical models data analysis and test plants*, 41.

[17] Prentice, R.L., Kalbfleisch, J.D., Peterson Jr, A.V., Flournoy, N., Farewell, V.T. and Breslow, N.E. (1978). The analysis of failure times in the presence of competing risks. *Biometrics*, 34(4), 541-554.

[18] Robert, C.P., Casella, G. and Casella, G. (1999). *Monte Carlo statistical methods*, volume 2. Springer.

[19] Shi, Y.M., Jin, L., Wei, C. and Yue, H.B. (2013). Constant-stress accelerated life test with competing risks under progressive type-ii hybrid censoring. *Advanced Materials Research*, 712, 2080-2083.

[20] Upadhyay, S. and Gupta, A. (2010). A bayes analysis of modified weibull distribution via markov chain monte carlo simulation. *Journal of Statistical Computation and Simulation*, 80(3), 241-254.

[21] Wu, M., Shi, Y. and Sun, Y. (2014). Inference for accelerated competing failure models from weibull distribution under type-i progressive hybrid censoring. *Journal of Computational and Applied Mathematics*, 263, 423-431.

[22] Xu, A. and Tang, Y. (2011). Objective bayesian analysis of accelerated competing failure models under type-i censoring. *Computational Statistics & Data Analysis*, 55(10), 2830-2839.

[23] Yao, H. and Gui, W. (2024). Inference on exponentiated rayleigh distribution with constant stress partially accelerated life tests under progressive type-ii censoring. *Journal of Applied Statistics*, 1-29. https://doi.org/10.1080/02664763.2024.2373930