Pak. J. Statist.
2024 Vol. 40(4), 403-413

A SCALE PARAMETERS AND MODIFIED RELIABILITY ESTIMATION FOR THE INVERSE EXPONENTIAL RAYLEIGH DISTRIBUTION

Shurooq A.K. AL-Sultany
Department of Mathematics, College of Science
Mustansiriyah University, Baghdad, Iraq
Email: shuroq80@uomustansiriyah.edu.iq

ABSTRACT

This paper present methods for estimating a scale parameters and modified reliability for the Inverse Exponential Rayleigh Distribution include Maximum Likelihood, rank set sampling and Cramér-von-Mises Estimations. In all the mentioned estimation methods, the Newton-Raphson iterative numerical method was used. Then a simulation was conducted to compare the three methods with six cases and different sample sizes. The comparisons between scale parameter estimates were based on values from Mean Square Error while it was based on values from Integrated Mean Square Error for the estimates of the modified reliability function. The results show that Cramér-von-Mises (MCV) estimators is the best among the other two methods for estimating the modified reliability function.

KEYWORDS

Maximum Likelihood Estimation; rank set sampling; Cramér-von-Mises Estimation; modified reliability. integral square error; Inverse Exponential Rayleigh Distribution.

1. INTRODUCTION

If a random variable X has Exponential Rayleigh distribution $ERD(a,b)$ with probability density function (pdf) given by **Hussein and et al. (2023)**:

$$f(x;a,b)_{ER} = \begin{cases} (a+bx)\, e^{-\left(ax+\frac{b}{2}x^2\right)}; x \geq 0; a,b > 0 \ldots \\ 0;\ \text{otherwise} \end{cases} \tag{1}$$

where a and b are scale parameters, then it is said that the random variable $Y = 1/X$ follows the inverse exponential Rayleigh distribution $IERD(a,b)$ with pdf is of the following form: Mohammed and Mohammed (2021)

$$g(y;a,b) = \begin{cases} (a+by^{-1})y^{-2}\, e^{-\left(ay^{-1}+\frac{b}{2}y^{-2}\right)}; y \geq 0; a,b > 0 \ldots \\ 0;\ \text{otherwise} \end{cases} \tag{2}$$

It is denoted by $Y \sim IERD(a,b)$,The Cumulative distribution function (CDF) of IERD is given by

$$G(y;a,b) = 1 - e^{-\left(ay^{-1}+\frac{b}{2}y^{-2}\right)}; y \geq 0; a,b > 0 \ldots \tag{3}$$

The reliability function $R(t)$ to some time $t \geq 0$, is defined as the probability that a component will be successfully operating without failure in the interval from 0 to time t,

$$R(t;a,b) = P(T > t) = \int_t^{\infty} g(y;a,b)dy$$

$$\Rightarrow R(t;a,b) = e^{-\left(at^{-1}+\frac{b}{2}t^{-2}\right)}; t \geq 0; a,b > 0 \ldots \tag{4}$$

And the hazard rate function for IERD to some time $t \geq 0$, is given by

$$H(t;a,b) = 1 - R(t;a,b)$$

$$\Rightarrow H(t;a,b) = (a + bt^{-1})t^{-2}; t \geq 0; a,b > 0 \ldots \tag{5}$$

2. MAXIMUM LIKELIHOOD ESTIMATION
Mohammed and Mohammed (2021)

Consider a random sample $\underline{y} = (y_1, y_2, \ldots, y_n)$ withdrawn from IERD with pdf given by equation (2). We want to express the complete data likelihood function, denoted as $L(a,b|\underline{y})$, for a given random sample

$$L(a,b|\,\underline{y}) = \prod_{i=1}^{n} g(y_i;a,b) = \prod_{i=1}^{n}\left[(a+by_i{}^{-1})y_i{}^{-2}\, e^{-\left(ay_i{}^{-1}+\frac{b}{2}y_i{}^{-2}\right)}\right]. \tag{6}$$

Now, we take the natural logarithm for equation (6), and then maximize the natural logarithm likelihood numerically.

$$\Lambda = \ln L(a,b|\,\underline{y}) = \sum_{i=1}^{n}\ln(a+by_i{}^{-1}) - 2\sum_{i=1}^{n} ln\, y_i - \sum_{i=1}^{n}(ay_i{}^{-1} + \frac{b}{2}\, y_i{}^{-2}) \ldots \tag{7}$$

Now,

$$\acute{\Lambda}_a = \frac{\partial\Lambda}{\partial a} = \sum_{i=1}^{n}\frac{1}{a+by_i{}^{-1}} - \sum_{i=1}^{n} y_i{}^{-1} = 0 \ldots \tag{8}$$

$$\acute{\Lambda}_b = \frac{\partial\Lambda}{\partial b} = \sum_{i=1}^{n}\frac{y_i{}^{-1}}{a+by_i{}^{-1}} - \frac{1}{2}\sum_{i=1}^{n} y_i^{-2} = 0 \ldots \tag{9}$$

Due to the non-linearity of the above equations, and by applying the Newton-Raphson Algorithm (NRA), we can find the parameters values that maximize the likelihood function. It is an Iterative approach used to find the roots of a real-valued function. **Kareem (2020).** The NRA is especially useful for solving nonlinear equations and finding the maximum or minimum of a function.

The solution in the NRA, to the likelihood equation at iteration $(k+1)$. It was obtained through the following iterative process. **Al-Sultany and Mohammed (2018)**.

$$\begin{bmatrix}\hat{a}^{k+1}\\ \hat{b}^{k+1}\end{bmatrix} = \begin{bmatrix}\hat{a}^{k}\\ \hat{b}^{k}\end{bmatrix} - J_{(k)}^{-1}.h\,;k = 0,1,2,\ldots;\ \ldots.(*)$$

where,

$$J_{(h)} = \begin{bmatrix}\acute{\Lambda}_{aa}\ \acute{\Lambda}_{ab}\\ \acute{\Lambda}_{ba}\ \acute{\Lambda}_{bb}\end{bmatrix}_{\substack{a=\hat{a}^k\\ b=\hat{b}^k}},\ h = \begin{bmatrix}\acute{\Lambda}_{a}\\ \acute{\Lambda}_{b}\end{bmatrix}_{\substack{a=\hat{a}^k\\ b=\hat{b}^k}}$$

The second partial derivatives by using equations (8) and (9) are given by,

$$\acute{\Lambda}_{aa} = \frac{\partial^2\Lambda}{\partial a^2} = \sum_{i=1}^{n}\frac{-1}{(a+by_i{}^{-1})^2}\ \ldots \quad (10)$$

$$\acute{\Lambda}_{bb} = \frac{\partial^2\Lambda}{\partial b^2} = \sum_{i=1}^{n}\frac{-y_i{}^{-2}}{(a+by_i{}^{-1})^2}\ \ldots \quad (11)$$

$$\acute{\Lambda}_{ab} = \frac{\partial^2\Lambda}{\partial a\partial b} = \sum_{i=1}^{n}\frac{-y_i{}^{-1}}{(a+by_i{}^{-1})^2}\ \ldots \quad (12)$$

where

$$\acute{\Lambda}_{ab} = \frac{\partial^2\Lambda}{\partial a\partial b} = \acute{\Lambda}_{ba} = \frac{\partial^2\Lambda}{\partial b\partial a}$$

3. RANK SET SAMPLING (RSS) [Hussein and et al. (2023)] Mohammed and Mohammed (2021)

Rank set sampling (RSS) is a sampling method that is used to estimate population parameters. It is particularly useful when the population in question has a natural ranking or classification's includes selecting groups of units from the population based on their ranks rather than their individual values.

According to increasing ordering random sampling $\left(y_{(1)}, y_{(2)}, \ldots, y_{(n)}\right)$, the pdf of the IERD is as follows:

$$g_i\big(y(i)\big) = \frac{n!}{(i-1)!\,(n-i)!}\left[F(y_{(i)})\right]^{i-1}\left[1-F(y_{(i)})\right]^{n-i} g\big(y_{(i)}\big)\,, c < y_{(i)} < d$$

then

$$g_i\big(y(i)\big) = Q\big(a+by_{(i)}^{-1}\big)\, y_{(i)}^{-2}\left[1-\ e^{-\left(ay_{(i)}^{-1}+\frac{b}{2}y_{(i)}^{-2}\right)}\right]^{i-1}\left[e^{-\left(ay_{(i)}^{-1}+\frac{b}{2}y_{(i)}^{-2}\right)}\right]^{n-i+1}\ \ldots \quad (13)$$

where

$$Q = \frac{n!}{(i-1)!\,(n-i)!}$$

For a given order sample $\left(y_{(1)}, y_{(2)}, \ldots, y_{(n)}\right)$, the full data likelihood function, denoted as $L(a,b|\,\underline{y})$, can be mentioned as follows:

$$L(a,b|\underline{y}), = Q^n \times \prod_{i=1}^{n}\left(a+by_{(i)}^{-1}\right)\prod_{i=1}^{n} y_{(i)}^{-2}$$
$$\prod_{i=1}^{n}\left[1-\ e^{-\left(ay_{(i)}^{-1}+\frac{b}{2}y_{(i)}^{-2}\right)}\right]^{i-1}\prod_{i=1}^{n}\left[\,e^{-\left(ay_{(i)}^{-1}+\frac{b}{2}y_{(i)}^{-2}\right)}\right]^{n-i+1}$$

The function that represents the natural log-likelihood is

$$\Lambda = \ln L(a,b|\underline{y}) = nlnQ + \sum_{i=1}^{n}\ln\left(a+by_{(i)}^{-1}\right)$$
$$-2\sum_{i=1}^{n} ln\, y_{(i)} - \sum_{i=1}^{n}(n-i+1)\left(ay_{(i)}^{-1}\ +\ \frac{b}{2}\ y_{(i)}^{-2}\right)$$
$$+\sum_{i=1}^{n}(i-1)\ln\left(1-e^{-\left(ay_{(i)}^{-1}+\frac{b}{2}y_{(i)}^{-2}\right)}\right)\ \ldots \qquad (14)$$

Now,

$$\acute{\Lambda}_a = \frac{\partial\Lambda}{\partial a} = \sum_{i=1}^{n}\frac{1}{a+by_{(i)}^{-1}} - \sum_{i=1}^{n}(n-i+1)y_{(i)}^{-1}$$
$$+\sum_{i=1}^{n}(i-1)\,\frac{y_{(i)}^{-1}\,.\,e^{-\left(ay_{(i)}^{-1}+\frac{b}{2}y_{(i)}^{-2}\right)}}{\left(1-e^{-\left(ay_{(i)}^{-1}+\frac{b}{2}y_{(i)}^{-2}\right)}\right)} = 0\ \ldots \qquad (15)$$

$$\acute{\Lambda}_b = \frac{\partial\Lambda}{\partial b} = \sum_{i=1}^{n}\frac{y_{(i)}^{-1}}{a+by_{(i)}^{-1}} - \frac{1}{2}\sum_{i=1}^{n}(n-i+1)y_{(i)}^{-2}$$
$$+\sum_{i=1}^{n}(i-1)\,\frac{\frac{y_{(i)}^{-2}}{2}\,.\,e^{-\left(ay_{(i)}^{-1}+\frac{b}{2}y_{(i)}^{-2}\right)}}{\left(1-e^{-\left(ay_{(i)}^{-1}+\frac{b}{2}y_{(i)}^{-2}\right)}\right)} = 0\ \ldots \qquad (16)$$

Since these equations do not have closed solutions, the NRA can be used to reach the solutions.

Where the second partial derivatives are given by, **Al-Sultany and Mohammed (2018)**

$$\acute{\Lambda}_{aa} = \frac{\partial^2\Lambda}{\partial a^2} = \sum_{i=1}^{n}\frac{-1}{\left(a+by_{(i)}^{-1}\right)^2} - \sum_{i=1}^{n}(i-1)\,\frac{y_{(i)}^{-2}\,.\,e^{-\left(ay_{(i)}^{-1}+\frac{b}{2}y_{(i)}^{-2}\right)}}{\left(1-e^{-\left(ay_{(i)}^{-1}+\frac{b}{2}y_{(i)}^{-2}\right)}\right)}$$
$$-\sum_{i=1}^{n}(i-1)\,\frac{y_{(i)}^{-2}\,.\,e^{-2\left(ay_{(i)}^{-1}+\frac{b}{2}y_{(i)}^{-2}\right)}}{\left(1-e^{-\left(ay_{(i)}^{-1}+\frac{b}{2}y_{(i)}^{-2}\right)}\right)^2}\ \ldots \qquad (17)$$

$$\acute{\Lambda}_{bb} = \frac{\partial^2 \Lambda}{\partial b^2} = \sum_{i=1}^{n} \frac{-y_{(i)}^{-2}}{\left(a + by_{(i)}^{-1}\right)^2} - \sum_{i=1}^{n} (i-1) \frac{\frac{y_{(i)}^{-4}}{4} \,.\, e^{-\left(ay_{(i)}^{-1} + \frac{b}{2} y_{(i)}^{-2}\right)}}{\left(1 - e^{-\left(ay_{(i)}^{-1} + \frac{b}{2} y_{(i)}^{-2}\right)}\right)}$$

$$- \sum_{i=1}^{n} (i-1) \frac{\frac{y_{(i)}^{-4}}{4} \,.\, e^{-2\left(ay_{(i)}^{-1} + \frac{b}{2} y_{(i)}^{-2}\right)}}{\left(1 - e^{-\left(ay_{(i)}^{-1} + \frac{b}{2} y_{(i)}^{-2}\right)}\right)^2} \ \ldots \qquad (18)$$

$$\acute{\Lambda}_{ab} = \frac{\partial^2 \Lambda}{\partial a \partial b} = \sum_{i=1}^{n} \frac{-y_{(i)}^{-1}}{\left(a + by_{(i)}^{-1}\right)^2} - \sum_{i=1}^{n} (i-1) \frac{\frac{y_{(i)}^{-3}}{2} \,.\, e^{-\left(ay_{(i)}^{-1} + \frac{b}{2} y_{(i)}^{-2}\right)}}{\left(1 - e^{-\left(ay_{(i)}^{-1} + \frac{b}{2} y_{(i)}^{-2}\right)}\right)}$$

$$- \sum_{i=1}^{n} (i-1) \frac{\frac{y_{(i)}^{-3}}{2} \,.\, e^{-2\left(ay_{(i)}^{-1} + \frac{b}{2} y_{(i)}^{-2}\right)}}{\left(1 - e^{-\left(ay_{(i)}^{-1} + \frac{b}{2} y_{(i)}^{-2}\right)}\right)^2} = \ \ldots \qquad (19)$$

where

$$\acute{\Lambda}_{ab} = \frac{\partial^2 \Lambda}{\partial a \partial b} = \acute{\Lambda}_{ba} = \frac{\partial^2 \Lambda}{\partial b \partial a}.$$

4. CRAMER-VON-MISES ESTIMATORS

MacDonald suggested Cramér von Mises in 1971 MacDonald (1971) as a way to reduce the next function.

$$E = (12n)^{-1} + \sum_{i=1}^{n} \left[F\left(y_{(i)}\right) - (2i-1)(2n)^{-1}\right]^2 \ \ldots \qquad (20)$$

After replacing $(y(i))$ in the above equation with its cumulative distribution function (c.d.f.), which was defined in equation (3), we get

$$E(a,b) = (12n)^{-1} + \sum_{i=1}^{n} \left[e^{-\left(ay_{(i)}{}^{-1} + \frac{b}{2} y_{(i)}{}^{-2}\right)} + (2i - 1 - 2n)(2n)^{-1}\right]^2 \ \ldots \qquad (21)$$

The Cramér-Von-Mises estimates can be determined by solving the following equations using Newton-Raphson algorithm

$$\frac{\partial E(a,b)}{\partial a} = \sum_{i=1}^{n} \left[e^{-\left(ay_{(i)}{}^{-1} + \frac{b}{2} y_{(i)}{}^{-2}\right)} + (2i - 1 - 2n)(2n)^{-1}\right]$$

$$.\ e^{-\left(ay_{(i)}{}^{-1} + \frac{b}{2} y_{(i)}{}^{-2}\right)} . y_{(i)}{}^{-1} = 0 \ \ldots \qquad (22)$$

$$\frac{\partial E(a,b)}{\partial b} = \sum_{i=1}^{n}\left[e^{-\left(ay_{(i)}{}^{-1}+\frac{b}{2}y_{(i)}{}^{-2}\right)} + (2i-1-2n)(2n)^{-1}\right] .\ e^{-\left(ay_{(i)}{}^{-1}+\frac{b}{2}y_{(i)}{}^{-2}\right)}.\, y_{(i)}{}^{-2} = 0 \ldots \quad (23)$$

the second partial derivatives are given by,

$$\frac{\partial^2 E(a,b)}{\partial a^2} = -\sum_{i=1}^{n}\left[2\, e^{-\left(ay_{(i)}{}^{-1}+\frac{b}{2}y_{(i)}{}^{-2}\right)} + (2i-1-2n)(2n)^{-1}\right] .\ e^{-\left(ay_{(i)}{}^{-1}+\frac{b}{2}y_{(i)}{}^{-2}\right)}.\, y_{(i)}{}^{-2} \ldots \quad (24)$$

$$\frac{\partial^2 E(a,b)}{\partial b^2} = -\sum_{i=1}^{n}\left[e^{-\left(ay_{(i)}{}^{-1}+\frac{b}{2}y_{(i)}{}^{-2}\right)} + (2i-1-2n)(4n)^{-1}\right] .\ e^{-\left(ay_{(i)}{}^{-1}+\frac{b}{2}y_{(i)}{}^{-2}\right)}.\, y_{(i)}{}^{-4} \ldots \quad (25)$$

$$\frac{\partial^2 E(a,b)}{\partial a \partial b} = -\sum_{i=1}^{n}\left[e^{-\left(ay_{(i)}{}^{-1}+\frac{b}{2}y_{(i)}{}^{-2}\right)} + (2i-1-2n)(4n)^{-1}\right] .\ e^{-\left(ay_{(i)}{}^{-1}+\frac{b}{2}y_{(i)}{}^{-2}\right)}.\, y_{(i)}{}^{-3} \ldots \quad (26)$$

where

$$\frac{\partial^2 E(a,b)}{\partial a \partial b} = \frac{\partial^2 E(a,b)}{\partial b \partial a}.$$

5. MODIFIED RELIABILITY FUNCTION [[Noaman, and et al. (2020)]]

Three different methods of estimation, Maximum Likelihood Estimation, Rank set sampling, and Cramer-von-Mises estimators, It is used for comparison the modified reliability function of IERD.

where,

$$R(t; a, b, c\,) = e^{-\left(act^{-1}+\frac{b}{2}c\,t^{-2}\right)};\, t \geq 0; a, b > 0, c > 0 \quad (27)$$

is a constant …

We used algorithm to find modified reliability function of IERD as follow:

Step 1: Initial guess $a^{(0)}, b^{(0)}, c, t$.
Step 2: Set $h = 0$.
Step 3: Generate $y_{(1)}, y_{(2)}, \ldots, y_{(n)}$ from IERD by using the inverse transformation method After arranging them in ascending order.
Step 4: Sol the equations (22) and (23) by using NRA.
Step 5: At iteration $(h+1)$, estimate the new value of a and b, as in the equation (*).
Step 6: Repeat step 5 until convergence occurs to find the $\hat{a}_{IERD}^{MCV}$ and $\hat{b}_{IERD}^{MCV}$ ...
Step 5: Compute $\hat{R}(t)_{IERD}^{MCV} = e^{-\left(\hat{a}_{IERD}^{MCV}ct^{-1}+\frac{\hat{b}_{IERD}^{MCV}}{2}c\,t^{-2}\right)}$.

6. SIMULATION

Simulation can be used to create random samples from a specified distribution or to simulate behavior Complex systems.

In this section, the emphasis was on comparing the estimations of the two scale parameters and the modified reliability function of the Inverse Exponential Rayleigh distribution (IERD) using three mentioned methods: Maximum Likelihood Estimation, Rank set sampling, and Cramér-von-Mises estimators.

The basic steps for simulation design can be summarized as the following seven steps:

1. An independent identically distributed random sample (i.i.d.) was generated according to the inverse exponential Rayleigh distribution (IERD) using the inverse transformation method.
2. Choose sample sizes $n = 10, 20, 50$ and 100 to take care of small, medium and large data sets.
3. The number of samples replicated chosen to be ($L = 1000$).
4. The default values for the two scale parameters shown below.

Parameters	**Case 1**	**Case 2**	**Case 3**	**Case 4**	**Case 5**	**Case 6**
a	0.2	0.5	1	0.2	0.5	1
b	0.5	0.5	0.5	1.5	1.5	1.5

5. The initial values required for proceeding chosen based on equating the median of the generated sample with the median of the distribution.
6. Choose the value of ($c = 0.5,1,10$) for modified reliability function.
7. The comparisons between parameter estimates were based on values from Mean Square Error (MSE)

$$MSE(a) = \sum_{m=1}^{L} \frac{(\hat{a}_m - a)^2}{L}, \tag{28}$$

$\hat{a}_m$ = is the estimate of a at the m^{th} replicate.

$$MSE(b) = \sum_{m=1}^{L} \frac{\left(\hat{b}_m - b\right)^2}{L} \tag{29}$$

$\hat{b}_m$ = is the estimate of b at the m^{th} replicate
while it was based on values from Integrated Mean Square Error (IMSE) for the estimates of the modified reliability function.

$$IMSE\left(\hat{R}(t)\right) = \sum_{m=1}^{L} \frac{\left(\sum_{i=1}^{r_t} \frac{\left(\hat{R}_m(t_i) - R(t_i)\right)^2}{r_t}\right)}{L}, \tag{30}$$

r_t = is the number of times chosen to be 4 (where $t = 1, 2, 3, 4$).
$\hat{R}_m(t_i)$ = is the estimate of $R(t)$ at the m^{th} replicateand i^{th} time.
The tables below display the outcomes of the estimators.

Table 1
MSE of the Estimates the Parameters a and b of IERD for Different Sample Sizes and for Six Cases

n	a	b	$\hat{a}_{IERD}^{MLE}$	$\hat{a}_{IERD}^{RSS}$	$\hat{a}_{IERD}^{MCV}$	$\hat{b}_{IERD}^{MLE}$	$\hat{b}_{IERD}^{RSS}$	$\hat{b}_{IERD}^{MCV}$
10	0.2	0.5	0.0458368	0.0458300	0.0458368	0.2388213	0.2388250	0.2388213
20	0.2	0.5	0.0409857	0.0409857	0.0409857	0.2337028	0.2337028	0.2337028
50	0.2	0.5	0.0324874	0.0324874	0.0324874	0.2282722	0.2282722	0.2282722
100	0.2	0.5	0.0322090	0.0322090	0.0322090	0.2280298	0.2280298	0.2280298
10	0.5	0.5	0.2948329	0.2948162	0.2948328	0.2948329	0.2948378	0.2948328
20	0.5	0.5	0.2711969	0.2711948	0.2711969	0.2711969	0.2711967	0.2711969
50	0.5	0.5	0.2171273	0.2171273	0.2171273	0.2171273	0.2171273	0.2171274
100	0.5	0.5	0.2119086	0.2119086	0.2119086	0.2119086	0.2119086	0.2119086
10	1	0.5	0.9901080	0.9901044	0.9901044	0.3464709	0.3464703	0.3464708
20	1	0.5	0.9689218	0.9689178	0.9689218	0.3160302	0.3160295	0.3160302
50	1	0.5	0.8768741	0.8768741	0.8768741	0.2396884	0.2396884	0.2396884
100	1	0.5	0.8561525	0.8561518	0.8561545	0.2031032	0.2031031	0.2031024
10	0.2	1.5	0.0637734	0.0637734	0.0637734	2.1736663	2.1736663	2.1736663
20	0.2	1.5	0.0315960	0.0315960	0.0315960	2.1721146	2.1721146	2.1721146
50	0.2	1.5	0.0312630	0.0312630	0.0312630	2.1598895	2.1598895	2.1598895
100	0.2	1.5	0.0305218	0.0305218	0.0305217	2.1567079	2.1567079	2.1567088
10	0.5	1.5	0.3194683	0.3194651	0.3194683	2.2050976	2.2050966	2.2050976
20	0.5	1.5	0.2673812	0.2673812	0.2673812	2.1246858	2.1246858	2.1246859
50	0.5	1.5	0.2250349	0.2250349	0.2250349	2.1083228	2.1083228	2.1083228
100	0.5	1.5	0.2059899	0.2059899	0.2059905	2.0984050	2.0984050	2.0984036
10	1	1.5	1.1818102	1.1817911	1.1818116	2.3109701	2.3109669	2.3109714
20	1	1.5	1.0872366	1.0872300	1.0872365	2.2048846	2.2048830	2.2048848
50	1	1.5	0.8710144	0.8710144	0.8710142	2.0048881	2.0048881	2.0048881
100	1	1.5	0.8424719	0.8424719	0.8424716	1.9952070	1.9952070	1.9952074

Table 2

IMSE of the Estimates the Modified Reliability Function $R(t)$ of IERD for Different Sample Sizes and $t = 1,2,3,4$

n	a	b	C=1			C=0.5			C=10		
			$\hat{R}(t)^{MLE}_{IERD}$	$\hat{R}(t)^{RSS}_{IERD}$	$\hat{R}(t)^{MCV}_{IERD}$	$\hat{R}(t)^{MLE}_{IERD}$	$\hat{R}(t)^{RSS}_{IERD}$	$\hat{R}(t)^{MCV}_{IERD}$	$\hat{R}(t)^{MLE}_{IERD}$	$\hat{R}(t)^{RSS}_{IERD}$	$\hat{R}(t)^{MCV}_{IERD}$
10	0.2	0.5	0.0362685	0.0362684	0.0362683	0.0112689	0.0112689	0.0112689	0.3991056	0.3991056	0.3991056
20	0.2	0.5	0.0357673	0.0357673	0.0357673	0.0110797	0.0110797	0.0110797	0.3859786	0.3859786	0.3859786
50	0.2	0.5	0.0346935	0.0346935	0.0346935	0.0105726	0.0105726	0.0105726	0.3744348	0.3744348	0.3744348
100	0.2	0.5	0.0345361	0.0345361	0.0345361	0.0105291	0.0105291	0.0105291	0.3696733	0.3696733	0.3696733
10	0.5	0.5	0.0844860	0.0844860	0.0844860	0.0291576	0.0291576	0.0291576	0.5282403	0.5282403	0.5282403
20	0.5	0.5	0.0829484	0.0829484	0.0829484	0.0287988	0.0287988	0.0287988	0.5002002	0.5002002	0.5002002
50	0.5	0.5	0.0798851	0.0798851	0.0798851	0.0272574	0.0272574	0.0272574	0.4711734	0.4711734	0.4711733
100	0.5	0.5	0.0792459	0.0792459	0.0792459	0.0269555	0.0269555	0.0269555	0.4567388	0.4567388	0.4567388
10	1	0.5	0.1684339	0.1684339	0.1684339	0.0667128	0.0667128	0.0667128	0.5199601	0.5199601	0.5199601
20	1	0.5	0.1680859	0.1680859	0.1680859	0.0665503	0.0665503	0.0665503	0.5033217	0.5033217	0.5033217
50	1	0.5	0.1600398	0.1600398	0.1600398	0.0635910	0.0635910	0.0635910	0.4478895	0.4478895	0.4478895
100	1	0.5	0.1581801	0.1581801	0.1581805	0.0627811	0.0627811	0.0627812	0.4226749	0.4226749	0.4226776
10	0.2	1.5	0.1037759	0.1037759	0.1037759	0.0384913	0.0384913	0.0384913	0.4951736	0.4951736	0.4951736
20	0.2	1.5	0.1029862	0.1029862	0.1029862	0.0379100	0.0379100	0.0379100	0.4900346	0.4900346	0.4900346
50	0.2	1.5	0.1013813	0.1013813	0.1013813	0.0373897	0.0373897	0.0373897	0.4708197	0.4708197	0.4708197
100	0.2	1.5	0.1006424	0.1006424	0.1006423	0.0371693	0.0371693	0.0371693	0.4591966	0.4591966	0.4591960
10	0.5	1.5	0.1540096	0.1540096	0.1540096	0.0617004	0.0617004	0.0617004	0.5674350	0.5674350	0.5674350
20	0.5	1.5	0.1494126	0.1494126	0.1494126	0.0602166	0.0602166	0.0602166	0.5257008	0.5257008	0.5257007
50	0.5	1.5	0.1473314	0.1473314	0.1473314	0.0592310	0.0592310	0.0592310	0.5004393	0.5004393	0.5004393
100	0.5	1.5	0.1459142	0.1459142	0.1459144	0.0586302	0.0586302	0.0586303	0.4882279	0.4882279	0.4882300
10	1	1.5	0.2282404	0.2282404	0.2282404	0.1027652	0.1027652	0.1027652	0.5255272	0.5255272	0.5255272
20	1	1.5	0.2227414	0.2227414	0.2227413	0.1009837	0.1009837	0.1009837	0.4848819	0.4848819	0.4848817
50	1	1.5	0.2158603	0.2158603	0.2158603	0.0977791	0.0977791	0.0977791	0.4438011	0.4438011	0.4438009
100	1	1.5	0.2146999	0.2146999	0.2146998	0.0973155	0.0973155	0.0973154	0.4261305	0.4261305	0.4261300

7. CONCLUDING REMARKS FROM SIMULATION

The most essential concluding observations of the simulation are:

Concluding from Table 1:

- For estimating the scale parameter a. The Rank set sampling (RSS) estimates give the best performance which gives the lowest value according to the MSE in comparison with other estimates with all cases and for all sample sizes except for large sample size with parameters, $a = 0.2, b = 1.5$, and for median and large sample sizes with parameters $a = 1, b = 1.5$, where Cramér-von-Mises (MCV) estimates introduced the best.

- For estimating the scale parameter b.The RSS estimates give the best performance which gives the lowest value according to the MSE in comparison with other estimates with all cases and for all sample sizes except for small sample size with parameters, $a = 0.2, b = 0.5$ and, $a = 0.5, b = 0.5$ and for large sample sizes with parameters $a = 1, b = 0.5$ and $a = 0.5, b = 1.5$, where Cramér-von-Mises (MCV) estimates introduced the best.

Concluding from Table 2:

- For estimating the modified Reliability Function $R(t)$ The Cramér-von-Mises Estimators (MCV) estimates give the best performance which gives the lowest value according to the IMSE in comparison with other estimates with all cases and for all sample sizes except for large sample size with parameters, $a = 1, b = 0.5$ and $a = 0.5, b = 1.5$ where Rank set sampling (RSS) estimates introduced the best.

- Accordingly, we recommend, in general using The Rank set sampling (RSS) estimates for estimating the two parameters of the $IERD$. Using the Cramér-von-Mises estimators (MCV) estimates for estimating the modified Reliability Function $R(t)$.

ACKNOWLEDGMENTS

We express our gratitude to the Department of Mathematics at Al-Mustansiriyah University for their valuable assistance in preparing this article.

REFERENCES

1. Al-Sultany, S.A. and Mohammed, S.A. (2018). Comparison between Bayesian and Maximum Likelihood Methods for parameters and the Reliability function of Perks Distribution. *Baghdad Science Journal*, 59(1B), 369-376.
2. Al-Sultany, S.A. and Mohammed, S.A. (2017). Mixture of two Inverse Exponential Distribution Based on Fuzzy Data, *Special Issus: 1st Scientific International Conference*, *College of Science*, *Al-Nahrain University*, Part II, pp. 73-81.
3. Hussein, L.K., Hussein, I.H. and Rasheed, H.A. (2023). Exponential Rayleigh distribution: Approximate non – Bayesian estimation parameters and survival function. *AIP Conf. Proc.* 2414, 040089, https://doi.org/10.1063/5.0114855.

4. Mohammed, M.J. and Mohammed, A.T. (2021), Parameter estimation of inverse exponential Rayleigh distribution based on classical methods. *Int. J. Nonlinear Anal. Appl*., 12(1), 935-944.
5. Macdonald, P. (1971). An estimation procedure for mixtures of distributions. *Journal of the Royal Statistical Society*, 20, 102-107.
6. Noaman, A.A., Ahmed, S.A. and Hawash, M.K. (2020). Constructing a new mixed probability distribution with fuzzy reliability estimation. *Periodicals of Engineering and Natural Sciences*, 8(2), 590-601.
7. Kareem, S.A. (2020). A Comparison between Two Shape Parameters Estimators for (Burr-XII) Distribution. *Baghdad Science Journal*, 17(3), 973-979.