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POLYNOMIAL-EXPONENTIAL MIXTURE OF GENERALISED POISSON DISTRIBUTION

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ABSTRACT

This proposed distribution is an outcome of mixing Generalised Poisson distribution of Consul and Jain with Polynomial-exponential distribution of Sah. The essential characteristics needed for sufficient analysis of this distribution have been defined as well as derived systematically. Two methods have been used to estimate parameter of this distribution. Comparing the theoretical frequency obtained using two methods with the theoretical frequency constructed by using Generalised Negative Binomial Distribution of Sah. The proposed distribution seems to be better alternative of generalised negative binomial distribution of Sah.

KEYWORDS

Probability distribution, Generalised Poisson distribution, Polynomial-exponential distribution, Over-dispersed data, Mixing, Distribution.

MSC:60E05

1. INTRODUCTION

Research is a continuous process through which some new, imaginative and unimaginative problems are being solved. Using an additional parameter θ, Consul and Jain, (see,[1]), derived the Generalised Poisson distribution (GPD) and found to be more useful and versatile in nature than Poisson distribution. The proposed distribution is named as Polynomial-exponential mixture of Generalised Poisson distribution (PEMGPD) whose particular case is Polynomial-exponential mixture of Poisson distribution (PEMPD), (see [2]), given by its probability mass function

$$P_1(z;\alpha)=\left(\frac{\alpha^4}{6+\pi\alpha^3}\right)\left(\frac{\pi(1+\alpha)^3+(1+z)(2+z)(3+z)}{(1+\alpha)^{z+4}}\right) \tag{1}$$

where $z=0,1,2,...$ and $\alpha>0$

The expression (1) has been obtained by mixing Poisson distribution (PD) with Polynomial-exponential distribution (PED), (see [3]), and its probability density function (pdf) was given by

$$f_2(z;\alpha) = \left(\frac{\alpha^4}{(6+\pi\alpha^3)}\right)(\pi + z^3)e^{-\alpha z} \ ; z > 0; \alpha > 0 \tag{2}$$

Generalised Negative Binomial Distribution (GNBD), (see [4]), was obtained by mixing GPD with two-parameter gamma distribution. It was observed that GNBD gives a better fit for modelling of over-dispersed count data than negative binomial distribution (NBD) as well as PD. In analysing this proposed distribution, what we observed that PEMGPD gives a better fit than GNBD [4]. The obtained distribution also gives a better fit to over-dispersed count-data than Poisson-Mishra distribution (PMD), (see [5]), has been obtained by mixing Poisson distribution with Mishra distribution [6]. We have been extracted many useful characteristics of this distribution by studying the following papers, (see, [7- 18]).

For a better physical appearance, work of this presented paper is kept under the following headings. The first section contains introduction and literature review required for this paper. The materials and methods needed for this paper have been placed in second section. Results which may also know as main body of scientific study has been placed in the third section. The fourth and the last section contains conclusion obtained.

There is no limit of knowledge and hence the authors of this paper belief that this work will play very important role for learning how a compound distribution has been obtained by mixing PD as well as GPD with a continuous distribution.

2. MATERIALS AND METHODS

Materials required for this paper are construction of theoretical concept and to test the validity of the constructed theoretical work by applying goodness of fit to some over-dispersed secondary count data.

3. RESULTS

The work done under this topic is divide into the following sub-headings.

3.1. Probability Mass Function of PEMGPD.
3.2 Moments about Origin and Central moment of PEMGPD
3.3 Methods of Estimating the Parameters of PEMGPD
3.4 Goodness of Fit and Applications of PEMGPD

3.1 Probability Mass Function of PEMGPD

It is a discrete compound probability distribution. It has two parameters α and θ. It is a mixture distribution of GPD and PED. GPD has two parameters λ and θ. λ is an original parameter of GPD and it is continuous in nature and in mixing process it acts as a variable which follows PED. θ is an additional parameter of GPD and it is versatile in nature and hence the GPD. The probability mass function of Polynomial-exponential mixture of Generalised Poisson distribution (PEMGPD) can be obtained as

$$P(z,\alpha,\theta)=\left[\frac{\alpha^4 e^{-\theta z}}{z!(6+\pi\alpha^3)}\right]\int_0^\infty\left[\lambda^z\left(1+\frac{\theta z}{\lambda}\right)^{z-1}(\pi+\lambda^3)\,e^{-\lambda(1+\alpha)}\right]d\lambda \tag{3}$$

where

$z=0,1,2,\ldots\ ;|\theta|\le 1\ ;\lambda>0;\alpha>0.$

$$=\left[\frac{\alpha^4 e^{-\theta z}}{z!(6+\pi\alpha^3)}\right]\int_0^\infty\left[\left\{\sum_{i=0}^{z-1}\binom{z-1}{i}\left(\frac{\theta z}{\lambda}\right)^i\right\}(\pi\lambda^z+\lambda^{z+3})\,e^{-\lambda(1+\alpha)}\right]d\lambda$$

$$P(z,\alpha,\theta)=\left[\frac{\alpha^4 e^{-\theta z}}{(6+\pi\alpha^3)}\right]\left[\sum_{i=0}^{z-1}\frac{\theta^i z^{i-1}(z-i)}{i!\,(1+\alpha)^{z-i+4}}\left\{\pi(1+\alpha)^3+(z-i+1)(z-i+2)(z-i+3)\right\}\right]$$

$$P(z,\alpha,\theta)=\left[\frac{\alpha^4 e^{-\theta z}}{(6+\pi\alpha^3)}\right]\left[\frac{\left\{\pi(1+\alpha)^3+(z+1)(z+2)(z+3)\right\}}{(1+\alpha)^{z+4}}\right]$$
$$+\left[\frac{\alpha^4 e^{-\theta z}}{(6+\pi\alpha^3)}\right]\left[\sum_{i=1}^{z-1}\frac{\theta^i z^{i-1}(z-i)}{i!\,(1+\alpha)^{z-i+4}}\left\{\pi(1+\alpha)^3+(z-i+1)(z-i+2)(z-i+3)\right\}\right] \tag{4}$$

Probability of $Z=z$, where $z=0,1,2,\ldots$, can be obtained by the expression (4) which is the obtained pmf of PEMGPD. When $\theta=0$, the expression (4) is converted into the pmf of PEMPD. When we put $z=0,1,2,3,4,5$ in the expression (4) the following expressions (5) to (10) of probability have been obtained which will be helpful while calculating theoretical frequencies for different discrete values of $Z=z$.

$$P(Z=0;\alpha,\theta)=\left(\frac{\alpha^4}{(6+\pi\alpha^3)}\right)\left[\frac{\left\{\pi(1+\alpha)^3+6\right\}}{(1+\alpha)^4}\right] \tag{5}$$

$$P(Z=1;\alpha,\theta)=\left(\frac{\alpha^4 e^{-\theta}}{(6+\pi\alpha^3)}\right)\left[\frac{\left\{\pi(1+\alpha)^3+24\right\}}{(1+\alpha)^5}\right] \tag{6}$$

$$P(Z=2;\alpha,\theta)=\left(\frac{\alpha^4 e^{-2\theta}}{(6+\pi\alpha^3)}\right)\left[\frac{\left\{\pi(1+\alpha)^3+60\right\}+\theta(1+\alpha)\left\{\pi(1+\alpha)^3+24\right\}}{(1+\alpha)^6}\right] \tag{7}$$

$$P(Z=3;\alpha,\theta)=\left(\frac{\alpha^4 e^{-3\theta}}{(6+\pi\alpha^3)}\right)\left[\frac{\left\{\pi(1+\alpha)^3+120\right\}+2\theta(1+\alpha)\left\{\pi(1+\alpha)^3+60\right\}+1.5\theta^2(1+\alpha)^2\left\{\pi(1+\alpha)^3+24\right\}}{(1+\alpha)^7}\right] \tag{8}$$

$$P(Z=4;\alpha,\theta)=\left(\frac{\alpha^4 e^{-4\theta}}{(6+\pi\alpha^3)}\right)\left[\frac{\begin{array}{l}\left\{\pi(1+\alpha)^3+210\right\}+3\theta(1+\alpha)\left\{\pi(1+\alpha)^3+120\right\}\\ +4\theta^2(1+\alpha)^2\left\{\pi(1+\alpha)^3+60\right\}\\ +(16/6)\theta^3(1+\alpha)^3\left\{\pi(1+\alpha)^3+24\right\}\end{array}}{(1+\alpha)^8}\right] \quad (9)$$

$$P(Z=5;\alpha,\theta)=\left(\frac{\alpha^4 e^{-5\theta}}{(6+\pi\alpha^3)}\right)\left[\frac{\begin{array}{l}\left\{\pi(1+\alpha)^3+336\right\}+4\theta(1+\alpha)\left\{\pi(1+\alpha)^3+210\right\}\\ +(15/2)\theta^2(1+\alpha)^2\left\{\pi(1+\alpha)^3+120\right\}+(50/6)\theta^3(1+\alpha)^3\\ \left\{\pi(1+\alpha)^3+60\right\}+(125/24)\theta^4(1+\alpha)^4\left\{\pi(1+\alpha)^3+24\right\}\end{array}}{(1+\alpha)^9}\right] \quad (10)$$

3.2 Moments about Origin and Central Moment of PEMGPD

To study the variation, shape and size of any probability distribution or frequency distribution, Statistical moments about the origin and hence about the mean are necessary. Let μ_r' denotes the r[th] moment about the origin which can be obtained as

$$\mu_r'=\left(\frac{\alpha^4}{(6+\pi\alpha^3)}\right)\int_0^\infty\left[\sum_{z=0}^\infty\frac{z^r\,\lambda(\lambda+\theta z)^{z-1}e^{-(\lambda+\theta z)}}{\Gamma(z+1)}\right](\pi+\lambda^3)\,e^{-\alpha\lambda}d\lambda \quad (11)$$

By substituting the value of $r=1,2,3,4$, the first four moments about the origin are obtained as follows. The expression (12) is the mean of the PEMGPD (4).

$$\mu_1'=\left(\frac{\alpha^4}{(6+\pi\alpha^3)}\right)\int_0^\infty\left[\sum_{z=0}^\infty\frac{z^1\,\lambda(\lambda+\theta z)^{z-1}e^{-(\lambda+\theta z)}}{\Gamma(z+1)}\right](\pi+\lambda^3)\,e^{-\alpha\lambda}d\lambda$$

$$=\left(\frac{\alpha^4}{(6+\pi\alpha^3)}\right)\int_0^\infty\left[\frac{\lambda}{(1-\theta)}\right](\pi+\lambda^3)\,e^{-\alpha\lambda}d\lambda=\left(\frac{\alpha^4}{(6+\pi\alpha^3)(1-\theta)}\right)\left(\frac{(\pi\alpha^3+24)}{\alpha^5}\right)$$

$$=\left(\frac{(\pi\alpha^3+24)}{\alpha(6+\pi\alpha^3)(1-\theta)}\right) \quad (12)$$

(a) If $\theta = 0$, the mean of PEMPD is equal to the mean of PEMDPD.
(b) If $0 < \theta < 1$, the mean of PEMPD is less than the mean of PEMDPD.
(c) If $0 > \theta > -1$, the mean of PEMPD is greater than the mean of PEMDPD.

$$\mu_2' = \left(\frac{\alpha^4}{(6+\pi\alpha^3)}\right)\int_0^\infty \left[\sum_{z=0}^{\infty} \frac{z^2 \lambda(\lambda+\theta z)^{z-1} e^{-(\lambda+\theta z)}}{\Gamma(z+1)}\right](\pi+\lambda^3)\, e^{-\alpha\lambda} d\lambda$$

$$= \left(\frac{\alpha^4}{(6+\pi\alpha^3)}\right)\int_0^\infty \left[\frac{\lambda}{(1-\theta)^3} + \frac{\lambda^2}{(1-\theta)^2}\right](\pi+\lambda^3)\, e^{-\alpha\lambda} d\lambda$$

$$= \left(\frac{(\pi\alpha^3+24)}{\alpha(6+\pi\alpha^3)(1-\theta)^3}\right) + \left(\frac{2(\pi\alpha^3+60)}{\alpha^2(6+\pi\alpha^3)(1-\theta)^2}\right) \tag{13}$$

$$\mu_3' = \left(\frac{\alpha^4}{(6+\pi\alpha^3)}\right)\int_0^\infty \left[\sum_{z=0}^{\infty} \frac{z^3 \lambda(\lambda+\theta z)^{z-1} e^{-(\lambda+\theta z)}}{\Gamma(z+1)}\right](\pi+\lambda^3)\, e^{-\alpha\lambda} d\lambda$$

$$= \left(\frac{\alpha^4}{(6+\pi\alpha^3)}\right)\int_0^\infty \left[\frac{\lambda(1+2\theta)}{(1-\theta)^5} + \frac{3\lambda^2}{(1-\theta)^4} + \frac{\lambda^3}{(1-\theta)^3}\right](\pi+\lambda^3)\, e^{-\alpha\lambda} d\lambda$$

$$= \left(\frac{(1+2\theta)(\pi\alpha^3+24)}{\alpha(6+\pi\alpha^3)(1-\theta)^5}\right) + \left(\frac{6(\pi\alpha^3+60)}{\alpha^2(6+\pi\alpha^3)(1-\theta)^4}\right) + \left(\frac{6(\pi\alpha^3+120)}{\alpha^3(6+\pi\alpha^3)(1-\theta)^3}\right) \tag{14}$$

$$\mu_4' = \left(\frac{\alpha^4}{(6+\pi\alpha^3)}\right)\int_0^\infty \left[\sum_{z=0}^{\infty} \frac{z^4 \lambda(\lambda+\theta z)^{z-1} e^{-(\lambda+\theta z)}}{\Gamma(z+1)}\right](\pi+\lambda^3)\, e^{-\alpha\lambda} d\lambda$$

$$= \left(\frac{\alpha^4}{(6+\pi\alpha^3)}\right)\int_0^\infty \left[\frac{(1+8\theta+6\theta^2)}{(1-\theta)^7} + \frac{(7+8\theta)\lambda^2}{(1-\theta)^6} + \frac{6\lambda^3}{(1-\theta)^5} + \frac{\lambda^4}{(1-\theta)^4}\right](\pi+\lambda^3)\, e^{-\alpha\lambda} d\lambda$$

$$= \left(\frac{(1+8\theta+6\theta^2)(\pi\alpha^3+24)}{\alpha(6+\pi\alpha^3)(1-\theta)^7}\right) + \left(\frac{(7+8\theta)(2)(\pi\alpha^3+60)}{\alpha^2(6+\pi\alpha^3)(1-\theta)^6}\right)$$
$$+ \left(\frac{36(\pi\alpha^3+120)}{\alpha^3(6+\pi\alpha^3)(1-\theta)^5}\right) + \left(\frac{24(\pi\alpha^3+210)}{\alpha^4(6+\pi\alpha^3)(1-\theta)^4}\right) \tag{15}$$

The first four moments about the mean of PEMGPD are obtained as follows. The first moment about the mean (μ_1) is always zero.

$$\mu_2 = E(z^2) - [E(z)]^2$$

$$= \left(\frac{(\pi\alpha^3+24)}{\alpha(6+\pi\alpha^3)(1-\theta)^3}\right) + \left(\frac{2(\pi\alpha^3+60)}{\alpha^2(6+\pi\alpha^3)(1-\theta)^2}\right) - \left(\frac{(\pi\alpha^3+24)}{\alpha(6+\pi\alpha^3)(1-\theta)}\right)^2$$

$$= \frac{\alpha(24+\pi\alpha^3)(6+\pi\alpha^3)+2(1-\theta)(60+\pi\alpha^3)(6+\pi\alpha^3)-(1-\theta)(24+\pi\alpha^3)^2}{(1-\theta)^3\alpha^2(6+\pi\alpha^3)^2}$$

$$= \frac{(\pi^2\alpha^7+\pi^2\alpha^6+30\pi\alpha^4+84\pi\alpha^3+144\alpha+144-\theta\pi^2\alpha^6-84\pi\theta\alpha^3-144\theta)}{(1-\theta)^3\alpha^2(6+\pi\alpha^3)^2} \quad (16)$$

$$\mu_3 = E(z^3) - 3[E(z^2)][E(z)] + 2[E(z)]^3$$

$$= \left[\left(\frac{(1+2\theta)(\pi\alpha^3+24)}{\alpha(6+\pi\alpha^3)(1-\theta)^5}\right) + \left(\frac{6(\pi\alpha^3+60)}{\alpha^2(6+\pi\alpha^3)(1-\theta)^4}\right) + \left(\frac{6(\pi\alpha^3+120)}{\alpha^3(6+\pi\alpha^3)(1-\theta)^3}\right)\right]$$

$$-3\left[\left(\frac{(\pi\alpha^3+24)}{\alpha(6+\pi\alpha^3)(1-\theta)^3}\right) + \left(\frac{2(\pi\alpha^3+60)}{\alpha^2(6+\pi\alpha^3)(1-\theta)^2}\right)\right]\left[\left(\frac{(\pi\alpha^3+24)}{\alpha(6+\pi\alpha^3)(1-\theta)}\right)\right]$$

$$+2\left[\left(\frac{(\pi\alpha^3+24)}{\alpha(6+\pi\alpha^3)(1-\theta)}\right)\right]^3$$

$$= \frac{\begin{array}{c}\alpha^2(1+2\theta)(\pi\alpha^3+24)(6+\pi\alpha^3)^2+6\alpha(1-\theta)(\pi\alpha^3+60)(6+\pi\alpha^3)^2 \\ +6(1-\theta)^2(\pi\alpha^3+120)(6+\pi\alpha^3)^2-3(1-\theta)\alpha(6+\pi\alpha^3)(\pi\alpha^3+24)^2 \\ -6(1-\theta)^2(6+\pi\alpha^3)(\pi\alpha^3+24)(\pi\alpha^3+60)+2(1-\theta)^2(\pi\alpha^3+24)^3\end{array}}{\alpha^3(1-\theta)^5(6+\pi\alpha^3)^3} \quad (17)$$

$$\mu_4 = E(z^4) - 4[E(z^3)][E(z)] + 6[E(z^2)][E(z)]^2 - 3[E(z)]^4$$

$$= \left[\begin{array}{l}\left(\frac{(1+8\theta+6\theta^2)(\pi\alpha^3+24)}{\alpha(6+\pi\alpha^3)(1-\theta)^7}\right) + \left(\frac{(7+8\theta)(2)(\pi\alpha^3+60)}{\alpha^2(6+\pi\alpha^3)(1-\theta)^6}\right) \\ +\left(\frac{36(\pi\alpha^3+120)}{\alpha^3(6+\pi\alpha^3)(1-\theta)^5}\right) + \left(\frac{24(\pi\alpha^3+210)}{\alpha^4(6+\pi\alpha^3)(1-\theta)^4}\right)\end{array}\right]$$

$$-4\left[\begin{array}{l}\left(\frac{(1+2\theta)(\pi\alpha^3+24)}{\alpha(6+\pi\alpha^3)(1-\theta)^5}\right) + \left(\frac{6(\pi\alpha^3+60)}{\alpha^2(6+\pi\alpha^3)(1-\theta)^4}\right) \\ +\left(\frac{6(\pi\alpha^3+120)}{\alpha^3(6+\pi\alpha^3)(1-\theta)^3}\right)\end{array}\right]\left[\left(\frac{(\pi\alpha^3+24)}{\alpha(6+\pi\alpha^3)(1-\theta)}\right)\right]$$

$$+6\left[\left(\frac{(\pi\alpha^3+24)}{\alpha(6+\pi\alpha^3)(1-\theta)^3}\right)+\left(\frac{2(\pi\alpha^3+60)}{\alpha^2(6+\pi\alpha^3)(1-\theta)^2}\right)\right]$$

$$\left[\frac{(\pi\alpha^3+24)}{\alpha(6+\pi\alpha^3)(1-\theta)}\right]^2-3\left[\frac{(\pi\alpha^3+24)}{\alpha(6+\pi\alpha^3)(1-\theta)}\right]^4$$

$$=\frac{\left\{\begin{array}{l}\alpha^3(1+8\theta+6\theta^2)(6+\pi\alpha^3)^3(\pi\alpha^3+24)+2\alpha^2(1-\theta)(7+8\theta)\\(6+\pi\alpha^3)^3(\pi\alpha^3+60)+36\alpha(1-\theta)^2(6+\pi\alpha^3)^3(\pi\alpha^3+120)\\+24(1-\theta)^3(6+\pi\alpha^3)^3(\pi\alpha^3+210)\\-4\alpha^2(1-\theta)(1+2\theta)(6+\pi\alpha^3)^2(\pi\alpha^3+24)^2\\-24\alpha(1-\theta)^2(6+\pi\alpha^3)^2(\pi\alpha^3+24)(\pi\alpha^3+60)\\-24(1-\theta)^3(6+\pi\alpha^3)^2(\pi\alpha^3+24)(\pi\alpha^3+120)\\+6\alpha(1-\theta)^2(6+\pi\alpha^3)(\pi\alpha^3+24)^3\\+12(1-\theta)^3(6+\pi\alpha^3)(60+\pi\alpha^3)(\pi\alpha^3+24)^2-3(\pi\alpha^3+24)^4\end{array}\right\}}{\left\{\alpha^4(1-\theta)^7(6+\pi\alpha^3)^4\right\}} \quad (18)$$

Study about Variability:

Let us suppose that the variance is greater than the mean. That is

$$\mu_2>\mu_1'$$

or,

$$\left[\frac{(\pi^2\alpha^7+\pi^2\alpha^6+30\pi\alpha^4+84\pi\alpha^3+144\alpha+144-\theta\pi^2\alpha^6-84\pi\theta\alpha^3-144\theta)}{(1-\theta)^3\alpha^2(6+\pi\alpha^3)^2}\right]$$

$$>\left(\frac{(\pi\alpha^3+24)}{\alpha(6+\pi\alpha^3)(1-\theta)}\right)$$

On simplification, we get

$$\left\{\begin{array}{l}(\pi^2\alpha^6+84\pi\alpha^3+144-\theta\pi^2\alpha^6-84\pi\theta\alpha^3-144\theta+60\pi\theta\alpha^4-30\pi\theta^2\alpha^4\\+2\theta\pi^2\alpha^7+288\theta\alpha-144\theta^2\alpha-\theta^2\pi^2\alpha^7)\end{array}\right\}>0 \quad (19)$$

The expression (19) will be true if $(1-\theta)<\alpha<\infty$, where $|\theta|<1$.

Study about shape:

$$\gamma_1 = \mu_3 / \mu_2^{3/2}$$

$$= \frac{\begin{array}{l}\alpha^2(1+2\theta)(\pi\alpha^3+24)(6+\pi\alpha^3)^2+6\alpha(1-\theta)(\pi\alpha^3+60)(6+\pi\alpha^3)^2\\ +6(1-\theta)^2(\pi\alpha^3+120)(6+\pi\alpha^3)^2-3(1-\theta)\alpha(6+\pi\alpha^3)(\pi\alpha^3+24)^2\\ -6(1-\theta)^2(6+\pi\alpha^3)(\pi\alpha^3+24)(\pi\alpha^3+60)+2(1-\theta)^2(\pi\alpha^3+24)^3\end{array}}{\alpha^3(1-\theta)^5(6+\pi\alpha^3)^3} \frac{\left[(1-\theta)^3\alpha^2(6+\pi\alpha^3)^2\right]^{3/2}}{(\pi^2\alpha^7+\pi^2\alpha^6+30\pi\alpha^4+84\pi\alpha^3+144\alpha+144-\theta\pi^2\alpha^6-84\pi\theta\alpha^3-144\theta)^{3/2}}$$

$$= \frac{\left[\begin{array}{l}\alpha^2(1+2\theta)(\pi\alpha^3+24)(6+\pi\alpha^3)^2+6\alpha(1-\theta)(\pi\alpha^3+60)(6+\pi\alpha^3)^2\\ +6(1-\theta)^2(\pi\alpha^3+120)(6+\pi\alpha^3)^2-3(1-\theta)\alpha(6+\pi\alpha^3)(\pi\alpha^3+24)^2\\ -6(1-\theta)^2(6+\pi\alpha^3)(\pi\alpha^3+24)(\pi\alpha^3+60)+2(1-\theta)^2(\pi\alpha^3+24)^3\end{array}\right]}{\left[(1-\theta)^{1/2}\right]\left[\begin{array}{l}(\pi^2\alpha^7+\pi^2\alpha^6+30\pi\alpha^4+84\pi\alpha^3+144\alpha\\ +144-\theta\pi^2\alpha^6-84\pi\theta\alpha^3-144\theta)^{3/2}\end{array}\right]} \tag{20}$$

The proposed distribution is positively skewed in shape because $1<\alpha<\infty$ provided that $-1<\theta<1$.

Study about size:

$$\beta_2 = \frac{\mu_4}{\mu_2^2}$$

$$= \frac{\dfrac{\left\{\begin{array}{l}\alpha^3(1+8\theta+6\theta^2)(6+\pi\alpha^3)^3(\pi\alpha^3+24)\\ +2\alpha^2(1-\theta)(7+8\theta)(6+\pi\alpha^3)^3(\pi\alpha^3+60)\\ +36\alpha(1-\theta)^2(6+\pi\alpha^3)^3(\pi\alpha^3+120)\\ +24(1-\theta)^3(6+\pi\alpha^3)^3(\pi\alpha^3+210)\\ -4\alpha^2(1-\theta)(1+2\theta)(6+\pi\alpha^3)^2(\pi\alpha^3+24)^2\\ -24\alpha(1-\theta)^2(6+\pi\alpha^3)^2(\pi\alpha^3+24)(\pi\alpha^3+60)\\ -24(1-\theta)^3(6+\pi\alpha^3)^2(\pi\alpha^3+24)(\pi\alpha^3+120)\\ +6\alpha(1-\theta)^2(6+\pi\alpha^3)(\pi\alpha^3+24)^3\\ +12(1-\theta)^3(6+\pi\alpha^3)(60+\pi\alpha^3)(\pi\alpha^3+24)^2-3(\pi\alpha^3+24)^4\end{array}\right\}}{\left\{\alpha^4(1-\theta)^7(6+\pi\alpha^3)^4\right\}}}{\left[\dfrac{(\pi^2\alpha^7+\pi^2\alpha^6+30\pi\alpha^4+84\pi\alpha^3+144\alpha+144-\theta\pi^2\alpha^6-84\pi\theta\alpha^3-144\theta)}{(1-\theta)^3\alpha^2(6+\pi\alpha^3)^2}\right]^2}$$

$$= \frac{\left\{\begin{array}{l} \alpha^3(1+8\theta+6\theta^2)(6+\pi\alpha^3)^3(\pi\alpha^3+24) \\ +2\alpha^2(1-\theta)(7+8\theta)(6+\pi\alpha^3)^3(\pi\alpha^3+60) \\ +36\alpha(1-\theta)^2(6+\pi\alpha^3)^3(\pi\alpha^3+120) \\ +24(1-\theta)^3(6+\pi\alpha^3)^3(\pi\alpha^3+210) \\ -4\alpha^2(1-\theta)(1+2\theta)(6+\pi\alpha^3)^2(\pi\alpha^3+24)^2 \\ -24\alpha(1-\theta)^2(6+\pi\alpha^3)^2(\pi\alpha^3+24)(\pi\alpha^3+60) \\ -24(1-\theta)^3(6+\pi\alpha^3)^2(\pi\alpha^3+24)(\pi\alpha^3+120) \\ +6\alpha(1-\theta)^2(6+\pi\alpha^3)(\pi\alpha^3+24)^3 \\ +12(1-\theta)^3(6+\pi\alpha^3)(60+\pi\alpha^3)(\pi\alpha^3+24)^2-3(\pi\alpha^3+24)^4 \end{array}\right\}}{\left[(1-\theta)\right]\left[\begin{array}{c}(\pi^2\alpha^7+\pi^2\alpha^6+30\pi\alpha^4+84\pi\alpha^3+144\alpha \\ +144-\theta\pi^2\alpha^6-84\pi\theta\alpha^3-144\theta\end{array}\right]^2} \quad (21)$$

The PEMGPD is Leptokurtic by size because $4.5<\beta_2<\infty$ provided that $-1<\theta<1$.

Graphical Representation of the mean and variance of PEMGPD:

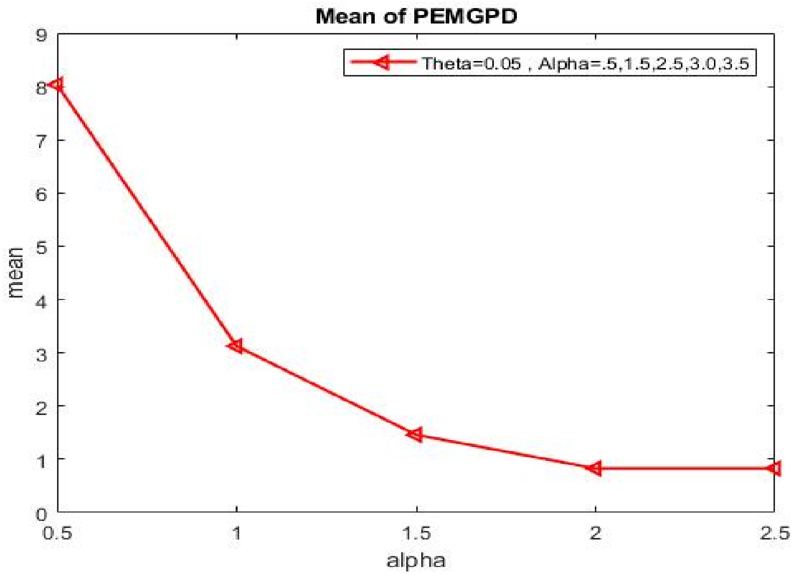

Figure 1: The Mean of PEMGPD at $\theta=.05$ **and** $\alpha=0.5,1.0,1.5,2.0,2.5.$

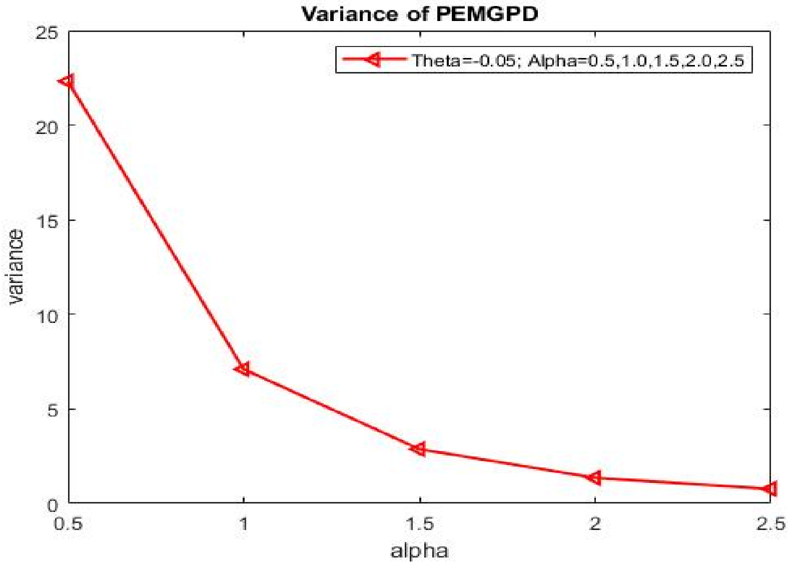

Figure 2: The variance of PEMGPD at $\theta = .05$ **and** $\alpha = 0.5, 1.0, 1.5, 2.0, 2.5.$

3.3 Estimation of the Parameter of PEMGPD

(a) By using μ_1' and μ_2': By using the mean and second moment about the origin of PEMGPD we get a Polynomial equation in term of α which is given as

$$f(\alpha) = \mu_2'(\pi\alpha^3 + 24)^2 - \mu_1'(6 + \pi\alpha^3)\left[\mu_1'\alpha^2(6 + \pi\alpha^3) + (120 + 2\pi\alpha^3)\right] = 0 \qquad (22)$$

Substituting the value of α in the expression (23) and solving for θ, we get an estimated value of θ.

$$(1 - \theta) = \frac{(24 + \pi\alpha^3)}{\mu_1'\alpha(6 + \pi\alpha^3)} \qquad (23)$$

(b) By using μ_1' and $P(z = 0)$: The estimated value of α can be obtained by using $P(z = 0)$ and solving it we get a Polynomial equation in terms of α as

$$f(\alpha) = P(z = 0)(\pi\alpha^3 + 6)(1 + \alpha)^4 - \alpha^4\left\{\pi(1 + \alpha)^3 + 6\right\} = 0 \qquad (24)$$

And θ can be obtained by using the expression (23).

3.4 Goodness of Fit and Applications of GPMMD

Examples (1) and (2) have been used to test goodness of fit of this distribution which were reported by Greenwood and Yule [19] and Garman [20] respectively.

Table 1
Example (1)
Distribution of Number of Accidents of 647 Women Working on H.E. Shells during 5 Weeks, Reported by Greenwood and Yule [19].

Number of Accidents	0	1	2	3	4	5^+
Observed Frequency	447	132	42	21	3	2

Table 2
Example (2)
Distribution of the Counts of the Number of European Red Mites on Apple Leaves, Reported by Garman [20].

Number of Red Mites per Leaf	0	1	2	3	4	5	6	7^+
Observed Frequency	70	38	17	10	9	3	2	1

Table 3
GNBD Verses PEMGPD-I and PEMGPD-II of Example (1)

Number of Accidents	**Observed Frequency**	**Expected Frequency**		
		GNBD	**PEMGPD-I**	**PEMGPD-II**
0	447	443.7	447.0	441.2
1	132	136.8	133.2	128.9
2	42	45.1	43.4	46.9
3	21	15.5	15.0	18.0
4	3	4.0	5.4	7.2
5^+	2	1.9	3.0	4.8
Total	647	647.0	647.0	647.0
μ_1'	0.4652241	μ_2	0.6919002	
$\hat{\beta}$	0.755768	-	-	
$\hat{\alpha}$		1.347201	2.705279175	2.63056137
$\hat{\theta}$		0.0641993	0.004267494	0.049901464
d.f.		1	2	2
χ^2		1.394	0.1653	1.196
P-value		0.238	0.921	0.55

Table 4
GNBD Verses PEMGPD-I and PEMGPD-II of Example (2)

Number of red mites per leaf	Observed Frequency	Expected Frequency		
		GNBD	**PEMGPD-I**	**PEMGPD-II**
0	70	68.2	70.0	68.3
1	38	37.9	32.7	38.5
2	17	20.8	19.0	19.9
3	10	11.2	11.4	11.2
4	9	6.7	7.0	6.1
5	3	2.7	4.2	3.6
6	2	2.7	3.0	1.6
7^+	1	1.0	2.7	0.8
Total	150.0	150.0	150.0	150
μ_1'	1.1466667	μ_2	2.2584888	
$\hat{\beta}$	0.672061	-	-	
$\hat{\alpha}$		1.983821	1.604074115	1.570230859
$\hat{\theta}$		0.156211	0.0.059642512	-0.105799179
d.f.		2	3	3
χ^2		1.146	1.455	1.295
P-value		0.56381	0.692695	0.731509

In the Table 3 and 4, we have used two words namely Expected frequency of PEMGPD-I and PEMGPD-II denote expected frequency due to Polynomial-exponential mixture of generalised Poisson distribution by using the mean and $P(Z = 0)$, and by using $\mu_1{}'$ and $\mu_2{}'$ respectively. The data used are secondary in nature as well as over-dispersed.

4. CONCLUSION

The Table 5 contains degrees of freedom (*d.f.)*, calculated value of Chi-Square at specific level of degrees of freedom and *P-Value* of example (1) and (2) on the basis of the theoretical concept of GNBD, PEMGPD-I and PEMGPD-II.

Table 5
GNBD Verses PEMGPD-I and PEMGPD-II

Table	GNBD			PEMGPD-I			PEMGPD-II		
	$d.f.$	$\chi^2_{d.f.}$	$P-Value$	$d.f.$	$\chi^2_{d.f.}$	$P-Value$	$d.f.$	$\chi^2_{d.f.}$	$P-Value$
3	1	1.39	0.238	2	0.165	0.921	2	1.196	0.55
4	2	1.17	0.56	3	1.455	0.69	3	1.295	0.73

It is clearly visible that the *P-Value* due to PEMGPD-I and PEMGPD-II are greater than GNBD [4]. Hence, it is suggested to apply the proposed distribution instead of GNBD [4] for the similar set of data-sets used in the Table 5.

CONFLICT OF INTEREST

The authors of this paper are selflessly credited for their contribution to the continuous mixture of generalised Poisson distribution,

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REFERENCES

[1] Consul, P.C. and Jain, G.C. (1973). A generalization of the Poisson distribution. *Technometrics*, 15(4), 791-799.

[2] Sah, B.K. and Sahani, S.K. (2023). Polynomial-Exponential Mixture of Poisson distribution. *Turkish Journal of Computer and Mathematics Education (TURCOMAT)*, 14(03), 505-516.

[3] Sah, B.K. and Sahani, S.K. (2022). Polynomial-exponential distribution. *Mathematical Statistician and Engineering Applications*, 71(4), 2474-2486.

[4] Sah, B.K. (2018). A generalised negative binomial distribution and its important features. *Bulletin of Mathematics and Statistics Research*, 6(3), 82-90.

[5] Sah, B.K. (2017). Poisson-Mishra distribution. *International Journal of Mathematics and Statistics Invention (IJMSI)*, 5(3), 25-30.

[6] Sah, B.K. (2015). Mishra distribution. *International Journal of Mathematics and Statistics Invention (IJMSI)*, 3(8), 14-17.

[7] Shoukri, M.M. (1980). *Estimation problems for some generalized discrete distributions*. Ph.D. Thesis. University of Calgary, Canada.

[8] Ahsanullah, M. (1991). Two characterizations of the generalized Poisson distribution. *Pakistan Journal of Statistics*, 7, 15-19.

[9] Kusumawati, A. and Wong, Y.D. (2015). The applications of generalised Poisson distribution in accident data analysis. *Journal of the Eastern Asia Society for Transportation Studies*, 11, 2189-2208.

[10] Mishra, A. (2009). A new generalised form of geometric distribution. *International Journal of Agricultural and Statistical Sciences*, 5(2), 623-627.

[11] Lehmann, E.L. and Romano, J.P. (2005). *Testing of Statistical Hypothesis.* Third Edition, Springer, New York.

[12] Mishra, A. (2011). A generalization of Feller distribution. *International Journal of Agricultural and Statistical Sciences*, 7(1), 47-51.

[13] Sah, B.K. (2012). *Generalisations of some countable and continuous mixtures of Poisson distribution and their applications*. Doctoral thesis, Patna University, Patna, India.

[14] Sah, B.K. and Mishra, A. (2022). On a generalised exponential-Lindley mixture of generalised Poisson distribution. *Nepalese Journal of Statistics*, 4, 33-42.

[15] Sah, B.K. and Sahani, S.K. (2023). Poisson-Modified Mishra Distribution. *Jilin Daxue Xuebao (Gongxueban) Journal of Jilin University (Engineering and Technology Edition)*, 42(1), 261-275.

[16] Sah, B.K. and Mishra, A. (2021). A generalised Neyman Type-A distribution. *International Journal for Research in Applied Science, Engineering Technology*, 9(1), 465-470.

[17] Sah, B.K. (2015). A two-parameter quasi-Lindley mixture of generalised Poisson distribution. *International Journal of Mathematics and Statistics Invention*, 3(7), 1-5.

[18] Jain, G.C. and Consul, P.C. (1971). A generalised negative binomial distribution. *SIAM J. Appl. Math.*, 21, 501-513.

[19] Greenwood, M. and Yule, G.U. (1920). An inquiry into the nature of frequency distribution representative of multiple happenings with particular reference to the occurrence of multiple attacks of disease or of repeated accidents. *Journal of the Royal Statistical Society*, *Series A*, 83, 288-298.

[20] Garman, P. (1923). The European red mites in Connecticut apple orchards. *Connecticut Agri. Exper. Station Bull.*, 252, 103-125.