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AN ANALYSIS OF COVID-19 DATA EMPLOYING A NOVEL WEIBULL-INVERSE NADARAJAH HAGHIGHI DISTRIBUTION

Aijaz Ahmad[1§], Muzamil Jallal[1], S. Qurat ul Ain[1] and **Afaq Ahmad[2]**
[1] Department of Mathematics, Bhagwant University, Ajmer, India
[2] Department of Mathematical Science, IUST, Awantipora, Kashmir, India
[§] Corresponding author Email: ahmadaijaz4488@gmail.com

ABSTRACT

Over numerous decades, academics have been attempting to develop a number of novel distributions to satisfy certain realistic demands. The rationale is that conventional distributions have generally been shown to lack fit in actual applications, such as medicinal research, engineering, hydrology, environmental science, and many more. Combining the Weibull and inverse Nadarajah Haghighi distributions generates a novel life-time distribution with four parameters, which is referred to as the Weibull-inverse Nadarajah Haghighi (WINH) distribution. Different structural characteristics of the formulated distribution have been determined and analysed. Distinct plots depict the behaviour of the probability density function (pdf) and the cumulative distribution function (cdf). The maximum likelihood estimation method is applied to estimate the stated distribution parameters. To assess and investigate the efficacy of estimators in terms of bias, variance, and mean square error (MSE), a simulation study was conducted. Lastly, the effectiveness of the stated distribution is proven by actual data sets.

KEYWORDS

Moments, moment generating function, reliability measures, mean deviations, maximum likelihood function.

Mathematics subject classification: 60-XX, 62-XX, 11-KXX.

1. INTRODUCTION

Over numerous decades, academics have been attempting to develop a number of novel distributions to satisfy certain realistic demands. The rationale is that conventional distributions have generally been shown to lack fit in actual applications, such as medicinal research, engineering, hydrology, environmental science, and many more. In particular, the objective of creating novel distributions or generalizations is to construct adaptable statistical models effective at dealing with complicated real-world data. This adaptability may be obtained in a straightforward manner by introducing new parameters to the standard distribution.

The Weibull distribution has been utilized in a variety of disciplines and applications. The hazard function of the Weibull distribution can only be monotonic in nature. As a result, it cannot be employed to simulate lifespan data with a bathtub-shaped hazard function.

Suppose T denotes a random variable that follows the Weibull distribution, then its probability density function is stated as

$$r(t;\alpha,\beta)=\alpha\beta t^{\beta-1}e^{-\alpha t^{\beta}};t>0,\alpha,\beta>0 \tag{1.1}$$

The exponential distribution is well-known for its constant hazard rate and memory less feature. This distribution cannot be used to analyze data with a monotonic hazard rate. Nadarajah and Haghighi (2011) developed a novel extension of the exponential distribution as a substitute model for the gamma and Exponentiated-exponential distributions. The probability density function of Nadarajah and Haghighi (NH) distribution is stated as

$$h(x;\theta,\lambda)=\theta\lambda(1+\lambda x)^{\theta-1}e^{1-(1+\lambda x)^{\theta}};x>0,\theta,\lambda>0 \tag{1.2}$$

The transformation $Y=\frac{1}{X}$, yields the inverse of the Nadarajah and Haghighi distribution. As a result, the probability density function (pdf) of Nadarajah and Haghighi (NH) distribution is stated as

$$g(y;\theta,\lambda)=\frac{\theta\lambda}{y^{2}}\left(1+\frac{\lambda}{y}\right)^{\theta-1}e^{1-\left(1+\frac{\lambda}{y}\right)^{\theta}};y>0,\theta,\lambda>0 \tag{1.3}$$

The associated cumulative distribution function of (1.3) is given as

$$G(y;\theta,\lambda)=e^{1-\left(1+\frac{\lambda}{y}\right)^{\theta}};y>0,\theta,\lambda>0 \tag{1.4}$$

The objective of this research is to generalize the inverse Nadarajah-Haghighi distribution by inserting two extra parameters. The generalised distribution is referred to as the Weibull-Inverse Nadarajah Haghighi distribution (WINHD). The extra parameters will provide us greater flexibility in evaluating the tail behaviour of the defined density function. Moreover, the explored distribution may be employed to manage various elements of the hazard rate function. The Nadarajah-Haghighi has been studied thoroughly and employed in range of aspects of research. Abdul-Moniem (2015), Yousof et al. (2017), Korkmaz et al. (2017), Tahir et al. (2018), Reyad et al. (2019), Ahmad et al. (2022), Jallal et al. (2022), Lone et al. (2022) and Shafiq et al. (2021).

In recent decades, researchers have concentrated on discovering novel generators from continuous conventional distributions. As an outcome, the resulting distribution enhances the efficacy and adaptability of data analysis. The following are some generated families of distribution: the beta-G family of distribution investigated by Eugene et al. (2002), the gamma-G family by Zografos and Balakrishnan (2009), the kumaraswamy-G family by Cordeiro et al. (2011), the transformed-transformer(T-X) by Alzaatrh et al. (2013), the Weibull-G by Bourguignon et al. (2014), Brito et al. (2017) created the Topp-Leone odd log-logistic family of distributions, Brito et al. (2017) constructed the Gompertz-G distribution family, and Alizadeh et al. (2017) established the inverse Weibull-G distribution.

T-X family of distributions defined by Alzaatreh et al. (2013) is given by

$$F(y) = \int_0^{W[G(y)]} v(t)\,dt \tag{1.5}$$

where $v(t)$ be the probability density function of a random variable T and $W\left[G(y)\right]$ be a function of cumulative density function of random variable Y .

Suppose $G(y,\zeta)$ denotes the baseline cumulative distribution function, which depends on parameter vector ζ . Now using T-X approach, the cumulative distribution function $F(y)$ of Weibull generator (WG) can be derived by replacing $r(t)$ in equation (1.4) with (1.1) and $W\left[G(y)\right] = \frac{G(y,\zeta)}{\bar{G}(y,\zeta)}$, where $\bar{G}(y,\zeta) = 1 - G(y,\zeta)$ which follows

$$F(y;\alpha,\beta,\zeta) = \int_0^{\frac{G(y,\zeta)}{\bar{G}(y,\zeta)}} \alpha\beta t^{\beta-1} e^{-\alpha t^{\beta}}\,dt$$

$$= 1 - e^{-\alpha\left(\frac{G(y,\zeta)}{\bar{G}(y,\zeta)}\right)^{\beta}} ;\; y > 0, \alpha, \beta, \zeta > 0 \tag{1.6}$$

The associated pdf of (1.6) becomes

$$f(y;\alpha,\beta,\zeta) = \alpha\beta g(y,\zeta) \frac{\left[G(y,\zeta)\right]^{\beta-1}}{\left[\bar{G}(y,\zeta)\right]^{\beta+1}} e^{-\alpha\left(\frac{G(y,\zeta)}{\bar{G}(y,\zeta)}\right)^{\beta}} ;\; y > 0, \alpha, \beta, \zeta > 0 \tag{1.7}$$

The survival $S(y)$, hazard rate function $h(y)$ and cumulative hazard function $H(y)$ are respectively given by

$$S(y) = 1 - F(y;\alpha,\beta,\zeta) = e^{-\alpha\left(\frac{G(y,\zeta)}{\bar{G}(y,\zeta)}\right)^{\beta}}$$

$$h(y) = \alpha\beta g(y,\zeta) \frac{\left[G(y,\zeta)\right]^{\beta-1}}{\left[\bar{G}(y,\zeta)\right]^{\beta+1}}$$

$$H(y) = -\log\left(\bar{F}(y)\right) = \alpha\left(\frac{G(y,\zeta)}{\bar{G}(y,\zeta)}\right)^{\beta}$$

1.1 Useful Expansion

We use Taylor's series to the exponential function of the pdf in equation (1.7), we have

$$e^{-\alpha\left(\frac{G(y,\zeta)}{\bar{G}(y,\zeta)}\right)^{\beta}} = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!}\alpha^p\left(\frac{G(y,\zeta)}{\bar{G}(y,\zeta)}\right)^{p\beta} \tag{1.8}$$

Using equation (1.8) in equation (1.7), we get

$$f(y;\alpha,\beta,\zeta) = \sum_{p=0}^{\infty}\frac{(-1)^p}{p!}\alpha^{p+1}\beta g(y,\zeta)\left(G(y,\zeta)\right)^{\beta(p+1)-1}\left(\bar{G}(y,\zeta)\right)^{-(\beta(p+1)+1)} \tag{1.9}$$

We know the generalized binomial expansion as

$$(1-z)^{-a} = \sum_{q=0}^{\infty}\binom{a+q-1}{q}z^q \qquad ;a>0,|z|<1$$

Now

$$\left(\bar{G}(y,\zeta)\right)^{-(\beta(p+1)+1)} = \left(1-G(y,\zeta)\right)^{-(\beta(p+1)+1)} = \sum_{q=0}^{\infty}\binom{\beta(p+1)+q}{q}\left(G(y,\zeta)\right)^q \tag{1.10}$$

Using equation (1.10) in equation (1.9), we have

$$\begin{aligned} f(y;\alpha,\beta,\zeta) &= \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(-1)^p}{p!}\binom{\beta(p+1)+q}{q}\alpha^{p+1}\beta g(y,\zeta)\left(G(y,\zeta)\right)^{\beta(p+1)+q-1} \\ &= \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\delta_{p,q}\, g(y,\zeta)\left(G(y,\zeta)\right)^{\beta(p+1)+q-1} \end{aligned} \tag{1.11}$$

where

$$\delta_{p,q} = \frac{(-1)^p}{p!}\binom{\beta(p+1)+q}{q}\alpha^{p+1}\beta$$

2. WEIBULL-INVERSE NADARAJAH HAGHIGHI (WINH) DISTRIBUTION

In this part, we construct the cumulative distribution function (cdf) and probability density function (pdf) of the Weibull-Inverse Nadarajah Haghighi distribution and analyze the behavior of the cdf and pdf employing different layouts. Using equation (1.4) in equation (1.6), the cumulative distribution function of (WINH) is given by

$$F(y;\alpha,\beta,\theta,\lambda) = 1-e^{-\alpha\left(e^{\left(1+\frac{\lambda}{y}\right)^{\theta}-1}-1\right)^{-\beta}} \quad ;y>0,\alpha,\beta,\theta,\lambda>0 \tag{2.1}$$

Figure (2.1) and (2.2) depicts a few of the most likely contours of the cdf for various parameter values of WINHD.

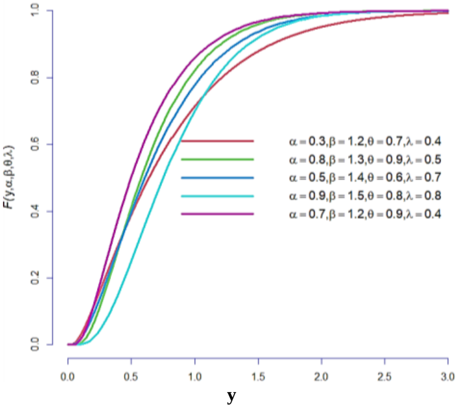

Figure 2.1: cdf of WINHD under Different Values of Parameters

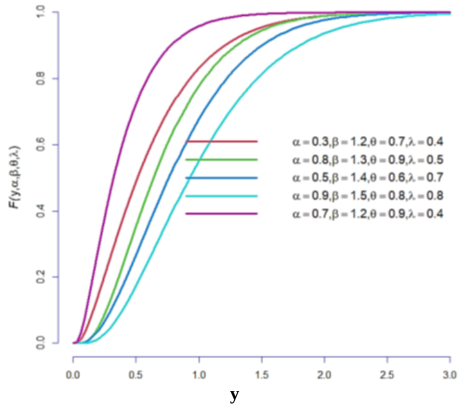

Figure 2.2: cdf of WINHD under Different Values of Parameters

The associated probability density function of (WINHD) is given by

$$f\left(y;\alpha,\beta,\theta,\lambda\right)=\frac{\alpha\beta\theta\lambda}{y^{2}}\left(1+\frac{\lambda}{y}\right)^{\theta-1}e^{\left(1+\frac{\lambda}{y}\right)^{\theta}-1}\left(e^{\left(1+\frac{\lambda}{y}\right)^{\theta}-1}-1\right)^{-\beta-1}e^{-\alpha\left(e^{\left(1+\frac{\lambda}{y}\right)^{\theta}-1}-1\right)^{-\beta}};$$

$$y>0,\alpha,\beta,\theta,\lambda>0 \quad (2.2)$$

Figure (2.3) and (2.4) depicts a few of the most likely contours of the pdf for various parameter values of WINHD.

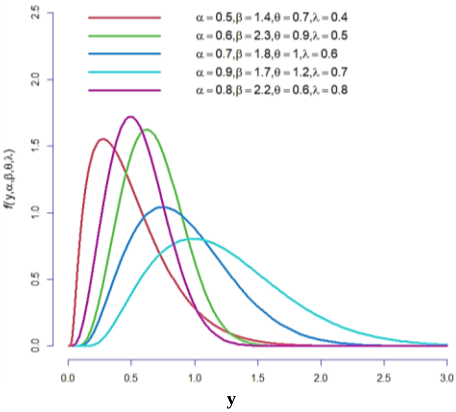

Figure 2.3: pdf of WINHD under Different Values of Parameters

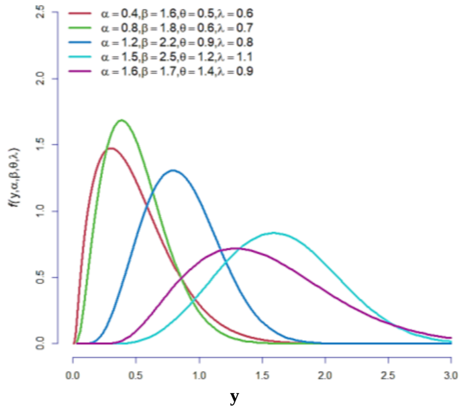

Figure 2.4: pdf of WINHD under Different Values of Parameters

3. RELIABILITY MEASURES OF WINH DISTRIBUTION

The reliability function is also known as survival function of a continuous random variable y having cdf $F(y)$, is defined as

$$S(y) = p_r(Y > y) = \int_y^{\infty} F(y)\,dy = 1 - F(y)$$

The survival function of WINH distribution is given as

$$S(y;\alpha,\beta,\theta,\lambda) = 1 - F(y;\alpha,\beta,\theta,\lambda)$$

$$S(y;\alpha,\beta,\theta,\lambda) = e^{-\alpha\left(e^{\left(1+\frac{\lambda}{y}\right)^{\theta}-1}-1\right)^{-\beta}} \tag{3.1}$$

The hazard rate function of a continuous random variable y is defined as

$$h(y;\alpha,\beta,\theta,\lambda) = \frac{f(y;\alpha,\beta,\theta,\lambda)}{S(y;\alpha,\beta,\theta,\lambda)} \tag{3.2}$$

Using equation (2.2) and (3.1) in equation (3.2), we obtain the hazard rate function of WINH distribution

$$h(y;\alpha,\beta,\theta,\lambda) = \frac{\alpha\beta\theta\lambda}{y^2}\left(1+\frac{\lambda}{y}\right)^{\theta-1} e^{\left(1+\frac{\lambda}{y}\right)^{\theta}-1}\left(e^{\left(1+\frac{\lambda}{y}\right)^{\theta}-1}-1\right)^{-\beta-1}$$

Figure (3.1) and (3.2) depicts a few of the most likely contours of the hazard rate function for various parameter values of WINHD.

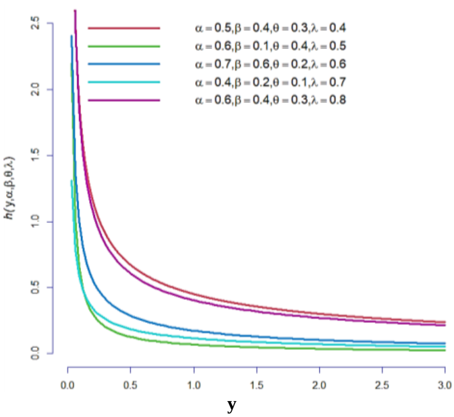

Figure 3.1: hrf of WINHD under Different Values of Parameters

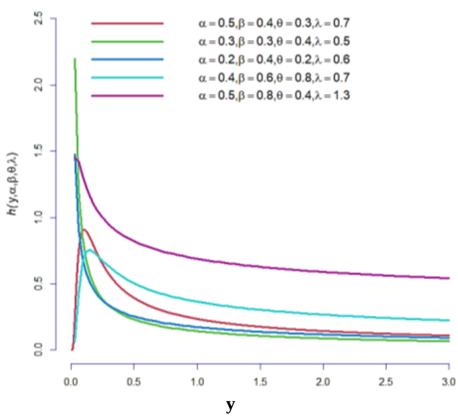

Figure 3.2: hrf of WINHD under Different Values of Parameters

The cumulative hazard rate function of a continuous random variable y is defined as

$$H(y) = -\log\left[\bar{F}(y;\alpha,\beta,\theta,\lambda)\right] \tag{3.3}$$

Using equation (2.1) in equation (3.3), we obtain the cumulative hazard rate function of WINH distribution

$$H(y;\alpha,\beta,\theta,\lambda) = \alpha\left(e^{\left(1+\frac{\lambda}{y}\right)^{\theta}-1}-1\right)^{-\beta}.$$

4. MATHEMATICAL PROPERTIES OF WINH DISTRIBUTION

4.1 Moments of WINH Distribution

Let Y denotes the random variable follows WINH distribution. Then the r^{th} moment denoted by μ'_r is stated as

$$\mu'_r = E\left(Y^r\right) = \int_0^{\infty} y^r f(y;\alpha,\beta,\theta,\lambda)\,dy$$

Using equation (1.11), we have

$$\mu'_r = \int_0^{\infty} y^r \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \delta_{p,q}\, g(y,\zeta)\left(G(y,\zeta)\right)^{\beta(p+1)+q-1} dy \tag{4.1}$$

Now using equations (1.3) and (1.4), in equation (4.1), we get

$$\mu'_r = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \delta_{p,q}\theta\lambda \int_0^{\infty} y^{r-2}\left(1+\frac{\lambda}{y}\right)^{\theta-1}\left(e^{-\left(1+\frac{\lambda}{y}\right)^{\theta}+1}\right)^{\beta(p+1)+q} dy$$

For convenience take $\beta(p+1)+q=\upsilon$, we have

$$\mu'_r = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \delta_{p,q}\theta\lambda e^{\upsilon} \int_0^{\infty} y^{r-2}\left(1+\frac{\lambda}{y}\right)^{\theta-1} e^{-\upsilon\left(1+\frac{\lambda}{y}\right)^{\theta}} dy$$

Making substitution $\upsilon\left(1+\frac{\lambda}{y}\right)^{\theta} = z$, so that $\upsilon < z < \infty$, we have

$$\mu'_r = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \delta_{p,q}\lambda^r \frac{e^{\upsilon}}{\upsilon} \int_{\upsilon}^{\infty}\left(\left(\frac{z}{\upsilon}\right)^{\frac{1}{\theta}}-1\right)^{-r} e^{-z} dz$$

$$= \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \delta_{p,q}\lambda^r \frac{e^{\upsilon}}{\upsilon}(-1)^{-r} \int_{\upsilon}^{\infty}\left(1-\left(\frac{z}{\upsilon}\right)^{\frac{1}{\theta}}\right)^{-r} e^{-z} dz \tag{4.2}$$

Using the following expansion in (4.2), we have

$$\left(1-w^{\frac{1}{a}}\right)^{-r}=1+\sum_{s=1}^{\infty}\frac{(-1)^{s+1}}{s!}\prod_{x=0}^{s-1}(r+x)w^{\frac{s}{a}}=\sum_{s=0}^{\infty}c_s(r)w^{\frac{s}{a}}$$

where $c_s(r)=\frac{(-1)^{s+1}}{s!}\prod_{x=0}^{s-1}(r+x)$

$$\mu'_r=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}c_s(r)\delta_{p,q}\lambda^r\frac{e^{\upsilon}}{\upsilon}(-1)^{-r}\int_{\upsilon}^{\infty}\left(\frac{z}{\upsilon}\right)^{\frac{s}{\theta}}e^{-z}dz$$

$$=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}c_s(r)\delta_{p,q}\lambda^r e^{\upsilon}\upsilon^{\frac{-(s+\theta)}{\theta}}(-1)^{-r}\int_{\upsilon}^{\infty}z^{\frac{s}{\theta}}e^{-z}dz$$

After solving the integral, we get

$$\mu'_r=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}c_s(r)\delta_{p,q}\lambda^r e^{\upsilon}\upsilon^{\frac{-(s+\theta)}{\theta}}(-1)^{-r}\Gamma\left(1+\frac{s}{\theta},\upsilon\right).$$

4.2 Moment Generating Function of WINH Distribution

Let Y be a random variable follows WINH distribution. Then the moment generating function of the distribution denoted by $M_Y(t)$ is given

$$M_Y(t)=E\left(e^{ty}\right)=\int_0^{\infty}e^{ty}f(y,\alpha,\beta,\theta,\lambda)dy$$

Using Taylor's series

$$=\int_0^{\infty}\left(1+ty+\frac{(ty)^2}{2!}+\frac{(ty)^3}{3!}+\dots\right)f(y,\alpha,\beta,\theta,\lambda)dy$$

$$=\sum_{r=0}^{\infty}\frac{t^r}{r!}\int_0^{\infty}y^r f(y,\alpha,\beta,\theta)dy$$

$$=\sum_{r=0}^{\infty}\frac{t^r}{r!}E\left(Y^r\right)$$

$$M_Y(t)=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}\sum_{r=0}^{\infty}\frac{t^r}{r!}c_s(r)\delta_{p,q}\lambda^r e^{\upsilon}\upsilon^{\frac{-(s+\theta)}{\theta}}(-1)^{-r}\Gamma\left(1+\frac{s}{\theta},\upsilon\right).$$

4.3 Incomplete Moments of WINH Distribution

The s^{th} incomplete moment of WINH about origin is given by

$$T_s\left(r\right)=\int_0^y y^r f\left(y,\alpha,\beta,\theta,\lambda\right)dy$$

From equations (1.11) and (1.4), we have

$$T_s\left(r\right)=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\delta_{p,q}e^{\upsilon}\theta\lambda\int_0^y y^{r-2}\left(1+\frac{\lambda}{y}\right)^{\theta-1}e^{-\upsilon\left(1+\frac{\lambda}{y}\right)^{\theta}}dy$$

Making substitution $\upsilon\left(1+\frac{\lambda}{y}\right)^{\theta}=z$ so that $\infty<z<\upsilon\left(1+\frac{\lambda}{y}\right)^{\theta}$, we have

$$T_s\left(r\right)=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\delta_{p,q}e^{\upsilon}\frac{\lambda^r}{\upsilon}\left(-1\right)^r\int_{\upsilon\left(1+\frac{\lambda}{y}\right)^{\theta}}^{\infty}\left(1-\left(\frac{z}{\upsilon}\right)^{\frac{1}{\theta}}\right)^{-r}e^{-z}dz$$

$$=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}c_s\left(r\right)\delta_{p,q}e^{\upsilon}\lambda^r\left(-1\right)^r\upsilon^{\frac{-(s+\theta)}{\theta}}\int_{\upsilon\left(1+\frac{\lambda}{y}\right)^{\theta}}^{\infty}z^{\frac{s}{\theta}}e^{-z}dz$$

After solving the integral, we get

$$T_s\left(r\right)=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}c_s\left(r\right)\delta_{p,q}e^{\upsilon}\lambda^r\left(-1\right)^r\upsilon^{\frac{-(s+\theta)}{\theta}}\Gamma\left(\frac{s+\theta}{\theta},\upsilon\left(1+\frac{\lambda}{y}\right)^{\theta}\right)$$

4.4 Quantile Function of WINH Distribution

The quantile function of any distribution may be described as follows:

$$Q\left(u\right)=Y_q=F^{-1}\left(u\right)$$

where $Q\left(u\right)$ denotes the quantile function of $F\left(y\right)$ for $u\in\left(0,1\right)$.

Let us suppose

$$F\left(y\right)=1-e^{-\alpha\left(e^{\left(1+\frac{\lambda}{y}\right)^{\theta}-1}-1\right)^{-\beta}}=u \tag{4.3}$$

After simplifying equation (4.3), we obtain quantile function of WINH distribution as

$$Q\left(u\right)=Y_q=\left[\frac{1}{\lambda}\left(1+\log\left(1+\left(\frac{-1}{\alpha}\log\left(1-u\right)\right)^{\frac{-1}{\beta}}\right)\right)^{\frac{1}{\theta}}-1\right]^{-1}.$$

5. MEAN DEVIATION FROM MEAN AND MEDIAN OF WINH DISTRIBUTION

The entirety of deviations is apparently a measure of amount of dispersion in a population. Let Y be a random variable from WINH distribution with mean μ. Then the mean deviation from mean is defined as.

$$\begin{aligned}
D(\mu) &= E\left(|Y-\mu|\right) \\
&= \int_0^\infty |Y-\mu| f(y)\,dy \\
&= \int_0^\mu (\mu-y) f(y)\,dy + \int_\mu^\infty (y-\mu) f(y)\,dy \\
&= 2\mu F(\mu) - 2\int_0^\mu y f(y)\,dy
\end{aligned} \tag{5.1}$$

Now

$$\int_0^\mu y f(y)\,dy = \sum_{p=0}^\infty \sum_{q=0}^\infty \delta_{p,q} e^{\upsilon}\theta\lambda \int_0^\mu \frac{1}{y}\left(1+\frac{\lambda}{y}\right)^{\theta-1} e^{-\upsilon\left(1+\frac{\lambda}{y}\right)^{\theta}} dy$$

Making substitution $\upsilon\left(1+\frac{\lambda}{y}\right)^{\theta} = z$ so that $\infty < z < \upsilon\left(1+\frac{\lambda}{\mu}\right)^{\theta}$, we have

$$\int_0^\mu y f(y)\,dy = \sum_{p=0}^\infty \sum_{q=0}^\infty \delta_{p,q} (-1)^{-1} e^{\upsilon} \frac{\lambda}{\upsilon} \int_{\upsilon\left(1+\frac{\lambda}{\mu}\right)^{\theta}}^\infty \left(1-\left(\frac{z}{\upsilon}\right)^{\frac{1}{\theta}}\right)^{-1} e^{-z} dz$$

$$= \sum_{p=0}^\infty \sum_{q=0}^\infty \sum_{s=0}^\infty c_s(1)\delta_{p,q} (-1)^{-1} \frac{\lambda e^{\upsilon}}{\upsilon} \int_{\upsilon\left(1+\frac{\lambda}{\mu}\right)^{\theta}}^\infty \left(\frac{z}{\upsilon}\right)^{\frac{s}{\theta}} e^{-z} dz$$

$$\int_0^\mu y f(y)\,dy = \sum_{p=0}^\infty \sum_{q=0}^\infty \sum_{s=0}^\infty c_s(1)\delta_{p,q} (-1)^{-1} \lambda e^{\upsilon} \upsilon^{\frac{-(s+\theta)}{\theta}} \int_{\upsilon\left(1+\frac{\lambda}{\mu}\right)^{\theta}}^\infty z^{\frac{s}{\theta}} e^{-z} dz$$

After solving the integral, we get

$$\int_0^\mu y f(y)\,dy = \sum_{p=0}^\infty \sum_{q=0}^\infty \sum_{s=0}^\infty c_s(1)\delta_{p,q} (-1)^{-1} \lambda e^{\upsilon} \upsilon^{\frac{-(s+\theta)}{\theta}} \Gamma\left(\frac{s}{\theta}+1, \upsilon\left(1+\frac{\lambda}{\mu}\right)^{\theta}\right) \tag{5.2}$$

Substituting value of equation (5.2) in (5.1), we get

$$D(\mu) = 2\mu - 2\mu e^{-\alpha\left(e^{\left(1+\frac{\lambda}{\mu}\right)^{\theta}-1}-1\right)_{-\beta}}$$

$$-2\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{s=0}^{\infty} c_s(1)\delta_{p,q}(-1)^{-1}\lambda e^{\upsilon}\upsilon^{\frac{-(s+\theta)}{\theta}}\Gamma\left(\frac{s}{\theta}+1, \upsilon\left(1+\frac{\lambda}{\mu}\right)^{\theta}\right)$$

Let Y be a random variable from WINH distribution with median M. Then the mean deviation from median is defined as.

$$D(M) = E(|Y-M|)$$

$$= \int_0^{\infty} |Y-M| f(y)\,dy$$

$$= \int_0^{M} (M-y) f(y)\,dy + \int_M^{\infty} (y-M) f(y)\,dy$$

$$= \mu - 2\int_0^{M} y f(y)\,dy \tag{5.3}$$

Now

$$\int_0^{M} y f(y)dy = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\delta_{p,q}e^{\upsilon}\theta\lambda\int_0^{M}\frac{1}{y}\left(1+\frac{\lambda}{y}\right)^{\theta-1} e^{-\upsilon\left(1+\frac{\lambda}{y}\right)^{\theta}} dy$$

Making substitution $\upsilon\left(1+\frac{\lambda}{y}\right)^{\theta} = z$ so that $\infty < z < \upsilon\left(1+\frac{\lambda}{M}\right)^{\theta}$, we have

$$\int_0^{M} y f(y)dy = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{s=0}^{\infty} c_s(1)\delta_{p,q}(-1)^{-1}\frac{\lambda e^{\upsilon}}{\upsilon}\int_{\upsilon\left(1+\frac{\lambda}{M}\right)^{\theta}}^{\infty}\left(\frac{z}{\upsilon}\right)^{\frac{s}{\theta}} e^{-z}dz$$

$$\int_0^{M} y f(y)dy = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{s=0}^{\infty} c_s(1)\delta_{p,q}(-1)^{-1}\lambda e^{\upsilon}\upsilon^{\frac{-(s+\theta)}{\theta}}\int_{\upsilon\left(1+\frac{\lambda}{M}\right)^{\theta}}^{\infty} z^{\frac{s}{\theta}} e^{-z}dz$$

After solving the integral, we get

$$\int_0^{M} y f(y)dy = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{s=0}^{\infty} c_s(1)\delta_{p,q}(-1)^{-1}\lambda e^{\upsilon}\upsilon^{\frac{-(s+\theta)}{\theta}}\Gamma\left(\frac{s}{\theta}+1, \upsilon\left(1+\frac{\lambda}{M}\right)^{\theta}\right) \tag{5.4}$$

Substituting value of equation (5.4) in (5.3), we get

$$D(M) = \mu - 2\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{s=0}^{\infty} c_s(1)\delta_{p,q}(-1)^{-1}\lambda e^{\upsilon}\upsilon^{\frac{-(s+\theta)}{\theta}}\Gamma\left(\frac{s}{\theta}+1,\upsilon\left(1+\frac{\lambda}{M}\right)^{\theta}\right).$$

6. RENYI ENTROPY OF WINH DISTRIBUTION

If Y denotes a continuous random variable having probability density function $f(y)$. Then Renyi entropy is defined as

$$T_R(\rho) = \frac{1}{1-\rho}\log\left\{\int_0^{\infty} f^{\rho}(y)\,dy\right\}, \text{ where } \rho > 0 \text{ and } \rho \neq 1$$

Thus, the Renyi entropy of WINH distribution is given as

$$T_R(\rho) = \frac{1}{1-\rho}\log\left\{\int_0^{\infty}\left(\alpha\beta g(y,\zeta)\frac{\left(G(y,\zeta)\right)^{\beta-1}}{\left(\bar{G}(y,\zeta)\right)^{\beta+1}} e^{-\alpha\left(\frac{G(y,\zeta)}{\bar{G}(y,\zeta)}\right)^{\beta}}\right)^{\rho} dy\right\}$$

$$= \frac{1}{1-\rho}\log\left\{(\alpha\beta)^{\rho}\int_0^{\infty}\left(g(y,\zeta)\right)^{\rho}\frac{\left(G(y,\zeta)\right)^{\rho(\beta-1)}}{\left(\bar{G}(y,\zeta)\right)^{\rho(\beta+1)}} e^{-\rho\alpha\left(\frac{G(y,\zeta)}{\bar{G}(y,\zeta)}\right)^{\beta}} dy\right\}$$

Apply Taylor's expansion for exponential function which is

$$e^{-mz} = \sum_{p=0}^{\infty}\frac{(-1)^p}{p!}(mz)^p$$

Now

$$T_R(\rho) = \frac{1}{1-\rho}\log$$
$$\left\{\sum_{p=0}^{\infty}\frac{(-1)^p}{p!}(\alpha\beta)^{\rho}(\alpha\rho)^{p}\int_0^{\infty}\left(g(y,\zeta)\right)^{\rho}\left(G(y,\zeta)\right)^{\beta(p+\rho)-\rho}\left(\bar{G}(y,\zeta)\right)^{-\beta(p+\rho)-\rho} dy\right\}$$

Using generalized binomial expansion, we have

$$(1-z)^{-a} = \sum_{q=0}^{\infty}\binom{a+q-1}{q}z^q \qquad ; a > 0, |z| < 1$$

So that

$$\left(\bar{G}\left(y,\zeta\right)\right)^{-\beta\left(p+\rho\right)-\rho}=\left(1-G\left(y,\zeta\right)\right)^{-\beta\left(p+\rho\right)-\rho}=\sum_{q=0}^{\infty}\binom{\beta\left(p+\rho\right)+\rho+q-1}{q}\left(G\left(y,\zeta\right)\right)^{q}$$

Applying above, we get

$$T_R\left(\rho\right)=\frac{1}{1-\rho}\log\left\{\begin{array}{l}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{\left(-1\right)^{p}}{p!}\binom{\beta\left(p+\rho\right)+\rho+q-1}{q}\left(\alpha\beta\right)^{\rho}\left(\alpha\rho\right)^{p}\\ \times\int_{0}^{\infty}\left(g\left(y,\zeta\right)\right)^{\rho}\left(G\left(y,\zeta\right)\right)^{\beta\left(p+\rho\right)-\rho+q}dy\end{array}\right\}$$

$$=\frac{1}{1-\rho}\log\left\{\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\omega_{p,q}\int_{0}^{\infty}\left(g\left(y,\zeta\right)\right)^{\rho}\left(G\left(y,\zeta\right)\right)^{\beta\left(p+\rho\right)-\rho+q}dy\right\} \quad (6.1)$$

where

$$\omega_{p,q}=\frac{\left(-1\right)^{p}}{p!}\binom{\beta\left(p+\rho\right)+\rho+q-1}{q}\left(\alpha\beta\right)^{\rho}\left(\alpha\rho\right)^{p}$$

Using equations (1.3) and (1.4) in equation (6.1), we have

$$T_R\left(\rho\right)=\frac{1}{1-\rho}\log\left\{\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\omega_{p,q}\left(\theta\lambda\right)^{\rho}\int_{0}^{\infty}\frac{1}{y^{2\rho}}\left(1+\frac{\lambda}{y}\right)^{\rho\left(\theta-1\right)}\left(e^{1-\left(1+\frac{\lambda}{y}\right)_{\theta}}\right)^{\beta\left(p+\rho\right)+q}dy\right\}$$

For convenience take $\beta\left(p+\rho\right)+q=\upsilon$, we have

$$T_R\left(\rho\right)=\frac{1}{1-\rho}\log\left\{\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\omega_{p,q}\left(\theta\lambda\right)^{\rho}e^{\upsilon}\int_{0}^{\infty}\frac{1}{y^{2\rho}}\left(1+\frac{\lambda}{y}\right)^{\rho\left(\theta-1\right)}e^{-\upsilon\left(1+\frac{\lambda}{y}\right)_{\theta}}dy\right\}$$

Making substitution $\upsilon\left(1+\frac{\lambda}{y}\right)^{\theta}=z$, so that $\upsilon<z<\infty$, we have

$$T_R\left(\rho\right)=\frac{1}{1-\rho}\log\left\{\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\omega_{p,q}e^{\upsilon}\frac{\left(\theta\lambda\right)^{\rho}\lambda^{1-2\rho}}{\upsilon\theta}\int_{\upsilon}^{\infty}\left(\left(\frac{z}{\upsilon}\right)^{\frac{1}{\theta}}-1\right)^{2\rho-2}\left(\frac{z}{\upsilon}\right)^{\frac{1}{\theta}-1}e^{-z}dz\right\}$$

$$=\frac{1}{1-\rho}\log\left\{\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\omega_{p,q}\left(-1\right)^{2\rho-2}\frac{e^{\upsilon}\left(\theta\lambda\right)^{\rho}\lambda^{1-2\rho}}{\upsilon\theta}\int_{\upsilon}^{\infty}\left(1-\left(\frac{z}{\upsilon}\right)^{\frac{1}{\theta}}\right)^{2\rho-2}\left(\frac{z}{\upsilon}\right)^{\frac{1}{\theta}-1}e^{-z}dz\right\}$$

Using $(1-x)^{a-1} = \sum_{s=0}^{\infty}(-1)^s \binom{a-1}{s} x^s$, we have

$$T_R(\rho) = \frac{1}{1-\rho}\log \left\{ \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{2\rho+s-2}\binom{2\rho-2}{s}\omega_{p,q}\frac{e^{\upsilon}(\theta\lambda)^{\rho}\lambda^{1-2\rho}}{\upsilon\theta}\int_{\upsilon}^{\infty}\left(\frac{z}{\upsilon}\right)^{\frac{s+1}{\theta}-1}e^{-z}dz \right\}$$

After solving the integral, we get

$$T_R(\rho) = \frac{1}{1-\rho}\log \left\{ \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{2\rho+s-2}\binom{2\rho-2}{s}\omega_{p,q}\frac{e^{\upsilon}(\theta\lambda)^{\rho}\lambda^{1-2\rho}\upsilon^{\frac{-(s+1)}{\theta}}}{\theta}\Gamma\left(\frac{s+1}{\theta},\upsilon\right) \right\}.$$

7. ORDER STATISTICS OF WINH DISTRIBUTION

Consider $Y_1, Y_2, \ldots Y_n$ be random samples from WINH distribution. Let $Y_{(1)} < Y_{(2)} < \ldots < Y_{(n)}$ be the corresponding order statistics. Then the pdf of the r^{th} order statistics of the WINH distribution, say $X = Y_{(r)}$

$$f_X(x) = \frac{n!}{(r-1)!(n-r)!}F^{r-1}(x)\left[1-F(x)\right]^{n-r} f(x)$$

$$= \frac{n!}{(r-1)!(n-r)!}\sum_{k=0}^{\infty}\binom{n-r}{k}(-1)^k F^{r+k-1}(x) f(x)$$

$$= \frac{n!\alpha\beta\theta\lambda}{(r-1)!(n-r)!}\sum_{k=0}^{\infty}\binom{n-r}{k}(-1)^k \left[1-e^{-\alpha\left(e^{\left(1+\frac{\lambda}{y_i}\right)^{\theta}-1}-1\right)^{-\beta}}\right]^{r+k-1}$$

$$\times\frac{1}{y_i^2}\left(1+\frac{\lambda}{y_i}\right)^{\theta-1}\left(e^{\left(1+\frac{\lambda}{y_i}\right)^{\theta}-1}-1\right)^{-\beta-1} e^{\left(1+\frac{\lambda}{y_i}\right)^{\theta}-1} e^{-\alpha\left(e^{\left(1+\frac{\lambda}{y_i}\right)^{\theta}-1}-1\right)^{-\beta}}$$

The corresponding cdf of X is given by

$$F_X(x)=\sum_{j=r}^{n} F^j(x)\left[1-F(x)\right]^{n-j}$$

$$F_X(x)=\sum_{j=r}^{n}\sum_{k=0}^{n-j}\binom{n}{j}\binom{n-j}{k}(-1)^k\left[1-e^{-\alpha\left(e^{\left(1+\frac{\lambda}{y_i}\right)^{\theta}-1}-1\right)}\right]^{j+k}.$$

8. MAXIMUM LIKELIHOOD ESTIMATION OF WINH DISTRIBUTION

Let $Y_1,Y_2,..,Y_n$ be a random sample of size n from WINH distribution then its likelihood function is given by

$$l=\prod_{i=1}^{n} f\left(y_i,\alpha,\beta,\theta,\lambda\right)$$

$$=\prod_{i=1}^{n}\frac{\alpha\beta\theta\lambda}{y_i^2}\left(1+\frac{\lambda}{y_i}\right)^{\theta-1}\left(e^{\left(1+\frac{\lambda}{y_i}\right)^{\theta}-1}-1\right)^{-\beta-1} e^{\left(1+\frac{\lambda}{y_i}\right)^{\theta}-1} e^{-\alpha\left(e^{\left(1+\frac{\lambda}{y_i}\right)^{\theta}-1}-1\right)_{-\beta}}$$

The log likelihood function is given as

$$\log l=n\log\alpha+n\log\beta+n\log\theta+n\log\lambda+\sum_{i=1}^{n}\log\left(\frac{1}{y_i^2}\right)+(\theta-1)\sum_{i=1}^{n}\log\left(1+\frac{\lambda}{y_i}\right)$$

$$-(\beta+1)\sum_{i=1}^{n}\left(e^{\left(1+\frac{\lambda}{y_i}\right)^{\theta}-1}-1\right)+\sum_{i=1}^{n}\left(\left(1+\frac{\lambda}{y_i}\right)^{\theta}-1\right)-\alpha\sum_{i=1}^{n}\left(e^{\left(1+\frac{\lambda}{y_i}\right)^{\theta}-1}-1\right)^{-\beta} \quad (8.1)$$

The partial derivatives of equation (8.1), with respect parameters are

$$\frac{\partial\log l}{\partial\alpha}=\frac{n}{\alpha}-\sum_{i=1}^{n}\left(e^{\left(1+\frac{\lambda}{y_i}\right)^{\theta}-1}\right)^{-\beta} \quad (8.2)$$

$$\frac{\partial\log l}{\partial\beta}=\frac{n}{\beta}-\sum_{i=1}^{n}\left(e^{\left(1+\frac{\lambda}{y_i}\right)^{\theta}-1}-1\right)+\alpha\sum_{i=1}^{n}\left(e^{\left(1+\frac{\lambda}{y_i}\right)^{\theta}-1}-1\right)^{-\beta}\log\left(e^{\left(1+\frac{\lambda}{y_i}\right)^{\theta}-1}-1\right) \quad (8.3)$$

$$\frac{\partial \log l}{\partial \theta}=\frac{n}{\theta}+\sum_{i=1}^{n}\log\left(1+\frac{\lambda}{y_i}\right)-(\beta+1)\sum_{i=1}^{n}e^{\left(1+\frac{\lambda}{y_i}\right)^{\theta}-1}\left(1+\frac{\lambda}{y_i}\right)^{\theta}\log\left(1+\frac{\lambda}{y_i}\right)$$
$$+\sum_{i=1}^{n}\left(1+\frac{\lambda}{y_i}\right)^{\theta}\log\left(1+\frac{\lambda}{y_i}\right)$$
$$+\alpha\beta\sum_{i=1}^{n}\left(e^{\left(1+\frac{\lambda}{y_i}\right)^{\theta}-1}-1\right)^{-\beta-1}e^{\left(1+\frac{\lambda}{y_i}\right)^{\theta}-1}\left(1+\frac{\lambda}{y_i}\right)^{\theta}\log\left(1+\frac{\lambda}{y_i}\right) \tag{8.4}$$

$$\frac{\partial \log l}{\partial \lambda}=\frac{n}{\lambda}+(\theta-1)\sum_{i=1}^{n}\frac{1}{(y_i+1)}-(\beta+1)\theta\sum_{i=1}^{n}e^{\left(1+\frac{\lambda}{y_i}\right)^{\theta}-1}\frac{(y_i+\lambda)^{\theta-1}}{y_i^{\theta}}$$
$$+\theta\sum_{i=1}^{n}\frac{(y_i+\lambda)^{\theta-1}}{y_i^{\theta}}+\alpha\beta\theta\sum_{i=1}^{n}\left(e^{\left(1+\frac{\lambda}{y_i}\right)^{\theta}-1}-1\right)^{-\beta-1}e^{\left(1+\frac{\lambda}{y_i}\right)^{\theta}-1}\frac{(y_i+\lambda)^{\theta-1}}{y_i^{\theta}} \tag{8.5}$$

Clearly the equations (8.2), (8.3), (8.4), and (8.5) are non-linear equations which cannot be expressed in compact form and it is difficult to solve them explicitly for α,β,θ and λ. By applying the iterative methods such as Newton–Raphson method, secant method, Regula-falsi method etc. The MLE of the parameters denoted as $\hat{\zeta}\left(\hat{\alpha},\hat{\beta},\hat{\theta},\hat{\lambda}\right)$ of $\zeta(\alpha,\beta,\theta,\lambda)$ can be obtained by using the above methods.

For interval estimation and hypothesis tests on the model parameters, an information matrix is required. The 4 by 4 observed matrix is

$$I^{-1}(\zeta)=-\frac{1}{n}\begin{bmatrix} E\left(\frac{\partial^2 \log l}{\partial \alpha^2}\right) & E\left(\frac{\partial^2 \log l}{\partial \alpha \partial \beta}\right) & E\left(\frac{\partial^2 \log l}{\partial \alpha \partial \theta}\right) & E\left(\frac{\partial^2 \log l}{\partial \alpha \partial \lambda}\right) \\ E\left(\frac{\partial^2 \log l}{\partial \beta \partial \alpha}\right) & E\left(\frac{\partial^2 \log l}{\partial \beta^2}\right) & E\left(\frac{\partial^2 \log l}{\partial \beta \partial \theta}\right) & E\left(\frac{\partial^2 \log l}{\partial \beta \partial \lambda}\right) \\ E\left(\frac{\partial^2 \log l}{\partial \theta \partial \alpha}\right) & E\left(\frac{\partial^2 \log l}{\partial \theta \partial \beta}\right) & E\left(\frac{\partial^2 \log l}{\partial \theta^2}\right) & E\left(\frac{\partial^2 \log l}{\partial \theta \partial \lambda}\right) \\ E\left(\frac{\partial^2 \log l}{\partial \lambda \partial \alpha}\right) & E\left(\frac{\partial^2 \log l}{\partial \lambda \partial \beta}\right) & E\left(\frac{\partial^2 \log l}{\partial \lambda \partial \theta}\right) & E\left(\frac{\partial^2 \log l}{\partial \lambda^2}\right) \end{bmatrix}$$

The elements of above information matrix can obtain by differentiating equations (8.2), (8.3), (8.4), and (8.5) again partially. Under standard regularity conditions when $n \to \infty$ the distribution of $\hat{\zeta}$ can be approximated by a multivariate normal $N\left(0, I\left(\hat{\zeta}\right)^{-1}\right)$ distribution to construct approximate confidence interval for the parameters.

Hence the approximate $100(1-\psi)\%$ confidence interval for α, β, θ and λ are respectively given by

$$\hat{\alpha} \pm Z_{\frac{\psi}{2}} \sqrt{I_{\alpha\alpha}^{-1}\left(\hat{\zeta}\right)},\ \hat{\beta} \pm Z_{\frac{\psi}{2}} \sqrt{I_{\beta\beta}^{-1}\left(\hat{\zeta}\right)},\ \hat{\theta} \pm Z_{\frac{\psi}{2}} \sqrt{I_{\theta\theta}^{-1}\left(\hat{\zeta}\right)} \text{ and } \hat{\lambda} \pm Z_{\frac{\psi}{2}} \sqrt{I_{\lambda\lambda}^{-1}\left(\hat{\zeta}\right)}$$

where $Z_{\frac{\psi}{2}}$ denotes the ζ^{th} percentile of the standard normal distribution.

9. SIMULATION STUDY OF WINH DISTRIBUTION

The mean value, bias, variance and MSE were all addressed to simulation analysis. From WINHD $N = 1000$, samples of size $n =$ 50, 100, 150, 250, 300 and 350 were obtained. The following expression has been used to produce random numbers.

$$Y = \left[\frac{1}{\lambda} \left(1 + \log\left(1 + \left(\frac{-1}{\alpha} \log(1-u) \right)^{\frac{-1}{\beta}} \right) \right)^{\frac{1}{\theta}} - 1 \right]^{-1}$$

where u is uniform random numbers with $u \in (0,1)$. For various parameter combinations, simulation results have been achieved. The mean value, bias, variance and MSE values are calculated and presented in Table 9.1 and 9.2. As the sample size increases, this becomes apparent that these estimates are relatively consistent and approximate the actual values of parameters. Interestingly, with all parameter combinations, the bias and MSE reduce as the sample size increases. As a result, it has been determined that the MLE technique is effective in estimating the WINHD parameters.

Table 9.1
The Mean Values, Average Bias and MSEs of 1,000 Simulations of WINHD for Parameter Values $\alpha = 1.2$, $\beta = 0.7$, $\theta = 0.8$ and $\lambda = 0.6$

n	**Parameters**	**Average**	**Bias**	**Variance**	**MSE**
50	α	2.0756	0.77565	0.9251	1.7386
	β	1.1637	0.6547	0.3368	0.7210
	θ	0.1104	-0.689	0.0047	0.4802
	λ	1.3474	0.7474	0.4732	1.0320
100	α	1.5694	0.3694	0.0833	0.2198
	β	1.2202	0.5202	0.1589	0.4296
	θ	0.1727	-0.627	0.0061	0.3995
	λ	0.7191	0.1191	0.1242	0.1384
150	α	1.4472	0.2472	0.0269	0.0880
	β	1.0457	0.3457	0.0544	0.1739
	θ	0.2082	-0.591	0.0065	0.3567
	λ	0.5455	-0.054	0.0753	0.0783
250	α	1.3714	0.1714	0.0137	0.0431
	β	0.9357	0.2357	0.0278	0.0834
	θ	0.2494	-0.550	0.0072	0.3103
	λ	0.4124	-0.187	0.0415	0.0767
300	α	1.3516	0.1516	0.0131	0.0361
	β	0.9072	0.2072	0.0267	0.0697
	θ	0.2642	-0.535	0.0074	0.2944
	λ	0.3783	-0.221	0.0378	0.0869
350	α	1.3379	0.1379	0.0082	0.0272
	β	0.8875	0.1875	0.0173	0.0525
	θ	0.2729	-0.527	0.0073	0.2851
	λ	0.3579	-0.242	0.0326	0.0912

Table 9.2
The Mean Values, Average Bias and MSEs of 1,000 Simulations of WINHD for Parameter Values $\alpha = 1.5$, $\beta = 0.5$, $\theta = 0.8$ and $\lambda = 0.7$

n	Parameters	Average	Bias	Variance	MSE
50	α	1.5258	0.3258	0.0808	0.1870
	β	1.1657	0.6657	0.3398	0.7830
	θ	0.1843	-0.615	0.0177	0.3967
	λ	0.6976	0.0976	0.3044	0.3139
100	α	1.3565	0.1565	0.0171	0.0417
	β	0.8110	0.3110	0.0733	0.1700
	θ	0.2540	-0.545	0.0152	0.3133
	λ	0.3572	-0.242	0.0648	0.1237
150	α	1.3123	0.1123	0.0095	0.0222
	β	0.7188	0.2188	0.0406	0.0885
	θ	0.2862	-0.513	0.0126	0.2765
	λ	0.2719	-0.328	0.0338	0.1414
250	α	1.2615	0.0615	0.0049	0.0087
	β	0.6146	0.1146	0.0204	0.0336
	θ	0.3450	-0.454	0.0104	0.2174
	λ	0.1871	-0.412	0.0150	0.1354
300	α	1.2496	0.0496	0.0037	0.0062
	β	0.5904	0.0904	0.0157	0.0239
	θ	0.3594	-0.440	0.0086	0.2027
	λ	0.1672	-0.432	0.0113	0.1186
350	α	1.2392	0.0392	0.0032	0.0047
	β	0.5694	0.0694	0.0132	0.0180
	θ	0.3762	-0.423	0.0083	0.1879
	λ	0.1539	-0.446	0.0098	0.1084

10. DATA ANALYSIS

This section assesses the effectiveness of the stated distribution using real-world data. We fitted the WINH distribution to many other models for comparative purposes, including modified Weibull distribution (MWD), additive Weibull distribution (AWD), Weibull distribution (WD), inverse Nadarajah-Haghighi distribution (INHD), Rayleigh distribution (RD) and exponential distribution.

We will use certain measures to evaluate which of the competitive models is the strongest, including AIC (Akaike Information Criterion), CAIC (Consistent Akaike Information Criterion), BIC (Bayesian Information Criterion) and HQIC (Hannan-Quinn Information Criterion). Such criteria can be represented mathematically by

$$AIC = 2k - 2\ln l \quad CAIC = \frac{2kn}{n-k-1} - 2\ln l$$

$$BIC = k\ln n - 2\ln l \text{ and } HQIC = 2k\ln\left(\ln\left(n\right)\right) - 2\ln l$$

We compute Anderson-Darling (A*), Cramer-Von Misses (W*), Kolmogorov-Smirnov Statistic, and P-value in addition to the aforementioned goodness of measures. The model with the lowest value of these indicators and the greatest p-value is considered the best among the competing models.

Data Set 1: The data shows the mortality rate due to covid-19 of France country which has been recorded from first January to 20 February 2021 [http://covid-19.who.int/]. The data follows

0.0995, 0.0525, 0.0615, 0.0455, 0.1474, 0.3373, 0.1087, 0.1055, 0.2235, 0.0633, 0.0565, 0.2577, 0.1345,0.0843, 0.1023, 0.2296, 0.0691, 0.0505, 0.1434, 0.2326, 0.1089, 0.1206, 0.2242, 0.0786, 0.0587, 0.1516, 0.2070, 0.1170, 0.1141, 0.2705, 0.0793, 0.0635, 0.1474, 0.2345, 0.1131, 0.1129, 0.2054, 0.0600, 0.0534, 0.1422, 0.2235, 0.0908, 0.1092, 0.1958, 0.0580, 0.0502, 0.1229, 0.1738, 0.0917, 0.0787, 0.1654.

Table 10.1
The Descriptive Statistics for Data Set 1

Min	Q_1	Med.	Mean	Q_3	Kurt.	Skew.	Max
0.04550	0.0738	0.1129	0.1299	0.1696	0.8817	3.0462	0.33730

Table 10.2
The ML Estimates (Standard Error in Parenthesis) for Data Set 1

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$
WINHD	1.241 (0.6611)	1.007 (0.2623)	4.487 (3.568)	0.018 (0.0171)
MWD	9.2169 (1.7278)	4.6125 (2.7231)	1.3982 (0.1518)	3.8229 (4.4800)
AWD	7.8109 (4.3140)	9.6561 (6.3205)	2.0168 (0.2793)	2.0184 (0.2854)
WD	46.753 (17.835)	2.0098 (0.2112)	------	------
INHD	0.0042 (0.0023)	16.15 (9.0638)	------	------
RD	46.113 (6.457)	------	------	------
EXD	7.6945 (1.0774)	------	------	------

Table 10.3
Comparison Criterion and Goodness of Fit Statistics for Data Set 1

Model	$-2\log l$	AIC	CAIC	BIC	HQIC
WINHD	-145.49	-137.49	-136.62	-129.76	-134.54
MWD	-131.52	-123.52	-122.65	-115.79	-120.57
AWD	-137.23	-129.23	-128.36	-121.50	-126.28
WD	-137.23	-133.23	-132.98	-129.36	-131.75
INHD	-125.03	-121.03	-120.78	-117.17	-119.56
RD	-66.527	-64.527	-64.445	-62.595	-63.789
EXD	-106.13	-104.13	-104.05	-102.19	-103.39

Table 10.4
Other Goodness of Fit Statistics Criterion for Data Set 1

Model	W^*	A^*	K-S value	p-value
WINHD	0.0603	0.4770	0.0974	0.7181
MWD	0.2131	0.4976	0.1386	0.2805
AWD	0.1416	0.9430	0.1086	0.5844
WD	0.1411	0.9404	0.1085	0.5851
INHD	0.1364	0.8930	0.2297	0.00916
RD	0.1406	0.9380	0.1061	0.6135
EXD	0.0911	0.6831	0.3008	0.00019

Data Set 2: The data shows the mortality rate due to covid-19 of Canada country which has been recorded from first November to 26 December 2020 [https://covid-19.who.int/]. The data follows

0.1622, 0.1159, 0.1897, 0.1260, 0.3025,0.2190, 0.2075, 0.2241, 0.2163, 0.1262, 0.1627, 0.2591, 0.1989, 0.3053, 0.2170, 0.2241, 0.2174, 0.2541, 0.1997, 0.3333, 0.2594, 0.2230, 0.2290, 0.1536, 0.2024, 0.2931, 0.2739, 0.2607, 0.2736, 0.2323, 0.1563, 0.2677, 0.2181, 0.3019, 0.2136, 0.2281, 0.2346, 0.1888, 0.2729, 0.2162, 0.2746, 0.2936, 0.3259, 0.2242, 0.1810, 0.2679, 0.2296, 0.2992, 0.2464, 0.2576, 0.2338, 0.1499, 0.2075, 0.1834, 0.3347, 0.2362.

Table 10.5
The Descriptive Statistics for Data Set 2

Min	Q_1	Med.	Mean	Q_3	Kurt.	Skew.	Max
0.1159	0.2017	0.2261	0.2305	0.2677	2.6537	-0.0872	0.3347

Table 10.6
The ML Estimates (Standard Error in Parenthesis) for Data Set 2

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$
WINHD	3.9226 (1.642)	3.5267 (2.1505)	0.7705 (1.1987)	0.3261 (1.3252)
MWD	85.734 (40.073)	7.9895 (0.4318)	3.5441 (0.3838)	12.170 (1.6038)
AWD	127.37 (120.21)	138.30 (110.27)	4.0197 (0.3775)	4.0453 (0.2053)
WD	119.25 (37.968)	3.4761 (0.2440)	------	------
INHD	0.0051 (0.0031)	32.516 (19.864)	------	------
RD	17.930 (2.396)	------	------	------
EXD	4.3391 (0.5798)	------	------	------

Table 10.7
Comparison Criterion and Goodness of Fit Statistics for Data Set 2

Model	$-2\log l$	AIC	CAIC	BIC	HQIC
WINHD	-173.64	-165.64	-164.85	-157.54	-162.50
MWD	-169.68	-161.68	-160.90	-153.58	-158.54
AWD	-168.83	-160.83	-160.04	-152.73	-157.69
WD	-161.66	-157.66	-157.43	-153.61	-156.09
INHD	-84.469	-80.469	-80.243	-76.419	-78.899
RD	-43.843	-41.843	-41.769	-39.817	-41.058
EXD	-52.380	-50.380	-50.306	-48.355	-49.595

Table 10.8
Other Goodness of Fit Statistics Criterion for Data Set 2

Model	W^*	A^*	K-S value	p-value
WINHD	0.0595	0.3334	0.0893	0.763
MWD	0.1314	0.7267	0.1346	0.2616
AWD	0.0792	0.4304	0.1115	0.489
WD	0.0668	0.3364	0.1384	0.2337
INHD	0.2886	0.7647	0.4729	2.627e-11
RD	0.0638	0.3963	0.3013	7.637e-05
EXD	0.0837	0.5359	0.4246	3.399e-09

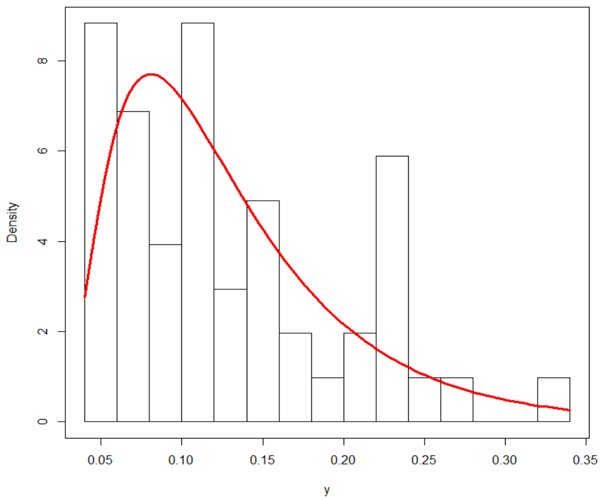

Figure 10.1: Estimated pdf of the Fitted Model for Data Set 1

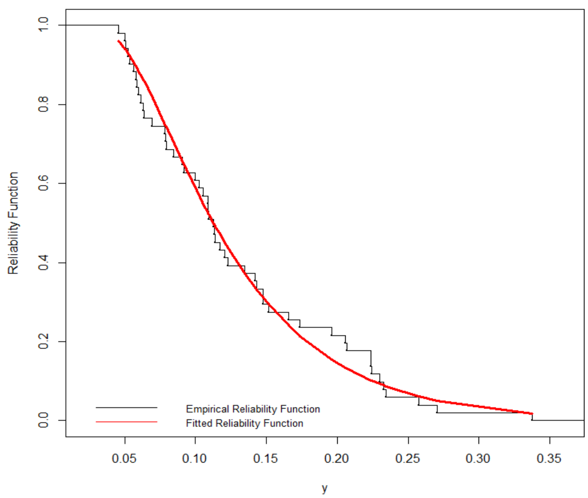

Figure 10.1: Empirical Reliability Function versus Fitted Reliability Function Data Set 1

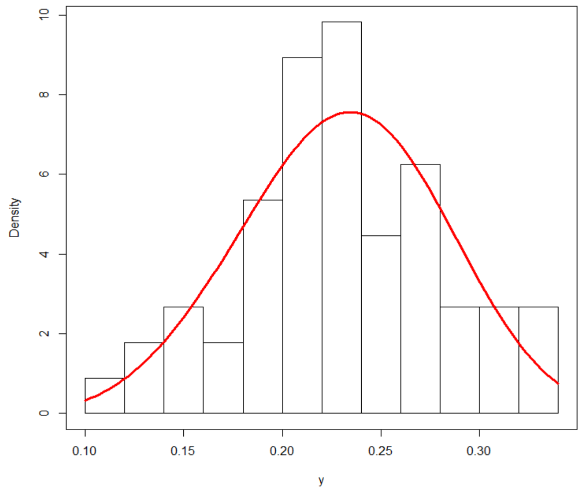

Figure 10.3: Estimated pdf of the Fitted Model for Data Set 2

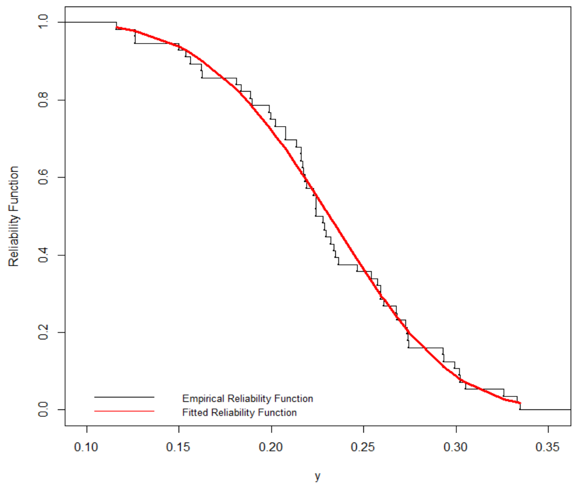

Figure 10.4: Empirical Reliability Function versus Fitted Reliability Function Data Set 2

CONCLUDING REMARKS

The focus of this research is to investigate prevailing pandemic-19 mortality data. In this study, we developed a novel flexible distribution known as the Weibull-inverse Nadarajah Haghighi distribution, renamed as (WINHD). Numerous mathematical characteristics are determined for this distribution, including moments, moment generating functions, incomplete moments, order statistics, Renyi entropy, mean deviations, and reliability analysis. The maximum likelihood estimation approach was used to estimate the distribution's parameters. From tables (10.3), (10.4), (10.7) and (10.8) it is evident that the formulated distribution leads to a better fit than the comparable ones.

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