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THE LOGISTIC EXPONENTIAL POISSON DISTRIBUTION WITH MATHEMATICAL PROPERTIES AND APPLICATION

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ABSTRACT

A new three-parameter distribution called logistic exponential-Poisson (LEP) distribution is proposed as a special sub-model of the newly LE power-series family. The failure rate of the LEP model can be increasing or decreasing. Explicit algebraic formulations of the LEP model such as quantile, ordinary moments and associated measures, mean residual life, moment generating function, and density of order statistics are derived. The LEP parameters are estimated using the maximum likelihood technique. To examine the behavior of maximum likelihood estimates, a complete Monte Carlo simulation analysis is employed. The applicability of the LEP distribution is evaluated using a real-life dataset, showing its more efficient results than well-known competing probability distributions.

KEYWORDS

Logistic-exponential, reliability analysis, moments, estimation, data analysis.

1. INTRODUCTION

Probability models are widely used and have considerable significance in a variety of fields, including income and wealth, management sciences, evolutionary biology, computer sciences, and actuarial science. For modeling natural issues, the probability distributions are suitable for prediction and forecast.

The classical probability distributions are often employed for modeling; however, they may not always produce adequate fits for heavy-tailed and skewed data. As a result, there is a need to improve the flexibility of existing probability models by introducing new parameters. There are several approaches in the literature for adding parameters to the parent model. Among them, combining power series distributions with continuous lifetime models and compounding of discrete models (Tahir and Cordeiro, 2015).

The exponential distribution is a popular model in literature however its limited characteristics. Hence, its use in reliability analysis is limited due to its constant hazard rate (hr). It is not suitable for modeling monotone and non-monotonic hr behaviors. These drawbacks motivated many authors to introduce extended versions of the exponential distribution to enhance its capabilities. For example, Lan and Leemis (2008) proposed the logistic exponential (LE) distribution. The cdf of the LE distribution has the form

$$G(x) = 1 - \frac{1}{1 + (e^{\lambda x} - 1)^{\alpha}}, x > 0. \quad (1)$$

where $\lambda > 0$ and $\alpha > 0$ are the scale and shape parameters.

The associated probability density function (pdf) is

$$g(x) = \frac{\alpha \lambda e^{\lambda x} \left(e^{\lambda x} - 1\right)^{\alpha - 1}}{[1 + (e^{\lambda x} - 1)^{\alpha}]^2}. \quad (2)$$

The random variable (*rv*) with pdf (2), is denoted by $X \sim \text{LE}(\alpha, \lambda)$.

There are two types of systems are the series system and parallel system and they are very important in the design of compound models. Assume that a system contains N subsystems at a given time supposed to be independent, identical and distributed (iid), where Y_i symbolized subsystem with a lifetime of the i^{th} and every subsystem contains α parallel units, so that if subsystem will fail all the system ultimately fails.

The random variables (*rv*s) $Y = \min(Y_1, Y_2, \dots, Y_N)$ or $Y = \max(Y_1, Y_2, \dots, Y_N)$ can be adopted in generating numerous models with identical components that arise in series or parallel systems and comprise various biological and industrial applications.

A discrete *rv* N is said to have a power-series family (which is truncated at zero) if its probability mass function (pmf) takes the form

$$P_n = P(N = n) = \frac{a_n \theta^n}{C(\theta)}, n = 1,2, \dots,$$

where $a_n \geq 0, C(\theta) = \sum_{n=1}^{\infty} a_n \theta^n$, for $\theta \in (0, S)$ is finite and its first three derivatives are $C'(.)$, $C''(.)$ and $C'''(.)$. More details about it are found in Noack (1950). This family of distributions comprises several distributions including the binomial, negative binomial, logarithmic, Poisson, and geometric distributions.

The power-series distributions are compounded with many continuous models to introduce several compounding families of distributions. For example, Morais and Barreto-Souza (2011) introduced the Weibull power-series distributions. Mahmoudi and Sepahdar (2013) derived the exponentiated Weibull-Poisson distribution which has some sub-models such as complementary exponential-Poisson, Rayleigh-Poisson, generalized exponential-Poisson, complementary Weibull-Poisson, and exponentiated Rayleigh-Poisson. Bagheri et al. (2016) introduced modified Weibull power-series distribution. Mendoza et al. (2016) introduced the exponentiated log-logistic geometric distribution. Gui et al. (2017) introduced the complementary Lindley-geometric distribution. Hassan and Abd-Allah (2019) introduced the inverse power Lomax-Poisson distribution. Elbatal et al. (2019) proposed the generalized Burr XII power series family.

The main motivation behind this work is to introduce a new logistic-exponential power-series (LEPS) family. The LEPS family is generated from the power-series of logistic-exponential distributions with a zero truncated-geometric distribution. Some mathematical properties are derived, including moments and associated measures, reliability properties, and expressions of order statistics. The three-parameter LE-Poisson (LEP) distribution is studied with some details. The LEP parameters are estimated via the maximum likelihood, and the estimators' behavior is demonstrated using a comprehensive simulation study. Additionally, a real-life dataset from medical field is utilized to explore the flexibility of the LEP distribution as compared to the parent and some competing probability distributions.

The article is organized in seven sections. Section 2 is devoted to the derivation of the LEPS model. Section 3 contains the derivation of the LEP model and its mathematical properties. The reliability characteristics are presented in Section 4. The LEPS parameters are estimated, and a simulation study is given in Section 5. The application of the LEPS distribution is given in Section 6. We conclude the paper in Section 7.

2. THE LEPS FAMILY

Let $Y_1, Y_2, \dots, Y_N$ be N independent LE *rv*s. The LEPS family of the $rv = \min\{Y_1, Y_2, \dots, Y_N\}$ is defined by the cdf

$$F(x) = 1 - \frac{C\left(\frac{\theta}{1+(e^{\lambda x}-1)^{\alpha}}\right)}{C(\theta)}, x > 0, \tag{3}$$

where $\lambda > 0$ and $\alpha > 0$ are the scale and shape parameters and $\theta > 0$.

The associated pdf of the LEPS family reduces to

$$f(x) = \frac{\theta\alpha\lambda e^{\lambda x}\left(e^{\lambda x}-1\right)^{\alpha-1}}{[1+(e^{\lambda x}-1)^{\alpha}]^2} \frac{C'\left(\frac{\theta}{1+(e^{\lambda x}-1)^{\alpha}}\right)}{C(\theta)}, x > 0. \tag{4}$$

The hf function (hrf) of X is

$$h(x) = \frac{\theta\alpha\lambda e^{\lambda x}\left(e^{\lambda x}-1\right)^{\alpha-1}}{[1+(e^{\lambda x}-1)^{\alpha}]^2} \frac{C'\left(\frac{\theta}{1+(e^{\lambda x}-1)^{\alpha}}\right)}{C\left(\frac{\theta}{1+(e^{\lambda x}-1)^{\alpha}}\right)}. \tag{5}$$

Proposition 1.

The limiting distribution of LEPS with parameters $(\alpha, \lambda, \theta)$ *when* $\theta \rightarrow 0+$.

$$\lim_{\theta\rightarrow 0+} F(x) = \lim_{\theta\rightarrow 0+}\left[1 - \frac{C[\theta-\theta G(x)]}{C(\theta)}\right]$$

$$= \lim_{\theta\rightarrow 0+} 1 - \lim_{\theta\rightarrow 0+} \frac{\sum_{n=1}^{\infty} a_n\left(\frac{\theta}{1+(e^{\lambda x}-1)^{\alpha}}\right)^n}{\sum_{n=1}^{\infty} a_n\theta^n}.$$

By using $L'Hopital's\ rule$, we obtain

$$\lim_{\theta \to 0+} F(x) = \lim_{\theta \to 0+} 1 - \lim_{\theta \to 0+} \frac{a_1\left(\frac{1}{1+(e^{\lambda x}-1)^{\alpha}}\right)^1 + \sum_{n=2}^{\infty} a_n \theta^{n-1}\left(\frac{1}{1+(e^{\lambda x}-1)^{\alpha}}\right)^n}{a_1 + \sum_{n=2}^{\infty} a_n \theta^{n-1}}.$$

Hence,

$$\lim_{\theta \to 0+} F(x) = [1 - G(x)] = \left[1 - \frac{1}{1+(e^{\lambda x}-1)^{\alpha}}\right],$$

which is the LE distribution with pdf (2).

Proposition 2.

The densities of the LEPS class is expressed as an infinite number mixture of the density of order statistics of the LE with pdf (2).

Proof:

We know that $C'(\theta) = \sum_{n=1}^{\infty} n a_n \theta^{n-1}$. Therefore,

$$f(x) = \theta g(x)\frac{C'(\theta[1-G(x)])}{C(\theta)} = \sum_{n=1}^{\infty} \frac{a_n \theta^n}{C(\theta)} \frac{n\theta\alpha\lambda e^{\lambda x}\left(e^{\lambda x}-1\right)^{\alpha-1}}{[1+(e^{\lambda x}-1)^{\alpha}]^2} \left[\frac{\theta}{1+(e^{\lambda x}-1)^{\alpha}}\right]^n = \sum_{n=1}^{\infty} P(N=n)\ g_{(1)}(x;n).$$

where $g_{(1)}(x;n)$ is the pdf of $X_{(1)} = \min(X_1, X_2, \dots, X_n)$, and it is specified by

$$g_{(1)}(x;n) = n g(x)[1-G(x)]^{n-1} = \frac{n\alpha\lambda e^{\lambda x}\left(e^{\lambda x}-1\right)^{\alpha-1}}{[1+(e^{\lambda x}-1)^{\alpha}]^{n-1}}\left[1+\left(e^{\lambda x}-1\right)^{\alpha}\right]^{-2}. \tag{6}$$

Equation (6) represents the pdf of the exponentiated-LE (ELE) model and it can adopt to derive mathematical properties of the LEPS distribution directly from those of the ELE distribution.

Proposition 3.

The limits of the hrf is

$$\lim_{\theta \to 0+} h(x) = \begin{cases} \infty & 0 < \alpha < 1, \\ \dfrac{\alpha\lambda\theta C'(\theta)}{C(\theta)} & \alpha = 1, \\ 0 & \alpha > 1. \end{cases}$$

Proof:

For $\lim_{\theta \to 0+} h(x)$, we have

$$\lim_{\theta \to 0+} C'\left(\frac{\theta}{1+(e^{\lambda x}-1)^{\alpha}}\right) = C'(\theta), \lim_{\theta \to 0+} C\left(\frac{\theta}{1+(e^{\lambda x}-1)^{\alpha}}\right) = C(\theta),$$

$$\lim_{\theta\to 0+} \theta\alpha\lambda e^{\lambda x} = \theta\alpha\lambda, \lim_{\theta\to 0+} \left[1+\left(e^{\lambda x}-1\right)^{\alpha}\right]^{2} = 1.$$

$$\lim_{\theta\to 0+} \left(e^{\lambda x}-1\right)^{\alpha-1} = \begin{cases} \infty & 0<\alpha<1, \\ 1 & \alpha=1, \\ 0 & \alpha>1. \end{cases}$$

Table 1 gives some special cases of the PS distributions.

Table 1
Some useful quantities of PS distributions

Distribution	a_n	$C(\theta)$	$C'(\theta)$	$C''(\theta)$	$C(\theta)^{-1}$	Θ
Binomial	$\binom{m}{n}$	$(\theta+1)^{m}-1$	$m(\theta+1)^{m-1}$	$\frac{m(m-1)}{(\theta+1)^{2-m}}$	$(\theta+1)^{\frac{1}{m}-1}$	$\theta\in(0,1)$
Poisson	$n!^{-1}$	$e^{\theta}-1$	e^{θ}	e^{θ}	$\ln(\theta+1)$	$\theta\in(0,\infty)$
Geometric	1	$\theta(1-\theta)^{-1}$	$(1-\theta)^{-2}$	$2(1-\theta)^{-3}$	$\theta(1+\theta)^{-1}$	$\theta\in(0,1)$
Logarithmic	n^{-1}	$-\ln(1-\theta)$	$(1-\theta)^{-1}$	$(1-\theta)^{-2}$	$1-e^{-\theta}$	$\theta\in(0,1)$

3. THE LEP DISTRIBUTION AND ITS PROPERTIES

The LEP distribution is a special case of the LEPS family with $a_n = n!^{-1}$ and $C(\theta) = e^{\theta}-1$. The cdf of LEP model is

$$F(x) = \frac{e^{\theta} - \theta e^{-1-\left(e^{\lambda x}-1\right)^{\alpha}}}{e^{\theta}-1}, x>0, \theta,\alpha,\lambda>0. \quad (7)$$

The pdf of the LEP modle reduces to

$$f(x) = \frac{\theta\alpha\lambda e^{\lambda x}\left(e^{\lambda x}-1\right)^{\alpha-1} e^{\left(\frac{\theta}{1+\left(e^{\lambda x}-1\right)^{\alpha}}\right)}}{[1+(e^{\lambda x}-1)^{\alpha}]^{2}(e^{\theta}-1)}. \quad (8)$$

We plot some pdf curves of the LEP model in Figure.

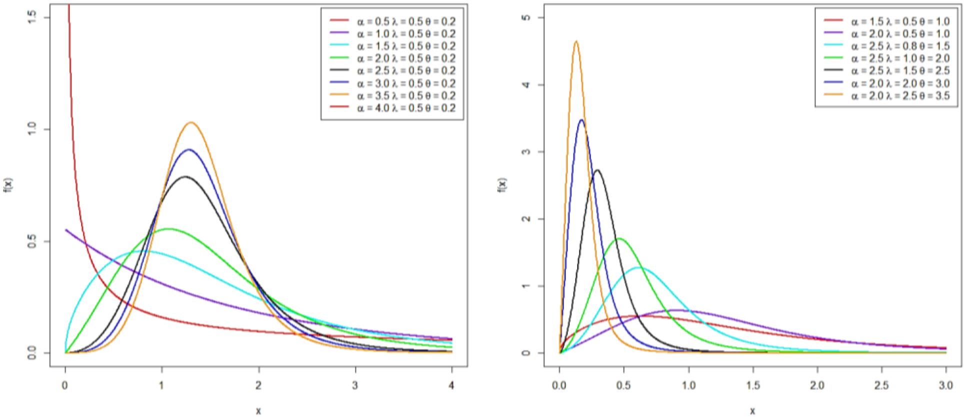

Figure 1: LEP density plots for some parameter values

It is found, from Figure 1, that the LEP model is flexible due to its shape behavior. The density function is mainly categorized into three subfamilies. In the first subfamily when $\alpha < 1$ all density curves are exponentially decreasing behavior and start from the infinite point. In the second subfamily $\alpha = 1$, the density curves show exponentially decreasing behavior but start from a specific point on the y-axis. In the third subfamily, the density curves start from the origin and display unimodal behavior for all combinations of parameters. It is also observed that all the curves approach zero when x becomes large for all combinations of parameters.

3.1 Mode

To determine the mode of the LEP model, firstly, we calculated the first derivative of the log of the pdf (2) as follows

$$\frac{d\log f(x)}{dx} = \frac{\lambda\left\{\left[1+\left(e^{\lambda x}-1\right)^{\alpha}\right]^{2} + \alpha e^{\lambda x}\left[\left(e^{\lambda x}-1\right)^{2\alpha} + \left(e^{\lambda x}-1\right)^{\alpha}\theta - 1\right]\right\}}{(1-e^{\lambda x})[(e^{\lambda x}-1)^{\alpha}+1]^{2}}.$$

Then, by setting $d\log f(x)/dx = 0$, and solving with respect to x, we have the mode of the LEP model. The exact solution of this equation is not simple, and it can be obtained numerically.

Table 2
The mode values of the LEP model for $\alpha = 1.2$ and some selected parameters

λ	θ						
	0.1	**0.2**	**0.5**	**0.7**	**0.9**	**1.5**	**3.0**
0.2	0.166197	0.171814	0.189932	0.202992	0.216767	0.261836	0.391411
0.5	0.415494	0.429535	0.474831	0.50748	0.541919	0.65459	0.978528
1.0	0.83098	0.85907	0.94966	1.01496	1.083838	1.30918	1.95706
1.5	1.24648	1.28860	1.42449	1.52244	1.62575	1.96377	2.93558
5.0	4.15494	4.29536	4.74832	5.0748	5.41919	6.5459	9.78528
7.5	6.23241	6.44304	7.12248	7.61221	8.12879	9.16425	14.6779
10.0	8.30988	8.59071	9.49663	10.1499	10.8384	13.0918	19.5706

Table 2 shows that the mode increases with the increase in θ and the mode is also increase with the increase in λ.

3.2 Linear Representation

For the LEP model, we introduce a linear representation for its cdf (7) and pdf (8) to simplify the calculations of its statistical properties. We have two cases of the cdf (7) as follows

$$F(x) = \frac{1}{e^{\theta}-1}\left[e^{\theta} - \sum_{k=0}^{\infty}\frac{\theta^k}{k!}\left(\left(e^{\lambda x}-1\right)^{\alpha}+1\right)^{-k}\right].$$

Then

$$F(x) = \begin{cases} \dfrac{1}{e^{\theta}-1}\left[e^{\theta} - \sum\limits_{k,j,m=0}^{\infty}\dfrac{(-1)^j\theta^k}{k!}\dbinom{k+j-1}{j}\dbinom{m+\alpha(j+k)-1}{\alpha(j+k)-1}e^{-x[m\lambda+\alpha\lambda(j+k)]}\right], \left(e^{\lambda x}-1\right)^{\alpha} > 1, \\ \dfrac{1}{e^{\theta}-1}\left[e^{\theta} - \sum\limits_{k,j,m=0}^{\infty}\dfrac{(-1)^{j+m}\theta^k}{k!}\dbinom{k+j-1}{j}\dbinom{\alpha j}{m}e^{-x[m\lambda-\alpha\lambda j]}\right], \left(e^{\lambda x}-1\right)^{\alpha} < 1. \end{cases}$$

Additionally, we have two cases of the pdf (8) which can be represented in terms of exponential (Ex) densities as follows

$$f(x) = \begin{cases} \sum\limits_{k,j,m=0}^{\infty}\phi'_{k,j,m}h_1(x), \left(e^{\lambda x}-1\right)^{\alpha} > 1, \\ \sum\limits_{k,j,m=0}^{\infty}\phi^{*}_{k,j,m}h_2(x), \left(e^{\lambda x}-1\right)^{\alpha} < 1, \end{cases}$$

where

$$\phi'_{k,j,m} = \frac{(-1)^j\theta^k}{(e^{\theta}-1)k!}\binom{k+j-1}{j}\binom{m+\alpha(j+k)-1}{\alpha(j+k)-1},$$

$$\phi^{*}_{k,j,m} = \frac{(-1)^{j+m}\theta^{k}}{(e^{\theta}-1)k!}\binom{k+j-1}{j}\binom{\alpha j}{m},$$

and $h_1(x)$ and $h_2(x)$ are two Ex densities with respective scale parameters $m\lambda + \alpha\lambda(j+k)$ and $m\lambda - \alpha\lambda j$.

3.3 Quantile Function and Moments

By Equating the cdf (7) to p and solving it with respect to x, we have the quantile function of the LEP model as follows

$$Q(p) = \frac{1}{\lambda}\log\left\{\left[\frac{\theta}{\log(e^{\theta} - e^{\theta}p + p)} - 1\right]^{\frac{1}{\alpha}} + 1\right\}, 0 < p < 1.$$

The ith moment of the LEP model is defined by

$$\mu_i = \int_0^{\infty} x^i\, f(x)dx = \begin{cases} \sum\limits_{k,j,m=0}^{\infty} \phi'_{k,j,m} \dfrac{i!}{[m\lambda + \alpha\lambda(j+k)]^i}, \left(e^{\lambda x}-1\right)^{\alpha} > 1, \\ \sum\limits_{k,j,m=0}^{\infty} \phi^{*}_{k,j,m} \dfrac{i!}{[m\lambda - \alpha\lambda j]^i}, \left(e^{\lambda x}-1\right)^{\alpha} < 1. \end{cases}$$

The moments can be used to calculate different measures such as the mean **(μ_1),** variance (Var), coefficients of skewness (CS), and kurtosis (CK). Table 3 reports some values of these measures for some selected values of the LEP parameters.

The moment-generating function of the LEP model is defined by

$$MG(t) = \int_0^{\infty} e^{tx}\, f(x)dx = \begin{cases} \sum\limits_{k,j,m=0}^{\infty} \phi'_{k,j,m} \dfrac{m\lambda + \alpha\lambda(j+k)}{m\lambda + \alpha\lambda(j+k) - t}, \left(e^{\lambda x}-1\right)^{\alpha} > 1, \\ \sum\limits_{k,j,m=0}^{\infty} \phi^{*}_{k,j,m} \dfrac{m\lambda - \alpha\lambda j}{m\lambda - \alpha\lambda j - t}, \left(e^{\lambda x}-1\right)^{\alpha} < 1. \end{cases}$$

By replacing each t by it in the previous equation, we obtain the LEP characteristic function.

3.4 Incomplete Moments

One of the most important properties of the statistical models is the ith incomplete moments, and it is determined for the LEP model as follows

$$l\Delta_i(t) = \int_0^{t} x^i\, f(x)dx$$

$$= \begin{cases} \sum\limits_{k,j,m=0}^{\infty} \phi'_{k,j,m} \dfrac{\gamma\left[r+1, \left(m\lambda + \alpha\lambda(j+k)\right)t\right]}{[m\lambda + \alpha\lambda(j+k)]^i}, \left(e^{\lambda x}-1\right)^{\alpha} > 1, \\ \sum\limits_{k,j,m=0}^{\infty} \phi^{*}_{k,j,m} \dfrac{\gamma[r+1, (m\lambda - \alpha\lambda j)t]}{[m\lambda - \alpha\lambda j]^i}, \left(e^{\lambda x}-1\right)^{\alpha} < 1, \end{cases}$$

where $\gamma(a, z) = \int_0^t t^{a-1} e^{-t} dt$.

The first incomplete moment, (Δ_1), can be used to determine some other statistical properties of the LEP model as follows:

The mean deviation about the mean is calculated as follows

$$\psi_1 = 2\mu_1 F(\mu_1) - 2\Delta_1(\mu_1).$$

The mean deviation about the median is calculated as follows

$$\psi_2 = 2\mu_1 - 2\Delta_1(M), M = Q(1/2).$$

The mean residual life function is calculated as follows

$$\psi_3(t) = \frac{1 - \Delta_1(t)}{S(t) - t}.$$

Mean inactivity time is calculated as follows

$$\psi_4(t) = t - \frac{\Delta_1(t)}{F(t)}.$$

Table 3
Mean, Var, CS, and CK values for some selected parameters

α	λ	θ	Mean	Var	CS	CK
0.15	0.1	0.25	4.832347	123.2820	2.498334	7.382903
	0.75	0.5	5.017194	69.42540	2.099458	5.928127
	1.5	2.0	1.345442	10.29580	3.598499	19.96240
	4.0	3.0	0.305365	0.852884	4.775595	35.79031
0.8	0.1	0.25	9.543069	105.5806	1.496145	0.687679
	0.75	0.5	1.306708	2.483134	2.297743	11.78684
	1.5	2.0	0.405654	0.354484	3.184870	25.34902
	4.0	3.0	0.108901	0.031343	3.944437	53.50582
1.5	0.1	0.25	8.036446	38.43936	1.699458	0.545613
	0.75	0.5	1.021835	0.662133	1.927931	23.67680
	1.5	2.0	0.372132	0.104644	2.426464	54.97953
	4.0	3.0	0.113708	0.010150	2.749437	162.3016
5	0.1	0.25	7.157168	9.299233	1.246459	3.623164
	0.75	0.5	0.912710	0.056738	0.833523	1430.925
	1.5	2.0	0.411543	0.011240	0.861423	3445.179
	4.0	3.0	0.113708	0.010150	2.749437	162.3016

3.5 Inequality Curves

The Lorenz, Bonferroni, and Zenga curves are key inequality curves, and they are used to analyze income and poverty in reliability, medicine, demography, insurance, and economics. They are determined for the LEP model, respectively, as follows

$$\boldsymbol{I_1} = \frac{\Delta_1(x_p)}{\mu_1}, \boldsymbol{I}_2 = \frac{\mathrm{I}_1}{p}, \boldsymbol{I}_3 = \frac{\mathrm{I}_1 - p}{p(1 - \mathrm{I}_1)} \text{ and } F(x_p) = p.$$

4. RELIABILITY ANALYSIS OF LEP DISTRIBUTION

In this section, we explore the reliability characteristics of the LEP distribution. The mathematical expressions of hrf, survival function (sf), reverse-hrf, and cumulative-hrf are derived. The pattern of failure rate is checked on the lower and upper limits of the distribution. We present the plots of the hrf.

The sf of the LEP distribution is

$$S(x) = \frac{e^{\frac{\theta}{1+(e^{\lambda x}-1)^{\alpha}}} - 1}{e^{\theta} - 1}. \tag{9}$$

The hrf of the LEP model is

$$h(x) = \frac{\theta\alpha\lambda e^{\lambda x}\left(e^{\lambda x}-1\right)^{\alpha-1} e^{\left(\frac{\theta}{1+(e^{\lambda x}-1)^{\alpha}}\right)}}{\left(e^{\frac{\theta}{1+(e^{\lambda x}-1)^{\alpha}}} - 1\right)(1+(e^{\lambda x}-1)^{\alpha})^{2}}. \tag{10}$$

The hrfr plots of the LEP model are given in Figure 2.

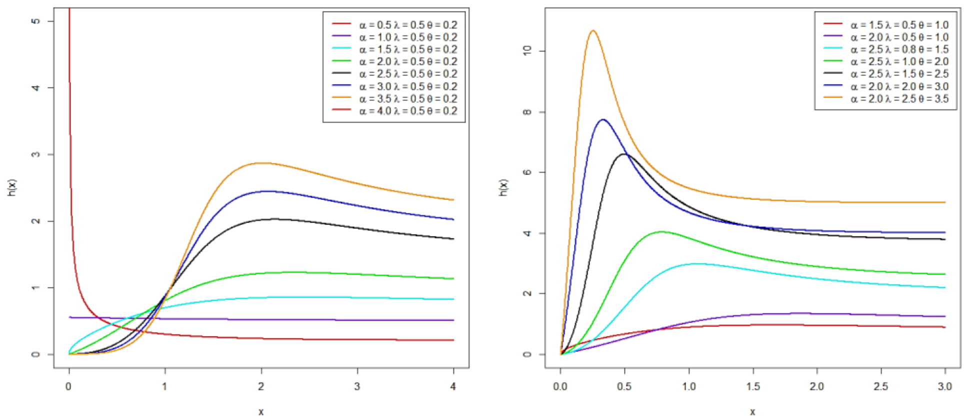

Figure 2: LEPD hrf plots for some selected parameter values

The shape of the failure rate is very important for analyzing lifetime data sets. From Figure 2, it is interesting to observe different shapes of the LEP hrf. In the first subfamily, for $\alpha \leq 1$, the failure rate of LEP shows exponentially decreasing behavior (starting from a specific point on the y-axis and moving towards zero). In the second subfamily, the hrf curves show inverted bathtub behavior.

The reverse-hrf and b cumulative-hrf take the forms

$$r_F(x) = \frac{\theta\alpha\lambda e^{\lambda x}\left(e^{\lambda x}-1\right)^{\alpha-1} e^{\left(\frac{\theta}{1+\left(e^{\lambda x}-1\right)^{\alpha}}\right)}}{\left(e^{\theta} - e^{\frac{\theta}{1+\left(e^{\lambda x}-1\right)^{\alpha}}}\right)(1+(e^{\lambda x}-1)^{\alpha})^2} \tag{11}$$

and

$$H(x) = -\log\left(\frac{e^{\frac{\theta}{1+\left(e^{\lambda x}-1\right)^{\alpha}}}-1}{e^{\theta}-1}\right). \tag{12}$$

5. ESTIMATION OF PARAMETERS

In this section, the maximum likelihood **(**ML) method is applied to estimate the unknown LEP parameters. Additionally, to examine the behavior of estimated parameters a Monte Carlo simulation results for different n at some parameter groupings are provided.

5.1 Maximum Likelihood Estimation

The log-likelihood function of the LEP distribution is

$$\begin{aligned} l = \ln(\alpha\lambda\theta) + \lambda\sum_{i=1}^{n} x_i + (\alpha-1)\sum_{i=1}^{n}\ln\left(e^{\lambda x_i}-1\right) + \sum_{i=1}^{n}\frac{\theta}{1+(e^{\lambda x_i}-1)^{\alpha}} \\ - n\ln\left(e^{\theta}-1\right) - 2\sum_{i=1}^{n}\ln\left(1+\left(e^{\lambda x}-1\right)^{\alpha}\right), \end{aligned} \tag{13}$$

The ML estimates (MLEs) of the LEP parameters are determined by equating the nonlinear equations $\frac{\partial \ln l}{\partial\alpha}=0, \frac{\partial \ln l}{\partial\lambda}=0, \frac{\partial \ln l}{\partial\theta}=0$, simultaneously.

$$\begin{aligned} \frac{\partial l}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\ln\left(e^{\lambda x_i}-1\right) - 2\sum_{i=1}^{n}\frac{\left(e^{\lambda x_i}-1\right)^{\alpha}\ln\left(e^{\lambda x_i}-1\right)}{1+(e^{\lambda x_i}-1)^{\alpha}} \\ -\sum_{i=1}^{n}\frac{\left(e^{\lambda x_i}-1\right)^{\alpha}\ln\left(e^{\lambda x_i}-1\right)}{(1+(e^{\lambda x_i}-1)^{\alpha})^2}, \end{aligned} \tag{14}$$

$$\begin{aligned} \frac{\partial l}{\partial\lambda} = \frac{n}{\lambda} + (\alpha-1)\sum_{i=1}^{n}\frac{x_i e^{\lambda x_i}}{\ln(e^{\lambda x_i}-1)} - 2\alpha\sum_{i=1}^{n}\frac{\left(e^{\lambda x_i}-1\right)^{\alpha-1}e^{\lambda x_i}x_i}{1+(e^{\lambda x_i}-1)^{\alpha}} \\ +\sum_{i=1}^{n}\left(x_i - \frac{\left(e^{\lambda x_i}-1\right)^{\alpha-1}e^{\lambda x_i}\alpha x_i}{(1+(e^{\lambda x_i}-1)^{\alpha})^2}\right) \end{aligned} \tag{15}$$

and

$$\frac{\partial l}{\partial\theta} = n\left(-1+\frac{1}{1-e^{\theta}}+\frac{1}{\theta}\right) + \sum_{i=1}^{n}\frac{1}{1+(e^{\lambda x_i}-1)^{\alpha}}. \tag{16}$$

5.2 Monte Carlo Simulation

Simulation experiments are provided to investigate the performance of the MLEs of the LEP parameters. The simulation results are replicated for N=10,000 times for each n (sample size) of 20, 50, 100, 200, and 300. Performance of the MLEs of the LEP parameters is checked based on the mean square errors (MSE).

The following sets of parameter values are used to derive random numbers from the LEP model using its qf.

- $\alpha = 0.5, \lambda = 0.5, \theta = 0.25$
- $\lambda = 0.5, \alpha = 0.5, \theta = 0.5$
- $\theta = 0.75, \alpha = 0.5, \lambda = 0.5$
- $\lambda = 0.5, \theta = 0.9, \alpha = 0.5$

The results of these simulations including the MLEs and their MSE are reported in Table 4. The MLEs are consistent. Table 4 elaborate that MSE of all parameters decrease with an increase in sample size. Also, the MSE for all parameters of the observed model increases with the increment in the parameters.

Table 4
The AE, and MSEs based on the simulation

n	Parameter	Set I		Set II		Set III		Set IV	
		MLEs	MSE	MLEs	MSE	MLEs	MSE	MLEs	MSE
20	α	0.5214	0.0174	0.5233	0.1675	0.5197	0.0182	0.5229	0.0169
	λ	0.6730	0.2429	0.6374	0.2106	0.6538	0.2033	0.6536	0.1887
	θ	0.4199	0.1126	0.4235	0.0891	0.4328	0.1803	0.4068	0.3328
50	α	0.5066	0.0063	0.5111	0.0073	0.5031	0.0063	0.5172	0.0069
	λ	0.5541	0.0524	0.5606	0.0573	0.5731	0.0620	0.5356	0.0506
	θ	0.4553	0.0859	0.4671	0.0462	0.4507	0.1380	0.4786	0.2250
100	α	0.5026	0.0032	0.5042	0.0031	0.5017	0.0031	0.5045	0.0033
	λ	0.5403	0.0299	0.5261	0.0247	0.5360	0.0267	0.5368	0.0284
	θ	0.4674	0.0732	0.4803	0.0267	0.4687	0.1045	0.4705	0.2115
200	α	0.5030	0.0015	0.5039	0.0018	0.5003	0.0017	0.5068	0.0017
	λ	0.5132	0.0117	0.5151	0.0125	0.5210	0.0123	0.5002	0.0109
	θ	0.4895	0.0697	0.4893	0.1367	0.4783	0.0875	0.4967	0.1753
300	α	0.5002	0.0011	0.4998	0.0012	0.5035	0.0010	0.5030	0.0010
	λ	0.5137	0.0082	0.5113	0.0070	0.5091	0.0080	0.5063	0.0081
	θ	0.4852	0.0635	0.4931	0.0078	0.4948	0.0732	0.4967	0.1719

6. APPLICATION OF LEPD

In this section, the LEP distribution is fitted to a real-life data set about cancer. The data is given in Lee and Wang (2003), and it represents remission times (in months) of a random sample of 128 bladder cancer patients. The observations are;

0.08, 3.48, 2.09, 4.87, 8.66, 6.94, 13.11, 0.20, 23.63, 2.23, 4.98, 3.52, 6.97, 13.29, 9.02, 0.40, 3.57, 2.26, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 25.82, 2.87, 5.62, 5.85, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 2.87, 5.62, 5.85, 8.26, 8.53, 12.03, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59 and 10.66.

For comparison purposes, we considered the LE, Weibull-geometric (WG), Marshal-Olkin logistic (MOL), and inverse Weibull (IW) distributions as competitor distributions. The fitted model parameters are estimated using the ML approach. The model selection criteria such as Akaike information criteria (AIC), Bayesian information criteria (BIC), Anderson -Darling (AD), Cramer von-Misses (CVM), and Kolmogorov-Smirnov (K-S) tests are used. The p-value (K-S p-value) of the K-S test is also provided.

The box and TTT plots are reported in Figure 3. The MLEs and goodness-of-fit measures are explored in Table 5. The fitted pdf, cdf, sf, and PP plots over the observed data set are displayed in Figure 4.

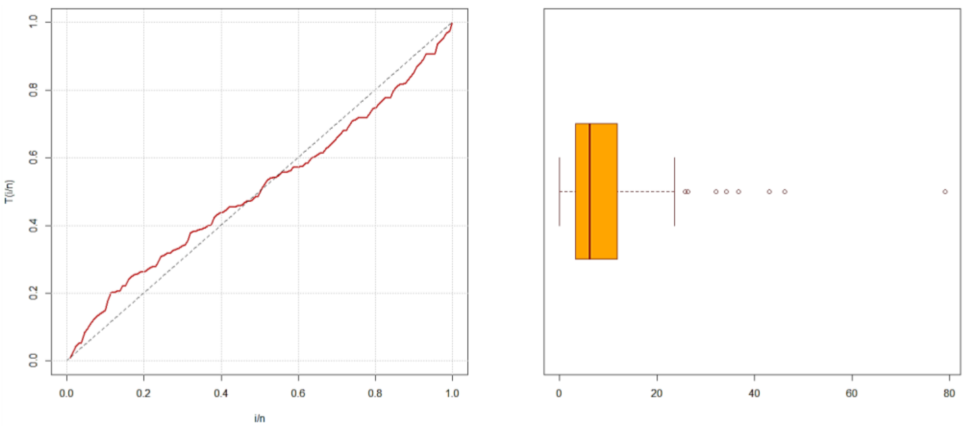

Figure 3: Box and TTT plots of bladder cancer data

Table 5
Findings from bladder cancer data

Model	MLE	AIC	BIC	CVM	AD	K-S	K-S p-value
LEP	$\alpha = 1.28280$ $\theta = 3.22035$ $\lambda = 0.04609$	**825.6480**	**834.2040**	**0.0166**	**0.1271**	**0.0324**	**0.9993**
MOL	$\alpha = 1.84726$ $\theta = 0.14207$ $\lambda = 0.03807$	**844.5723**	**857.0134**	**0.0169**	**0.1275**	**0.0325**	**0.9962**
WG	$\alpha = 2.60284$ $p = 0.93530$ $\lambda = 0.02878$	**826.1844**	**834.7405**	**0.2045**	**0.1716**	**0.0323**	**0.9991**
LE	$\alpha = 1.16342$ $\lambda = 0.01007$	**829.2507**	**834.9548**	**0.1133**	**0.6274**	**0.1975**	**0.0554**
IW	$\alpha = 1.75207$ $\lambda = 0.04310$	**892.0015**	**897.7056**	**0.9787**	**6.1183**	**0.1408**	**0.0125**

Based on the model selection measures given in Table 5, the LEP model has smaller values of AIC and BIC, ADF, CVM, and KS measures and largest p-value. Hence, the proposed LEP distribution provides the best fit as compared to competing distributions. Further, Figure 4 also shows and supports the findings in Table 5. Hence, it is observed that the LEP distribution provides better fits to bladder cancer data.

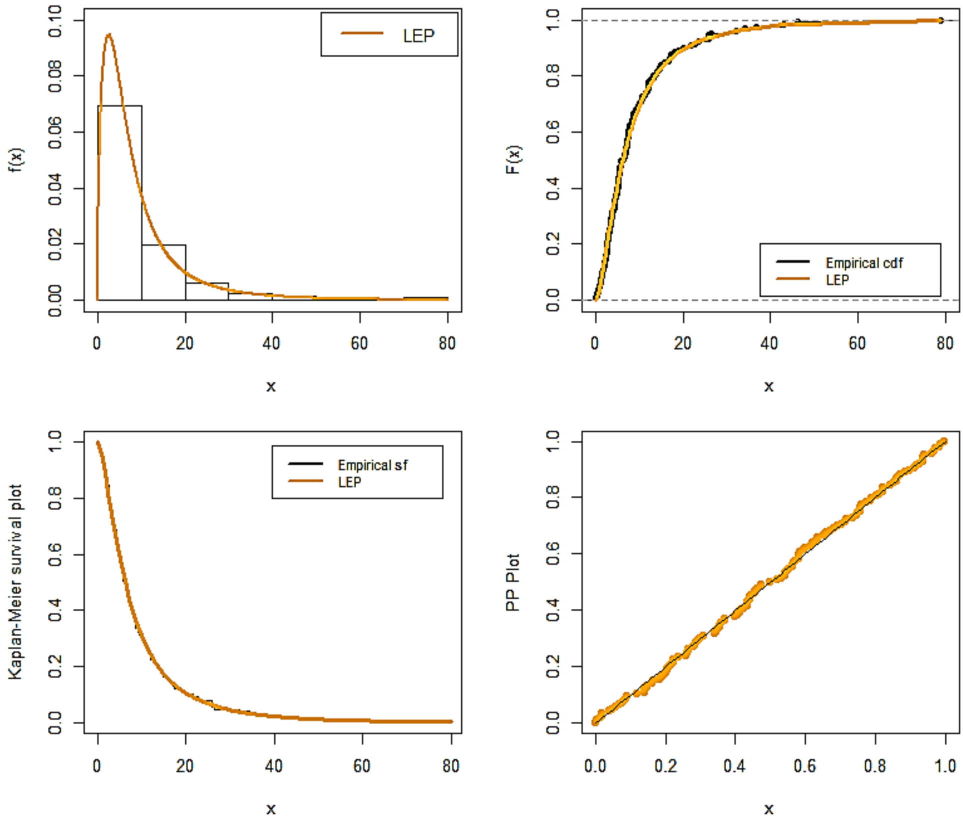

Figure 4: Empirical and pdf, cdf, sf, and PP plots for bladder cancer data

7. CONCLUSION

A new three-parameter logistic exponential Poisson (LEP) distribution is proposed as a special case of the newly introduced logistic exponential power-series family. Some mathematical properties are derived. The parameters of LEP model are estimated using a common approach, the maximum likelihood estimation. The behavior of estimators is assessed using a comprehensive simulation study. We draw samples considering the different combinations of parameters and different sample sizes. For the application of LEP distribution, we fit a real-life data set related to bladder cancer data. The proposed distribution provides better fit as compared to competing distributions.

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