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A NEW TRANSFORMED G CLASS OF DISTRIBUTIONS WITH THEORY AND APPLICATIONS

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ABSTRACT

In this paper, we develop a new class that generates flexible optimal models. The appropriate features of the new class and a special member are studied by utilizing analytical, graphical, and numerical methodologies. For the estimation of unknown parameters, the maximum likelihood, least-square, and percentile methods are discussed and selection is based on bias and mean square error via an extensive simulation study. Five-life time data sets are evaluated, revealing that the new class has a significant advantage over well-known competitors.

KEY WORDS

Transformed family, probability distribution, power function distribution.

1. INTRODUCTION

Applied researchers and practitioners from diverse fields often require a model that is capable of modeling a wider range of datasets. Scientists took the initiative to develop new generalizations (or G-classes), extensions, and modifications to counter complex random phenomena commonly observed in the physical and natural sciences, as well as bathtub-shaped failure rate problems, in order to investigate hidden characteristics of baseline models. In all these scenarios, Pearson (1895) was the first scientist who developed the lifetime models using the system of differential equations. Burr (1942) later developed a novel approach for developing lifetime models using differential equations. Meanwhile, Hastings et al. (1947) presented a lifetime model based on quantile method.

The era of generated families was attributed to Lehmann (1953) who proposed two classes acknowledged as Lehmann type I & Lehmann type II (LI, LII) and Gupta et al. (1998) efforts were acknowledged for exponentiated generated class. A wide range of well-known generators are Marshall-Olkin-G class by Marshall and Olkin (1997), beta normal generated class by Eugene et al. (2002), An odd log-logistic generated class by Gleaton and Lynch (2006), A new methodology of generating symmetric models known as quadratic rank transmutation maps (QRTM) by Shaw and Buckley (2009), gamma-generated class by Zografos and Balakrishnan (2009), Kumaraswamy generalized generated class by Cordeiro and De Castro (2011), gamma exponentiated exponential generated class by Ristic and Balakrishnan (2012), exponentiated generalized class by

Cordeiro et al. (2013), odd Weibull generated class by Bourguignon et al. (2014), beta Marshall Olkin generated class by Alizadeh et al. (2015), Logistic–X class by Tahir et al. (2016), alpha power transformation (APT) and new APT methodology of generating new models by Mahdavi and Kundu (2017), and Elbatal et al. (2019), respectively, beta transformed–H class referenced by Afify et al. (2017), transmuted transmuted–G class by Mansour et al. (2018), generalized odd half-logistic class by Altun et al. (2019), odd Lomax class by Cordeiro et al. (2019), Marshall-Olkin alpha power class by Nassar et al. (2019), a new modified Kies class by Al-Babtain et al. (2020), Gull alpha power Weibull generated class by Ijaz et al. (2020), new Kumaraswamy generated class by Tahir et al. (2020), new logarithmic generated class by Aslam et al. (2020), a new generalized class by Aldahlan et al. (2021), beta Topp–Leone generated class by Elbatal (2021), generalized DUS transformation by Irshad et al. (2021), DUS Kumaraswamy by Karakaya et al. (2021), a flexible Burr X–G class by Al-Babtain et al. (2021), extended Burr–R class by Aldahlan et al. (2021), Marshall–Olkin odd Burr III–G class by Afify et al. (2021), DUS Weibull by Chaudhry and Shareef (2021), Marshall–Olkin Weibull–H class by Afify et al. (2022), a new class for power function by Mutairi et al. (2022), odd Fréchet LII generated class by Mutairi and Arshad (2022).

Recently, Iqbal et al. (2021) developed a modified version of LII that was further discussed as Kumaraswamy size-biased MLII by Balogun et al. (2021). This time, Balogun et al. (2021) developed a MLII generated (MLII–G) class with cumulative distribution function (cdf) is given as follows:

$$Py|_{\psi} = 1 - \left[\frac{1 - G_{(y;\,\psi)}}{1 - aG_{(y;\,\psi)}}\right]^{b}, y \in \mathbb{R}. \tag{1.1}$$

where, $G_{(y;\,\psi)}$ is cdf of the arbitrary baseline model, $1 - a > 0$ and $b > 0$ are scale and shape parameters, respectively.

To widen the scope and improve the fit, we are interested to develop a new class that is capable of generating flexible models to address the monotonic increasing, decreasing, and upside–down bathtub–shaped failure rate problems. For this, in Equation (1.1) we modify (1–a) as (1+α). Further, a transformation known as DUS–transformation attributed to Kumar et al. (2015), with cdf

$$Fy|_{\psi} = \frac{e^{G(y;\psi)} - 1}{e - 1}, y \in \mathbb{R},$$

and pdf

$$fy|_{\psi} = \frac{1}{e - 1} e^{G(y;\,\psi)} \mathrm{g}_{(y;\,\psi)}\,, \tag{1.2}$$

is utilized.

In addition, Alzaatreh et al. (2013) “transformed-transfer” T–X approach is used and new class known as the DUS modified Lehmann type–II generated (DUSMLII–G) class of distributions is developed (see definition 1.1).

The cdf of the T–X class is defined as follows:

$$Fy|_{\psi} = \int_{a}^{W\left[G_{(y;\psi)}\right]} r(t)dt = R\left[W\left(G_{(y;\psi)}\right)\right], y \in \mathbb{R}, \tag{1.3}$$

where $r(t)$ is a pdf corresponding to the cdf $R(t)$ of any random variable $T \in (a, b)$ for $(-\infty < a < b < \infty)$, ψ is a parameters vector and $W\left[G_{(y;\psi)}\right]$ is a function of $G_{(y;\psi)}$ must satisfy three conditions:

i) $W\left[G_{(y;\psi)}\right] \epsilon\ (a, b)$,

ii) $W\left[G_{(y;\psi)}\right]$ is differentiable and monotonically increasing (non-decreasing), and

iii) $\lim\limits_{n \to -\infty} W\left[G_{(y;\psi)}\right] = a$ and $\lim\limits_{n \to +\infty} W\left[G_{(y;\psi)}\right] = b$.

1.1 Definition

The New class is proposed by placing the pdf (Equation (1.2)) in the cdf (Equation (1.3)) and substituting the $W\left[G_{(y;\psi)}\right]$ with $1 - \left[\left(1 - G_{(y;\,\psi)}\right)/\left(1 + \alpha G_{(y;\psi)}\right)\right]^{\beta}$.

Let Y~DUSMLII–G $(y; \psi)$ *with scale* $(1 + \alpha > 0)$, *shape* $(\beta > 0)$, *and* ψ *is a parameters vector. The cdf of random variable y is obtained as*

$$Fy|_{\psi} = \frac{1}{e-1} \int_{0}^{1-\left[\left(1-G_{(y;\,\psi)}\right)/\left(1+\alpha G_{(y;\psi)}\right)\right]^{\beta}} e^{G_{(y;\,\psi)}} \mathrm{g}_{(y;\,\psi)} dy,$$

$$Fy|_{\psi} = \frac{e^{1-\left[\left(1-G_{(y;\,\psi)}\right)/\left(1+\alpha G_{(y;\psi)}\right)\right]^{\beta}} - 1}{e-1}, y \,\epsilon\, \mathbb{R}, \tag{1.4}$$

where $\left[1 - G_{(y;\psi)}\right]$ and $\mathrm{g}_{(y;\,\psi)} = dG_{(y;\,\psi)}/dy$ are survival function and pdf of the arbitrary baseline model, respectively. Note that, $\lim\limits_{y \to 0} G_{(y;\psi)} = 0$, and $\lim\limits_{y \to \infty} G_{(y;\psi)} = 1$. Hence, cdf at $\lim\limits_{y \to -\infty} Fy|_{\psi} = 0$, and cdf at $\lim\limits_{y \to \infty} Fy|_{\psi} = 1$.

The pdf, survival function (sf), hazard rate function (hrf), and quantile function (qf) of the DUSMLII–G class are respectively, given by

$$fy|_{\psi} = \frac{e\beta(1+\alpha)}{e-1} \mathrm{g}_{(y;\psi)} \frac{\left[1 - G_{(y;\,\psi)}\right]^{\beta-1}}{\left[1 + \alpha G_{(y;\,\psi)}\right]^{1+\beta}} e^{-\left[\frac{1-G_{(y;\,\psi)}}{1+\alpha G_{(y;\psi)}}\right]^{\beta}}, \tag{1.5}$$

$$sfy|_{\psi} = \frac{e}{e-1}\left[1 - e^{-\left[\frac{1-G_{(y;\psi)}}{1+\alpha G_{(y;\psi)}}\right]^{\beta}}\right],$$

$$hrfy|_{\psi} = \beta(1+\alpha)\mathrm{g}_{(y;\psi)} \frac{\left[1 - G_{(y;\psi)}\right]^{\beta-1}}{\left[1 + \alpha G_{(y;\psi)}\right]^{1+\beta}} \left[e^{-\left[\frac{1-G_{(y;\,\psi)}}{1+\alpha G_{(y;\psi)}}\right]^{\beta}} - 1\right]^{-1},$$

and

$$qfu|_{\psi} = G^{-1}\left[\frac{1-\left[1-log[1+u(e-1)]\right]^{1/\beta}}{1+\alpha\left[1-log[1+u(e-1)]\right]^{1/\beta}}\right]. \tag{1.6}$$

Note that, if U follows the uniform distribution $U\sim(0,1)$, then $Y=Q(U)$ follows DUSMLII–G $(y;\psi)$. Henceforward the random variable Y with pdf provided in Equation (1.5) is denoted by Y~DUSMLII–G $(y;\psi)$. The key motivations behind the development of the DUSMLII–G class are: to provide a simple and appropriate approach to transform the existing models; to advance the features and enhance the applicability of the existing models; to introduce the closed-form functions for new models; to improve the fits than the other existing models.

The proposed study is discussed in the following sections. Section 2 illustrates general properties along with useful analytical expressions of a special member. Section 3 illustrates different estimation techniques of parameter. Section 4 presents the real life application and Section 5 reports conclusion finally.

2. SPECIAL MEMBER

This section illustrates some valuable characteristics for a special member of the DUSMLII–G class. For this, we suppose, that Y is a random variable that follows to the power function distribution (PFD) with cdf $G_y=(y/M)^{\theta}$, and pdf $g_y=(\theta/M)(y/M)^{\theta-1}$, having $y>0,\theta>0$, and $M\geq y$ are given by respectively.

Then the analytical expressions for the DUSMLII–PF distribution cdf

$$F|_y = \frac{e^{1-\left[\left(1-(y/M)^{\theta}\right)/\left(1+\alpha(y/M)^{\theta}\right)\right]^{\beta}}-1}{e-1}, \tag{2.1}$$

pdf, hrf, and qf are written as follows, respectively

$$f|_y = \left[\frac{e\beta\theta(\alpha+1)}{(e-1)M}\right]\left[\frac{y}{M}\right]^{\theta-1}\frac{\left[1-(y/M)^{\theta}\right]^{\beta-1}}{[1+\alpha(y/M)^{\theta}]^{\beta+1}}e^{-\left[\frac{1-(y/M)^{\theta}}{1+\alpha(y/M)^{\theta}}\right]^{\beta}}, \tag{2.2}$$

$$hrf|_y = \frac{e\beta\theta(\alpha+1)y^{\theta-1}\left[1-(y/M)^{\theta}\right]^{\beta-1}e^{-\left[\left(1-(y/M)^{\theta}\right)/\left(1+\alpha(y/M)^{\theta}\right)\right]^{\beta}}}{eM^{\theta}[1+\alpha(y/M)^{\theta}]^{\beta+1}\left[1-e^{-\left[\left(1-(y/M)^{\theta}\right)/\left(1+\alpha(y/M)^{\theta}\right)\right]^{\beta}}\right]}, \tag{2.3}$$

$$qf|_y = M\left[\frac{1-\left[1-log[1+u(e-1)]\right]^{1/\beta}}{1+\alpha\left[1-log[1+u(e-1)]\right]^{1/\beta}}\right]^{1/\theta} ;q\in(0,1]. \tag{2.4}$$

Here discuss all conceivable shapes of pdf and hrf for the DUSMLII–PF distribution. Note that, Figure 1 explores right-skewed, left-skewed, symmetric, and reversing bathtub-shaped curves with different combinations of the parameters. Figure 2 explores increasing, and bathtub-shaped curves with different combinations of the parameters.

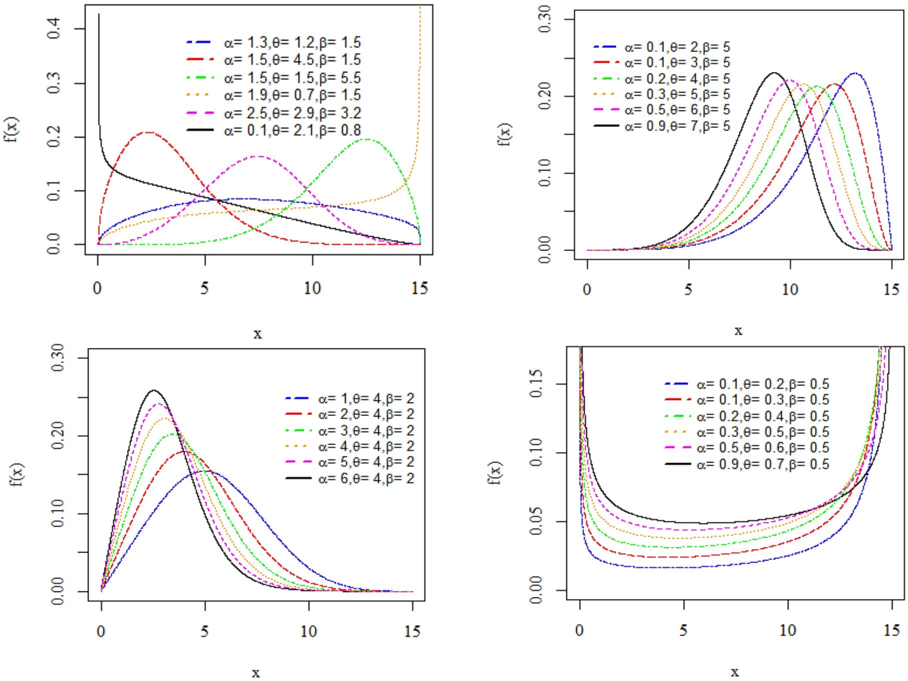


Figure 1: Density Function Plots for Different Parameter Combinations

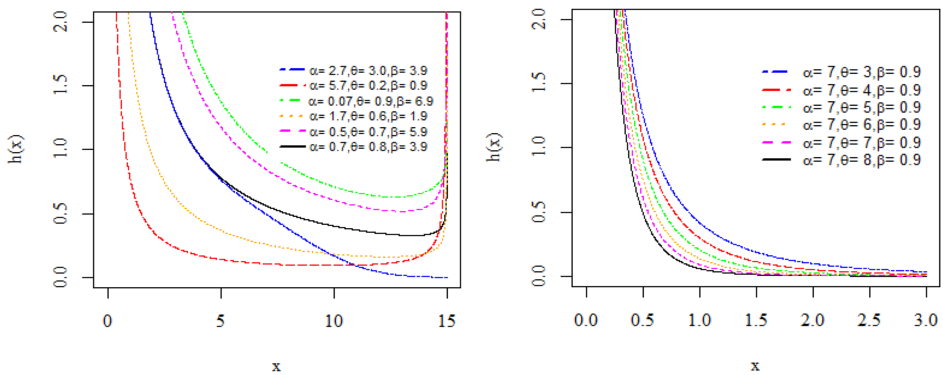


Figure 2: Hazrad Rate Function Plots for Different Parameter Combinations

2.1 Moments and Associated Measures

Theorem 1:

If Y ~ DUSMLII–G $(y; \alpha, \theta, \beta)$ *with scale* $(1 + \alpha > 0)$, *shape* $(\theta, \beta > 0)$ *parameters with* $M \geq y$. *Then the s-th moments about zero* (μ_s') *of Y is given by*

$$\mu_s' = \frac{e\beta\theta(\alpha + 1)}{(e - 1)} \sum_{i,j,k=0}^{\infty} \Psi_{s,\eta,\zeta,i+k} \frac{M^S}{s + \theta(j + k + 1)},$$

Proof:

The s-th moments about zero (μ_s') is defined as

$$\mu_s' = \int_0^M y^s f|_y dy.$$

By placing the information from Equation (2.2) in μ_s' , μ_s' can be written as follows:

$$\mu_s' = \left[\frac{e\beta\theta(\alpha+1)}{(e-1)M}\right] \int_0^M y^s \left[\frac{y}{M}\right]^{\theta-1} \frac{\left[1-(y/M)^\theta\right]^{\beta-1}}{\left[1+\alpha(y/M)^\theta\right]^{\beta+1}} e^{-\left[\frac{1-(y/M)^\theta}{1+\alpha(y/M)^\theta}\right]^\beta} dy. \quad (2.5)$$

First, we apply exponential series expansion, we have

$$e^{-\left[\frac{1-(y/M)^\theta}{1+\alpha(y/M)^\theta}\right]^\beta} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left[\frac{1-(y/M)^\theta}{1+\alpha(y/M)^\theta}\right]^{\beta i}.$$

Further, we simplify the terms $\frac{\left[1-(y/M)^\theta\right]^{\beta-1}}{\left[1+\alpha(y/M)^\theta\right]^{\beta+1}}$ and $\sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left[\frac{1-(y/M)^\theta}{1+\alpha(y/M)^\theta}\right]^{\beta i}$ and the expression can be written as follows:

$$\sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left[1-(y/M)^\theta\right]^{\beta i+\beta-1} \left[1+\alpha(y/M)^\theta\right]^{-(\beta i+\beta+1)},$$

then, we can present the last expression in terms of linear representation as

$$\sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+k}}{i!} \alpha^k \binom{\eta}{j} \binom{\zeta}{k} \left[\frac{y}{M}\right]^v, \quad (2.6)$$

Hence, by placing the information (Equation (2.6)) in Equation (2.5) and integral w.r.t "y", the s-th moments about zero is written as follows:

$$\mu_s' = \frac{e\beta\theta(\alpha+1)}{(e-1)} \sum_{i,j,k=0}^{\infty} \Psi_{\eta,\zeta,i+k} \frac{M^s}{s+\theta(j+k+1)}, \quad (2.7)$$

where $\Psi_{\eta,\zeta,i+k} = \frac{(-1)^{i+j}}{i!} \alpha^k \binom{\eta}{j} \binom{\zeta}{k}$, $\eta = \beta i + \beta - 1$ and $\zeta = -(\beta i + \beta + 1)$, $v = \theta(j+k)$, s = 1,2,3, and 4.

The s-th moments about zero (μ_s') has an important role in providing descriptive information for a given data set that has extensive use in multidisciplinary areas of science. The necessary statistics may easily be deduced by following Equation (2.7).

Corollary 1:

The mean μ_1', negative moments μ_{-v}', and variance $\aleph^2 = \mu_2' - (\mu_1')^2$ of Y can be deduced by replacing s of μ_s' with 1, $(-v)$, and 2, respectively.

The analytical expressions of mean, negative moments, and variance are:

$$\mu_1' = \frac{e\beta\theta(1+\alpha)}{e-1}\sum_{i,j,k=0}^{\infty}\Psi_{\eta,\zeta,i+k}\frac{M}{1+\theta(j+k+1)},$$

$$\mu_{-v}' = \frac{e\beta\theta(1+\alpha)}{e-1}\sum_{i,j,k=0}^{\infty}\Psi_{\eta,\zeta,i+k}\frac{M^{-v}}{-v+\theta(j+k+1)},$$

and

$$\aleph^2 = \frac{e\beta\theta(1+\alpha)}{e-1}\left[\begin{array}{c}\sum_{i,j,k=0}^{\infty}\Psi_{\eta,\zeta,i+k}\frac{M^2}{2+\theta(j+k+1)} - \\ \frac{e\beta\theta(1+\alpha)}{e-1}\left(\sum_{i,j,k=0}^{\infty}\Psi_{\eta,\zeta,i+k}\frac{M}{1+\theta(j+k+1)}\right)^2\end{array}\right],$$

respectively.

Note that, skewness and kurtosis are very useful characteristics to discuss the tail and peak behaviour of a given data set, respectively. The coefficient of skewness (∇_{sk}) and coefficient of kurtosis (∇_{kur}) of Y can be calculated by

$$\nabla_{sk} = \frac{\mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3}{\sqrt[3]{\mu_2' - \mu_1'^2}}, \quad \text{and} \quad \nabla_{kur} = \frac{\mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4}{(\mu_2' - \mu_1'^2)^2}.$$

respectively.

Corollary 2:

The factorial generating function (FGF) is derived by following

$$E(1+p)^y = E\left[e^{y\log(1+p)}\right] = \sum_{s=0}^{\infty}\frac{[\log(1+p)]^s}{s!}\mu_s'$$

and the analytical expression of FGF of Y is given as follows:

$$FGF(p) = \frac{e\beta\theta(1+\alpha)}{e-1}\sum_{s=0}^{\infty}\frac{(\log(1+p))^s}{s!}\sum_{i,j,k=0}^{\infty}\Psi_{\eta,\zeta,i+k}\frac{M^s}{s+\theta(j+k+1)}.$$

Theorem 2:

If Y ~ DUSMLII–G $(y;\alpha,\theta,\beta)$ with scale $(1+\alpha>0)$, shape $(\theta,\beta>0)$ parameters with $M \geq y$. Then the moment generating function (MGF) $M_Y(p)$ of Y is given by

$$MGF(p) = \frac{e\beta\theta(1+\alpha)}{e-1}\sum_{s=0}^{\infty}\frac{p^s}{s!}\sum_{i,j,k=0}^{\infty}\Psi_{\eta,\zeta,i+k}\frac{M^s}{s+\theta(j+k+1)}.$$

Proof:

The MGF is derived by following

$$MGF(p) = \int_0^M e^{py} f|_y dy.$$

We know that $e^{py} = \sum_{s=0}^{\infty} \frac{(py)^s}{s!}$, then the analytical expression of MGF of Y is written as follows:

$$M_Y(p) = \frac{e\beta\theta(1+\alpha)}{e-1} \sum_{s=0}^{\infty} \frac{p^s}{s!} \int_0^M y^s (y/M)^{\theta-1} \frac{\left[1-(y/M)^\theta\right]^{\beta-1}}{[1+\alpha(y/M)^\theta]^{\beta+1}} e^{-\left[\frac{1-(y/M)^\theta}{1+\alpha(y/M)^\theta}\right]^\beta} dy.$$

Hence, the MGF of Y is given as

$$M_Y(p) = \frac{e\beta\theta(1+\alpha)}{e-1} \sum_{s=0}^{\infty} \frac{p^s}{s!} \sum_{i,j,k=0}^{\infty} \Psi_{\eta,\zeta,i+k} \frac{M^s}{s+\theta(j+k+1)}.$$

Theorem 3:

If Y ~ DUSMLII–G $(y;\alpha,\theta,\beta)$ *with scale* $(1+\alpha>0)$, *shape* $(\theta,\beta>0)$ *parameters with* $M \geq y$. *Then the characteristic function (CF)* $\vartheta_Y(p)$ *of Y is given by*

$$CF(p) = \frac{e\beta\theta(1+\alpha)}{e-1} \sum_{s=0}^{\infty} \frac{(ip)^s}{s!} \sum_{i,j,k=0}^{\infty} \Psi_{\eta,\zeta,i+k} \frac{M^s}{s+\theta(j+k+1)}, i = \sqrt{-1}.$$

Proof:

The CF is derived by following

$$CF(p) = \int_0^M e^{ipy} f|_y dy.$$

We know that $e^{ipy} = \sum_{s=0}^{\infty} \frac{(ipy)^s}{s!}, i = \sqrt{-1}$, then the analytical expression of CF of Y is given as follows:

$$\vartheta_Y(p) = \frac{e\beta\theta(1+\alpha)}{e-1} \sum_{s=0}^{\infty} \frac{(ip)^s}{s!} \int_0^M y^s (y/M)^{\theta-1} \frac{\left[1-(y/M)^\theta\right]^{\beta-1}}{[1+\alpha(y/M)^\theta]^{\beta+1}} e^{-\left[\frac{1-(y/M)^\theta}{1+\alpha(y/M)^\theta}\right]^\beta} dy.$$

Hence, the CF of Y is given as

$$\vartheta_Y(p) = \frac{e\beta\theta(1+\alpha)}{e-1} \sum_{s=0}^{\infty} \frac{(ip)^s}{s!} \sum_{i,j,k=0}^{\infty} \Psi_{\eta,\zeta,i+k} \frac{M^s}{s+\theta(j+k+1)}, i = \sqrt{-1}.$$

Here, we analyze the DUSMLII-PF distribution numerically with first four moments (m_1', m_2', m_3', m_4') about zero, variance ($\aleph^2$), skewness (∇_{sk}), and kurtosis (∇_{kur}) for different values of model parameter.

Table 1
Numerical Analysis with Moments, Variance, Skewness, and Kurtosis

Parameters			m'_1	m'_2	m'_3	m'_4	$\aleph^2$	∇_{sk}	∇_{kur}
M=5	α=2.1 & θ=2.1	β=0.1	0.0140	0.0157	0.0331	0.0897	0.0156	277.502	360.034
		β=0.3	0.1432	0.1684	0.3511	0.9429	0.1600	23.7118	32.3060
		β=0.5	0.3716	0.4950	1.0554	2.8431	0.4476	7.3434	10.7022
		β=0.7	0.6383	0.9685	2.1467	5.8563	0.8427	3.4342	5.3757
		β=0.9	0.9069	1.5446	3.5807	9.9518	1.3053	1.9814	3.3219

Remarks 1. For fixed $\boldsymbol{\alpha = 2.1, \theta = 2.1, M = 5}$, with $\boldsymbol{0.1 < \beta < 0.9}$: Increase in moments, and variance alongside decrease in skewness, and kurtosis is observed.

M=3	β=1.1 & θ=1.1	α=1.1	1.3677	2.5468	5.4363	12.5319	1.7957	0.1879	0.6006
		α=1.3	1.3222	2.4183	5.1066	11.6888	1.7305	0.2277	0.6710
		α=1.5	1.2808	2.3036	4.8160	10.9526	1.6690	0.2681	0.7407
		α=1.7	1.2429	2.2003	4.5577	10.3039	1.6112	0.3086	0.8094
		α=1.9	1.2080	2.1069	4.3265	9.7277	1.5569	0.3493	0.8773

Remarks 2. For fixed $\boldsymbol{\beta = 1.1, \theta = 1.1, M = 3}$, with $\boldsymbol{1.1 < \alpha < 1.9}$: Increase in moments, decrease in variance alongside increase in skewness, and kurtosis is observed.

M=2	α=2 & β=3	θ=1.1	1.3280	1.9292	2.9735	4.7791	0.4617	0.0124	0.1319
		θ=2.3	1.0535	1.2319	1.5518	2.0695	0.2922	0.0064	0.1994
		θ=3.5	0.9141	0.9294	1.0214	1.1934	0.1269	0.0404	0.0887
		θ=4.7	0.8264	0.7596	0.7553	0.7994	0.0338	0.1089	-0.1161
		θ=4.9	0.8147	0.7382	0.7236	0.7550	0.0224	0.1243	-0.1566

Remarks 3. For fixed $\boldsymbol{\alpha = 2, \beta = 3, M = 2}$, with $\boldsymbol{1.1 < \alpha < 4.9}$: Increase in moments, decrease in variance alongside decrease and then increase in skewness, and kurtosis is observed.

Theorem 4:

If Y ~ DUSMLII–G $(y; \alpha, \theta, \beta)$ with scale $(1 + \alpha > 0)$, shape $(\theta, \beta > 0)$ parameters with $M \geq y$. Then the s-th incomplete moment $\Phi_s(t)$ of Y is given by

$$\tau_s(t) = \frac{e\beta\theta(1+\alpha)}{e-1} \sum_{i,j,k=0}^{\infty} \Psi_{\eta,\zeta,i+k} \frac{t^s}{s + \theta(j+k+1)}.$$

Proof:

The s-th incomplete moment is derived by following

$$\tau_s(t) = \int_0^t y^s f|_y dy.$$

The analytical expression of s-ICM for Y is given by

$$\tau_s(t) = \frac{e\beta\theta(1+\alpha)}{e-1} \sum_{i,j,k=0}^{\infty} \Psi_{\eta,\zeta,i+k} \frac{t^s}{s + \theta(j+k+1)}. \quad (2.8)$$

Corollary 3:

The first ICM is obtained by replacing s with 1 in Equation (2.8) and the analytical expression of the first ICM (1st – ICM) of Y is given by

$$\tau_1(t) = \frac{e\beta\theta(1+\alpha)}{e-1}\sum_{i,j,k=0}^{\infty}\Psi_{\eta,\zeta,i+k}\frac{t}{1+\theta(j+k+1)}.$$

2.2 Conditional Moments

The conditional moments (CMs) of Y are derived by following

$$CM(y) = E(y^s|_{Y>u}) = \frac{1}{\bar{F}(u)}\int_u^M y^s f_y dy$$

and the analytical expression of CMs is given as follows:

$$CM(y) = \frac{1}{1-F(u)}\left[\frac{e\beta\theta(1+\alpha)}{e-1}\sum_{i,j,k=0}^{\infty}\Psi_{\eta,\zeta,i+k}\frac{M^s-u^s}{s+\theta(j+k+1)}\right].$$

2.3 Residual Functions and Associated Measures

2.3.1 Residual Life Function (RLF)

The RLF is discussed as $R_{u(y)} = \frac{S(y+u)}{S(u)}$ and for Y it can be written as follows:

$$R_y = \frac{e^{1-\left[\left(1-(y+u/M)^{\theta}\right)/\left(1+\alpha(y+u/M)^{\theta}\right)\right]^{\beta}}-1}{e-e^{1-\left[\left(1-(u/M)^{\theta}\right)/\left(1+\alpha(u/M)^{\theta}\right)\right]^{\beta}}}.$$

Furthermore, cdf of RLF ($F_{u(y)}$) can be written as follows:

$$F_{u(y)} = \frac{e-e^{1-\left[\left(1-(u/M)^{\theta}\right)/\left(1+\alpha(u/M)^{\theta}\right)\right]^{\beta}}-e^{1-\left[\left(1-(y+u/M)^{\theta}\right)/\left(1+\alpha(y+u/M)^{\theta}\right)\right]^{\beta}}+1}{e-e^{1-\left[\left(1-(u/M)^{\theta}\right)/\left(1+\alpha(u/M)^{\theta}\right)\right]^{\beta}}},$$

2.3.2 Reversed Residual Life Function (R-RLF)

The R-RLF is discussed as $\underline{R}_{u(y)} = \frac{S(y-u)}{S(u)}$ and for Y it can be written as follows:

$$\underline{R}_y = \frac{e^{1-\left[\left(1-(y-u/M)^{\theta}\right)/\left(1+\alpha(y-u/M)^{\theta}\right)\right]^{\beta}}-1}{e-e^{1-\left[\left(1-(u/M)^{\theta}\right)/\left(1+\alpha(u/M)^{\theta}\right)\right]^{\beta}}}.$$

Furthermore, cdf of RLF ($\underline{F}_{u(y)}$) can be written as follows:

$$\underline{F}_{u(y)} = \frac{e-e^{1-\left[\left(1-(u/M)^{\theta}\right)/\left(1+\alpha(u/M)^{\theta}\right)\right]^{\beta}}-e^{1-\left[\left(1-(y-u/M)^{\theta}\right)/\left(1+\alpha(y-u/M)^{\theta}\right)\right]^{\beta}}+1}{e-e^{1-\left[\left(1-(u/M)^{\theta}\right)/\left(1+\alpha(u/M)^{\theta}\right)\right]^{\beta}}},$$

2.3.3 Mean Residual Life Function (M-RLF)

The M-RLF is discussed as M-RLF $= \frac{1-\tau_1(u)}{S(u)-u}$ and for Y it can be written as follows:

$$\mathrm{M-RLF} = \frac{1}{S(u)-u}\left[1-\frac{e\beta\theta(1+\alpha)}{e-1}\sum_{i,j,k=0}^{\infty}\Psi_{\eta,\zeta,i+k}\frac{u}{1+\theta(j+k+1)}\right].$$

2.3.4 Mean Inactivity Time (M-IT)

The MIT is discussed as M-IT $= u-\frac{\tau_1(u)}{F(u)}$ and for Y it can be written as follows:

$$\mathrm{M-IT} = u-\frac{\left[e\beta\theta(1+\alpha)\sum_{i,j,k=0}^{\infty}\Psi_{\eta,\zeta,i+k}u\right]}{\frac{1}{[e-1]}\left[e^{1-\left[\left(1-(u/M)^{\theta}\right)/\left(1+\alpha(u/M)^{\theta}\right)\right]^{\beta}}-1\right][1+\theta(j+k+1)]}.$$

2.3.5 Strong Mean Inactivity Time (S-MIT)

The S-MIT is discussed as S-MIT $= v^2-\frac{\tau_2(v)}{F(v)}$ and for Y it can be written as follows:

$$\mathrm{S-MIT} = v^2-\frac{e\beta\theta(1+\alpha)\sum_{i,j,k=0}^{\infty}\Psi_{\eta,\zeta,i+k}v^2}{\frac{1}{[e-1]}\left[e^{1-\left[\left(1-(v/M)^{\theta}\right)/\left(1+\alpha(v/M)^{\theta}\right)\right]^{\beta}}-1\right][1+\theta(j+k+1)]}$$

2.4 Bonferroni and Lorenz Curves

The application of 1st – ICM is frequently discussed in the Lorenz $L(p)$ and Bonferroni $B(p)$ inequalities context. These curves have found some useful applications, particularly in demography, insurance, income, and poverty events. The $L(p)$ and $B(p)$ curves are defined as follows respectively

$$L(p) = pB(p) = \frac{1}{\mu_1'}\int_0^q yf|_y dy, \quad \text{and} \quad B(p) = \frac{\tau_1(q)}{p\mu_1'} = \frac{\int_0^q yf|_y dy}{p\mu_1'}.$$

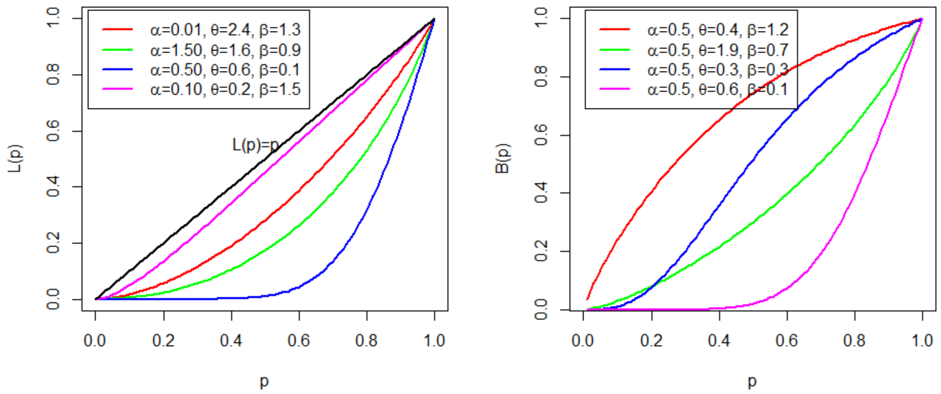
$\tau_1(q)$ is first incomplete moments and it is defined under quantile function. The expression is defined as

$$\tau_1(q) = \frac{e\beta\theta(1+\alpha)}{e-1}\sum_{i,j,k=0}^{\infty}\Psi_{\eta,\zeta,i+k}\frac{q(p)}{1+\theta(j+k+1)},$$

$$\mu_1' = \frac{e\beta\theta(1+\alpha)}{e-1}\sum_{i,j,k=0}^{\infty}\Psi_{\eta,\zeta,i+k}\frac{M}{1+\theta(j+k+1)},$$

with $\quad q(p) = y_{\mathrm{g}}\left[\frac{1-\left[1-log[1+p(e-1)]\right]^{1/\beta}}{1+\alpha\left[1-log[1+p(e-1)]\right]^{1/\beta}}\right]^{1/\theta}$; $q \in (0,1]$,

where $\Psi_{\eta,\zeta,i+k} = \frac{(-1)^{i+k}}{i!}\binom{\eta}{j}\binom{-\zeta}{k}\alpha^k$. Figure 3 presents Bonferroni (a) and Lorenz (b) curves of Y for different combinations of the parameters.



(a) (b)

Figure 3: Bonferroni (a) and Lorenz (b) Curves for Different Parameter Combinations

3. INFERENCE

In this section, we compare the performance of three well-known estimation methods. For this, we use the maximum likelihood estimate (MLE), least-square estimate (LSE), and percentile (P) methods. A simulation study-based experiment is conducted to compare these three methods and the performance is assessed in terms of bias and mean square error (MSE) of the estimators. For this, we generate N = 10,000 replicates from Equation (2.3) and simulate each sample of size $n = 30, 40, \dots, 500$. Plots against the Set-I to Set-III for MLE, LSE, and P at different combinations of the parameters (α, θ, β) are presented in Figures 4 to 6.

3.1 Maximum Likelihood Estimate (MLE)

The MLE is considered one of the most popular and widely used methods of parameter estimation due to its joyful properties of unbiasedness, efficiency, invariance under parameter transformation, consistency, and asymptotically normal distribution. In the case of the DUSMLII-PF distribution, we draw a random sample $\boldsymbol{z} = (z_1, \dots, (z_n)$, to estimate the unknown parameters $\beth = (\alpha|_{MLE}, \theta|_{MLE}, \beta_{MLE})^T$. The log-likelihood function (say $l_y(\beth)\big|_{MLE}$) can be defined as follows:

$$(z_1 l_y(\beth)\big|_{MLE} = \begin{bmatrix} nlog(1+\alpha) + nlog(\beta) + nlog(\theta) - nlog(M) + nlog\left(\frac{e}{e-1}\right) + \\ (\theta-1)\sum_{j=1}^{n} log\,(y/M) + (\beta-1)\sum_{j=1}^{n} log\left[1-(y/M)^{\theta}\right] - \\ (\beta+1)\sum_{j=1}^{n} log\left[1+\alpha(y/M)^{\theta}\right] - \sum_{j=1}^{n}\left[\frac{1-(y/M)^{\theta}}{1+\alpha(y/M)^{\theta}}\right]^{\beta} \end{bmatrix}. \quad (3.1)$$

The ML estimators $\hat{ב} = \left(\hat{\alpha}|_{MLE}, \hat{\theta}\big|_{MLE}, \hat{\beta}\big|_{MLE}\right)^T$ can be derived by maximizing the $l_y(ב)\big|_{MLE}$ or solving the nonlinear likelihood equations simultaneously by differentiating Equation (3.1) (see appendix A)

3.2 Least Square Estimation (LSE)

The LS estimates $\hat{\alpha}|_{LS}, \hat{\theta}\big|_{LS}$, and $\hat{\beta}\big|_{LS}$ for parameters $\alpha|_{LS}, \theta|_{LS}$, and $\beta|_{LS}$ can be derived by minimizing the function $l_y(ב)\big|_{LS}$ as

$$l_y(ב)\big|_{LS} = \sum_{j}^{n}\left[\frac{e^{1-\left[\left(1-\left(y|_{j:n}/M\right)^{\theta}\right)/\left(1+\alpha\left(y|_{j:n}/M\right)^{\theta}\right)\right]^{\beta}}-1}{e-1} - \frac{j}{n+1}\right]^2,$$

w.r.t. $\alpha|_{LS}, \theta|_{LS}$, and $\beta|_{LS}$. Otherwise, these estimates can be derived by solving the following nonlinear equations (see appendix B).

3.3. Percentile Method (P)

The P estimates can be derived for the DUSMLII-PF distribution if it has closed form cdf. The P estimates $\hat{\alpha}|_P, \hat{\theta}\big|_P$, and $\hat{\beta}\big|_P$ for parameters $\alpha|_P, \theta|_P$, and $\beta|_P$ can be derived by equating the sample P points with corresponding population P points. We can derive the estimates of the DUSMLII-PF distribution by minimizing the function $l_y(ב)\big|_P$ as

$$l_y(ב)\big|_P = \sum_{j}^{n}\left[y|_{j:n} - y_g\left[\frac{1-\left[1-log\left[1+u_j(e-1)\right]\right]^{1/\beta}}{1+\alpha\left[1-log\left[1+u_j(e-1)\right]\right]^{1/\beta}}\right]^{1/\theta}\right]^2,$$

w.r.t. $\alpha|_P, \theta|_P$, and $\beta|_P$. Note that, $u_j = \frac{j}{n+1}$ is the estimate $F(y|_{j:n})$.

The nonlinear equations are unable to offer the analytical solution for the MLEs. Hence, the statistical software R may provide the solution appropriately. The following are the plots for MLE, LSE, and P for different combinations of parameters.

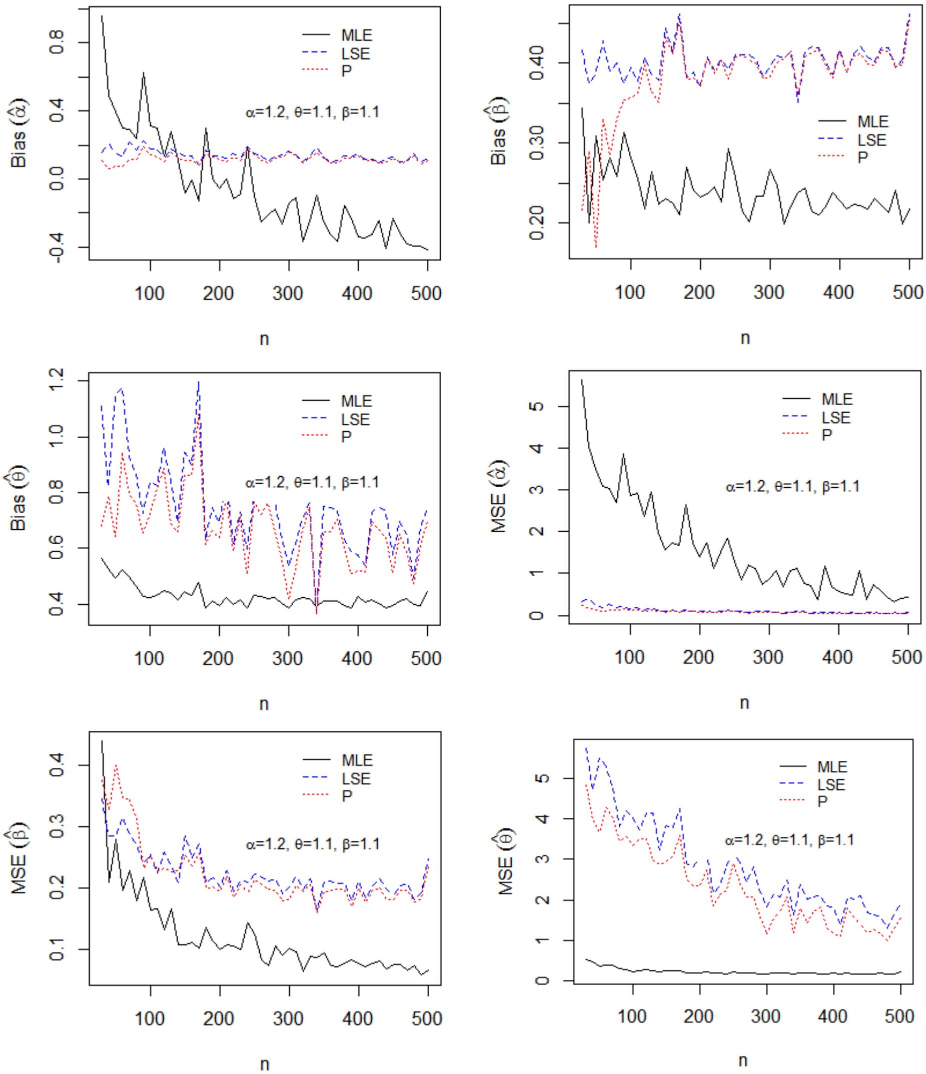


Figure 4: MLE, LSE, and P plots of Bias and MSE for Set-I
$(\boldsymbol{\alpha = 1.2, \theta = 1.1, \beta = 1.1})$

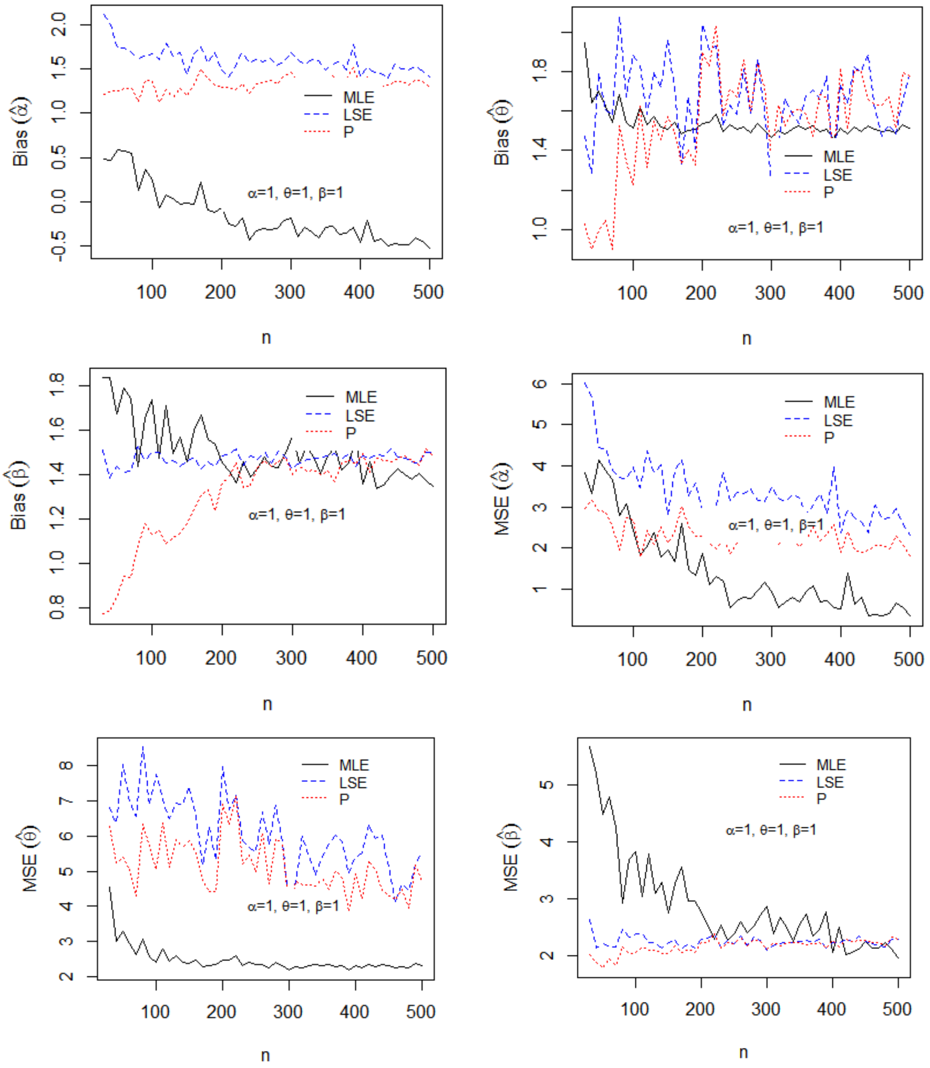


Figure 5: MLE, LSE, and P plots of Bias and MSE for Set-II
$(\boldsymbol{\alpha = 1, \theta = 1, \beta = 1})$

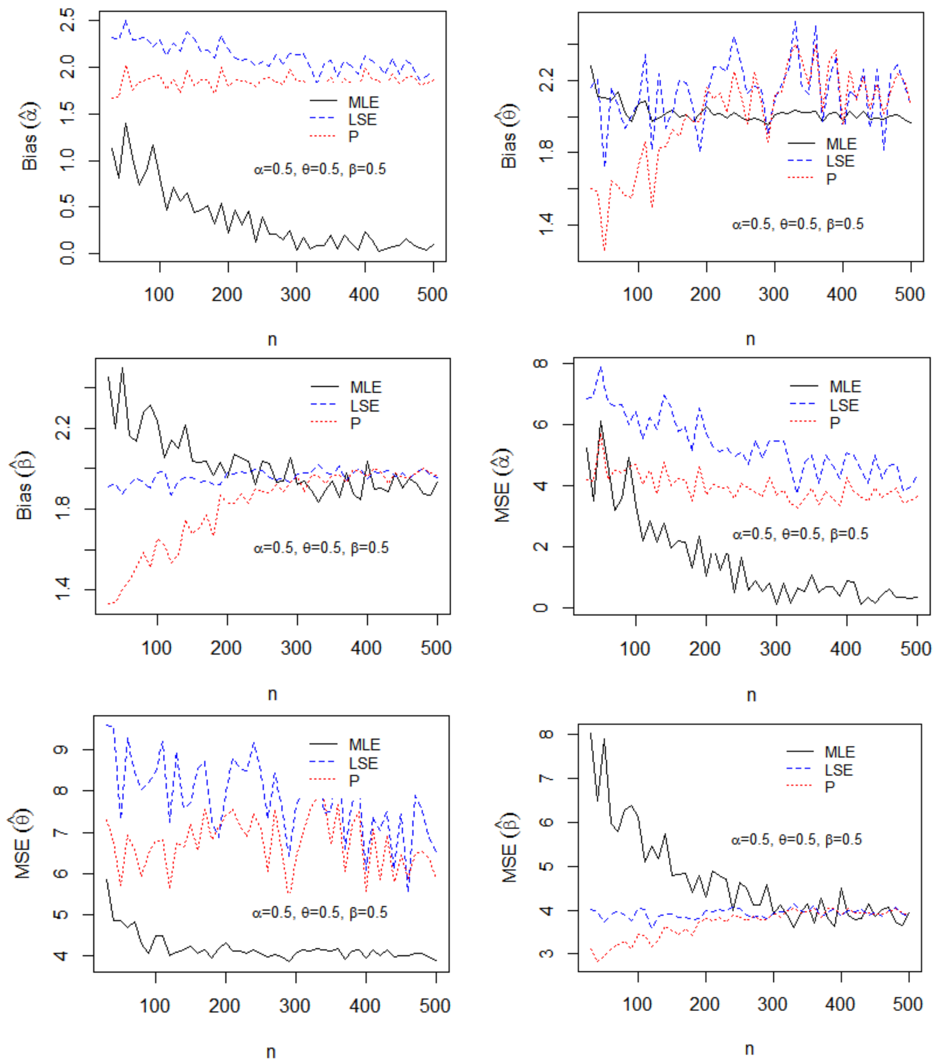


Figure 6: MLE, LSE, and P plots of Bias and MSE for Set-III
$(\boldsymbol{\alpha = 0.5, \theta = 0.5, \beta = 0.5})$

The average Bias and average MSE curves of MLE, LSE, and P, in most cases, converge to zero with the increase of sample sizes nevertheless the MLE performance is substantially better and more consistent than the LSE and P since it has a reduced MSE in all circumstances.

4. APPLICATION TO LIFETIME DATA

In this section, we compare the DUSLII-PF with those of modified Lehmann type-II power function (MLII-PF)(New), DUS-power function(DUS-PF)(New), power function

(PF) (Ahsanullah and Kabir (1974)), generalized power function (Gen-PF) (Saran and Pandey (2004)), Weibull power function (W-PF) (Tahir et al. (2014)), transmuted power function (Tr-PF) (Ahsan-ul-Haq et al. (2016)), Marshall Olkin power function (MO-PF) (Okorie et al. (2017)), Kumaraswamy power function (Kum-PF) (Abdul-Moniem (2017)), power function Poisson (PF-Poi) (Hassan and Assar (2021)), and zero truncated Poisson power function (ZTP-PF) (Okorie et al. (2021)).

Some well-known statistics of the data sets including mean, standard deviation, skewness, kurtosis, lower control limit (LCL), and upper control limits (UCL) are presented in Table 2. Anderson- Darling (AD), Cramer von Mises (CVM), and Kolmogorov-Smirnov (KS) with their *p*-values are presented in Tables 3 to 7. Note that the better fit model criteria follow the lowest values of these statistics with the highest value of KS *p*-value.

For this, five-lifetime data sets were explored from the multidisciplinary areas of science. The first dataset was discussed by Aarset (1987), which provides information about the failure times of fifty devices. The second dataset was discussed by Meeker and Escobar (1998), which provides information about the failure times of thirty electronic devices. The third dataset was discussed by Barlow et al. (1984), which provides information about the fatigue fracture of Kevlar 373/epoxy. The fourth dataset was discussed by Cook and Weisberg (1994), which analyzes the lean body mass of Australian athletes and finally the fifth dataset provides information about the daily death count owing to COVID-19 in China from 23rd January to 28th March. The COVID-19 data can be downloaded free from the official website. "(https://www.worldometers.info/coronavirus/country/china/)". The data sets are provided in the Appendix C.

Table 2
Some Well-Known Statistics

Data	Mean	Standard Deviation	Skewness	Kurtosis	LCL	UCL
Fifty devices	45.660	32.8353	-0.1319	1.4135	36.3342	54.9977
Thirty devices	177.03	114.992	-0.2699	1.4536	134.094	219.972
Fatigue fracture	1.9592	1.5739	1.9406	8.1607	1.5995	2.3189
Body mass	54.899	6.9221	-0.3023	3.4494	53.5213	56.2684
COVID-19 death	49.742	43.8730	0.8176	2.4502	38.9570	60.5277

It's worth noting that the competing models MLII-PF and DUS-PF are brand new, and to our knowledge, these models have never been mentioned before. Furthermore, both models are being considered for future projects.

The cdfs of proposed and competitive models are given as follows:

Proposed DUSMLII-PF:

$$F|_y = \left(e^{1-\left[\left(1-(y/M)^{\theta}\right)/\left(1+\alpha(y/M)^{\theta}\right)\right]^{\beta}} - 1\right)(e-1)^{-1}\Big|_{\alpha,\theta,\beta>0,1+\alpha>0,M\geq y}.$$

MLII-PF (New):

$$P_{y|_I} = \left(1 - \left(1 - (y/M)^{\theta}\right)^{\beta}\left(1 + \alpha(y/M)^{\theta}\right)^{-\beta}\right)\Big|_{\theta,\beta>0,1+\alpha>0,M\geq y}.$$

Kum-PF:

$$P_{y|_{II}} = 1 - \left(1 - (y/M)^{\alpha\theta}\right)^{\beta}\Big|_{\alpha,\theta,\beta>0,M\geq y}.$$

MO-PF:

$$P_{y|_{III}} = 1 - \left(\alpha\left(1 - (y/M)^{\beta}\right)\right)/\left((y/M)^{\beta} + \alpha\left(1 - (y/M)^{\beta}\right)\right)\Big|_{\alpha,\beta>0,M\geq y}.$$

PF-Poi:

$$P_{y|_{IV}} = \left(e^{\alpha(y/M)^{\beta}} - 1\right)/\left(e^{\beta} - 1\right)\Big|_{\alpha,\beta>0,M\geq y}.$$

DUS-PF(New):

$$P_{y|_V} = \left(e^{(y/M)^{\alpha}} - 1\right)/(e - 1)\Big|_{\alpha>0,M\geq y}.$$

Gen-PF:

$$P_{y|_{VI}} = 1 - (M - y)^{\alpha}(M - y_m)^{-\alpha}|_{\alpha>0,y_m\leq y\leq M}.$$

Tr-PF:

$$P_{y|_{VII}} = (1 + \lambda)(y/M)^{\alpha} - \lambda(y/M)^{2\alpha}\ |_{\alpha>0,|\lambda|\leq 1,M\geq y}.$$

PF:

$$P_{y|_{VIII}} = (y/M)^{\alpha}|_{\alpha>0,M\geq y}.$$

ZTP-PF:

$$P_{y|_{IX}} = \left(1 - e^{-\alpha(y/M)^{\beta}}\right)(1 - e^{-\alpha})^{-1}\Big|_{\alpha,\beta>0,M\geq y}.$$

W-PF:

$$P_{y|_X} = 1 - e^{-\alpha\left(y^{\theta}/\left(M^{\theta} - y^{\theta}\right)\right)^{\beta}}\Big|_{\alpha,\theta,\beta>0,M\geq y}.$$

Table 3
MLEs with SEs of Parameters of the DUSMLII-PF Distribution and Fitted Measures for Failure Times of Fifty Devices

Model	MLEs with SEs in (.)				Fitted Measures			
	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\lambda}$	CVM	AD	KS	*p*-Value
DUSMLII-PF	1.2324 (2.1672)	0.4924 (0.1060)	0.4855 (0.1879)	-	0.0735	0.5976	0.0930	0.7799
MLII-PF(*New*)	-0.1330 (1.0618)	0.4733 (0.1149)	0.4556 (0.1806)	-	0.0778	0.6261	0.0983	0.7189
Kum-PF	0.9823 (47.0761)	0.4435 (21.2546)	0.4847 (0.0802)	-	0.0779	0.6270	0.0993	0.7077
MO-PF	7.1925 (5.2427)	0.2629 (0.1545)	-	-	0.1171	0.8684	0.1677	0.1203
PF-Poi	2.1017 (0.9987)	0.4564 (0.1474)	-	-	0.0886	0.7005	0.2037	0.0315
DUS-PF(*New*)	0.5968 (0.0934)	-	-	-	0.0796	0.6413	0.2111	0.0232
Gen-PF	0.6754 (0.0955)	-	-	-	0.0847	0.6440	0.2148	0.0198
Tr-PF	0.5978 (0.1233)	-	-	-0.4468 (0.2435)	0.0815	0.6533	0.2150	0.0197
PF	0.7238 (0.1024)	-	-	-	0.0793	0.6376	0.2350	0.0080
ZTP-PF	1.8634 (0.2734)	0.9487 (0.1195)	-	-	-	-	0.3119	0.0001

Table 4
MLEs with SEs of Parameters of the DUSMLII-PF Distribution and Fitted Measures for Failure Times of Thirty Electronic Devices

Model	MLEs with SEs in (.)				Fitted Measures			
	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\lambda}$	CVM	AD	KS	*p*-Value
DUSMLII-PF	13.1682 (19.6991)	0.2942 (0.0699)	1.0086 (0.3979)	-	0.0691	0.6398	0.1597	0.4283
MLII-PF(New)	-5.6148 (8.6331)	0.2674 (0.0702)	0.9254 (0.3755)	-	0.0712	0.6485	0.1599	0.4273
Kum-PF	0.1761 (7.9660)	2.6101 (118.081)	0.3387 (0.0694)	-	0.1010	0.8767	0.1787	0.2937
Gen-PF	0.4233 (0.0773)	-	-	-	0.1078	0.8802	0.2008	0.1779
MO-PF	11.3315 (12.5805)	0.2926 (0.2699)	-	-	0.2043	1.5292	0.2612	0.0334
PF-Poi	2.0797 (1.2097)	0.6441 (0.2548)	-	-	0.1312	1.0777	0.2641	0.0304
DUS-PF(New)	0.8302 (0.1685)	-	-	-	0.1159	0.9757	0.2675	0.0273
Tr-PF	0.8327 (0.2191)	-	-	-0.4317 (0.0362)	0.1119	0.9503	0.2704	0.0249
PF	0.9949 (0.1816)	-	-	-	0.0999	0.8691	0.2828	0.0165
ZTP-PF	1.8089 (0.3396)	1.2647 (0.2043)	-	-	-	-	0.3283	0.0031

Table 5
MLEs with SEs of Parameters of the DUSMLII-PF Distribution and Fitted Measures for Fatigue Fracture of Kevlar 373/Epoxy

Model	MLEs with SEs in (.)				Fitted Measures			
	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\lambda}$	CVM	AD	KS	*p*-Value
DUSMLII-PF	17.5448 (15.7567)	1.4278 (0.4180)	1.6541 (0.2636)	-	0.1076	0.6395	0.0861	0.5957
MLII-PF(New)	-13.9691 (12.4432)	1.3794 (0.4357)	1.7451 (0.2595)	-	0.1163	0.6924	0.0872	0.5803
MO-PF	0.0310 (0.0142)	1.9167 (0.2209)	-	-	0.1352	0.8035	0.0896	0.5446
ZTP-PF	7.3429 (1.2897)	1.3259 (0.1138)	-	-	0.1274	0.7518	0.1107	0.2879
PF-Poi	-7.2519 (1.3271)	1.3194 (0.1162)	-	-	0.1409	0.8234	0.1108	0.2868
W-PF	7.2308 (1.2823)	14.6137 (14.6532)	0.0901 (0.0893)	-	0.1339	0.7847	0.1110	0.2846
Gen-PF	3.7183 (0.4265)	-	-	-	0.2798	1.6096	0.1439	0.0777
Kum-PF	1.1560 (41.9093)	0.9345 (33.8788)	4.1117 (0.7810)	-	0.2700	1.5388	0.1517	0.0544
Tr-PF	0.7749 (0.0673)	-	-	0.0382 (0.0514)	0.3109	1.7823	0.2225	0.0009
OGE-PF	1.4109 (0.6392)	0.5519 (0.2923)	2.8444 (1.1105)	-	0.6044	3.3845	0.2370	0.0003
PF	0.5199 (0.0596)	-	-	-	0.2373	1.3686	0.3162	0.0000
DUS-PF(New)	0.3569 (0.0465)	-	-	-	0.2691	1.5362	0.3149	0.0000

Table 6
MLEs with SEs of Parameters of the DUSMLII-PF Distribution and Fitted Measures for Lean Body Mass of Australian Athletes

Model	MLEs with SEs in (.)				Fitted Measures			
	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\lambda}$	CVM	AD	KS	*p*-Value
DUSMLII-PF	28.2122 (24.0845)	1.3449 (0.3630)	11.6735 (1.6300)	-	0.1110	0.5774	0.0880	0.4206
MLII-PF(New)	-21.9715 (18.4399)	1.3119 (0.3831)	12.2321 (1.5890)	-	0.1183	0.6178	0.0910	0.3791
W-PF	8.2208 (1.3175)	81.9114 (50.1174)	0.1076 (0.0649)	-	0.1662	0.8230	0.0938	0.3429
MO-PF	0.0219 (0.0089)	13.2606 (1.2732)	-	-	0.1381	0.7499	0.0944	0.3354
ZTP-PF	8.3120 (1.3186)	8.8670 (0.6582)	-	-	0.1571	0.7733	0.0945	0.3342
PF-Poi	-8.2652 (1.3381)	8.8494 (0.6656)	-	-	0.1690	0.8387	0.0946	0.3330
Kum-PF	1.8559 (105.648)	3.9582 (225.32)	4.7851 (0.8091)	-	0.3013	1.6150	0.1277	0.0765
Tr-PF	4.9473 (0.3718)	-	-	-0.0346 (0.0373)	-	-	0.1937	0.0011
OGE-PF	7.4655 (2.9707)	0.7331 (0.3414)	2.7506 (0.9588)	-	0.7232	4.1595	0.2050	0.0004
Gen-PF	1.1895 (0.1189)	-	-	-	0.5261	2.9759	0.2774	0.0000
PF	3.3343 (0.3334)	-	-	-	0.2462	1.2977	0.3189	0.0000
DUS-PF(New)	2.2762 (0.2593)	-	-	-	0.2864	1.5294	0.3229	0.0000

Table 7
MLEs with SEs of Parameters of the DUSMLII-PF Distribution and Fitted Measures for Daily Deaths Count Owing to COVID-19 in China

Model	MLEs with SEs in (.)				Fitted Measures			
	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\lambda}$	CVM	AD	KS	p-Value
DUSMLII-PF	32.1375 (31.966)	0.5881 (0.1339)	1.4510 (0.2786)	-	0.0406	0.3008	0.0520	0.9940
MLII-PF(New)	-18.2305 (19.269)	0.5628 (0.1502)	1.4387 (0.2843)	-	0.0433	0.3173	0.0553	0.9877
W-PF	2.7164 (0.4623)	5.6399 (2.2049)	-	-	0.0700	0.4962	0.0642	0.9482
PF-Poi	-1.8807 (0.6820)	0.9174 (0.1290)	-	-	0.1080	0.7253	0.0790	0.8050
MO-PF	0.2715 (0.1148)	1.0906 (0.1821)	-	-	0.0757	0.5369	0.0843	0.7368
Tr-PF	0.7928 (0.0992)	-	-	0.5687 (0.2034)	0.1396	0.9095	0.1028	0.4887
ZTP-PF	3.2235 (0.4618)	1.0893 (0.1069)	-	-	0.0766	0.6053	0.1142	0.3553
Kum-PF	0.2128 (3.5868)	3.2835 (55.340)	1.2064 (0.1952)	-	0.1785	1.1510	0.1296	0.2179
PF	0.6235 (0.0767)	-	-	-	0.1750	1.1304	0.1524	0.0933
OGE-PF	8.6530 (0.0039)	0.0684 (0.0084)	0.4427 (0.0039)	-	0.2009	1.3538	0.1628	0.0606
DUS-PF(New)	0.4523 (0.0638)	-	-	-	0.2281	1.4413	0.1677	0.0488
Gen-PF	1.6176 (0.1991)	-	-	-	0.0718	0.4995	0.2114	0.0055

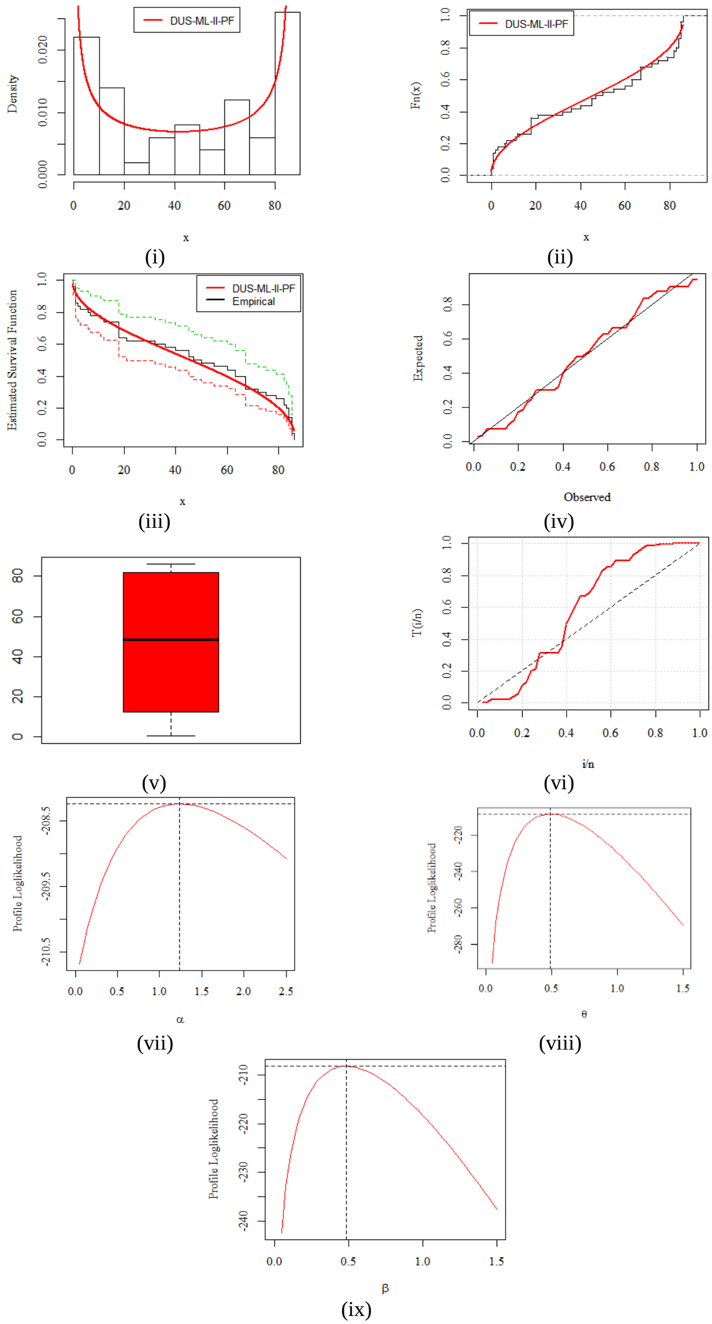


Figure 7: Fitted Plots for Times to the First Failure of Fifty Devices

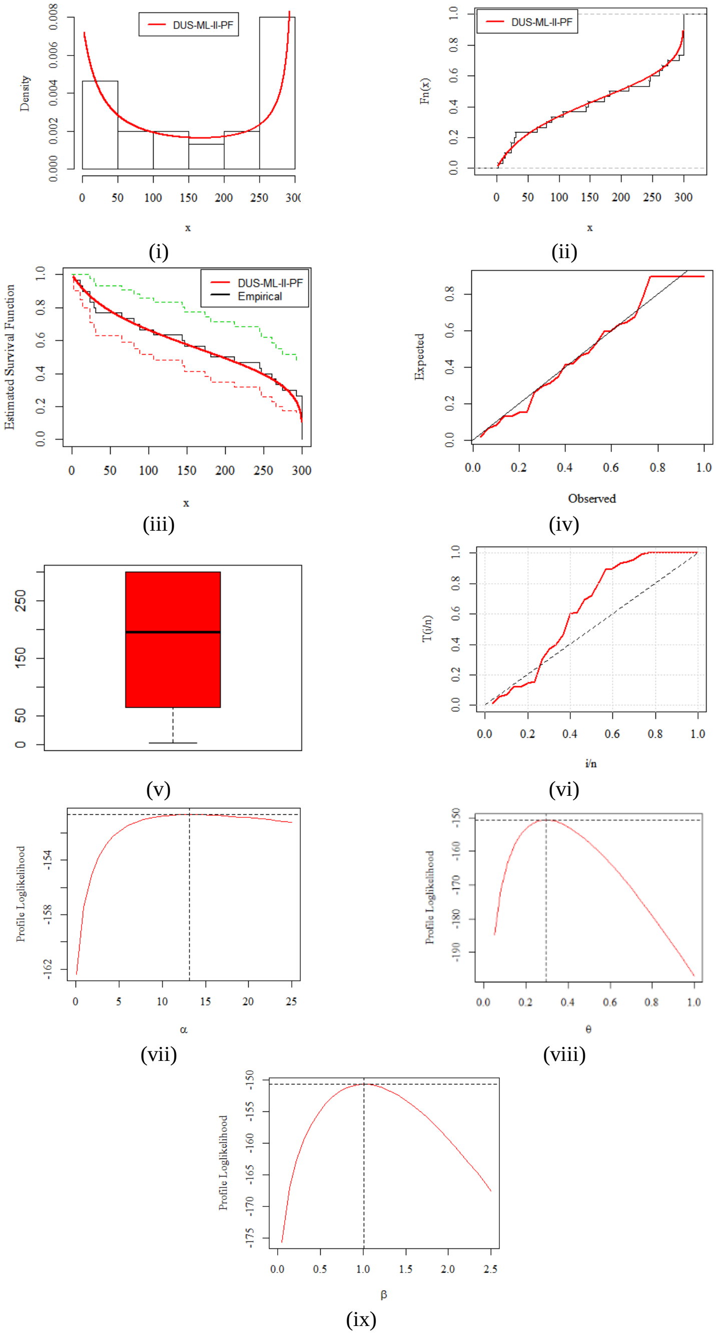


Figure 8: Fitted Plots for Failure Times of 30 Electronic Components

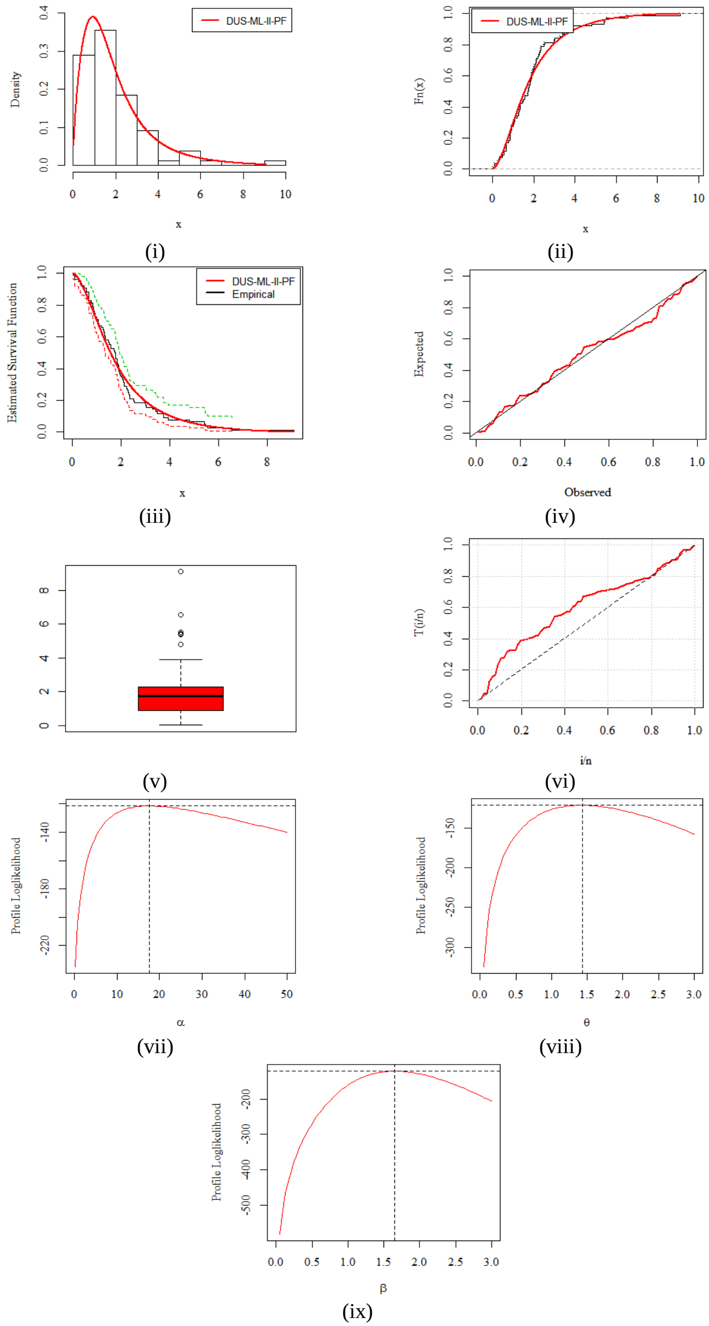
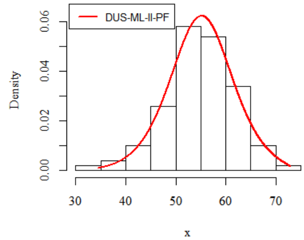
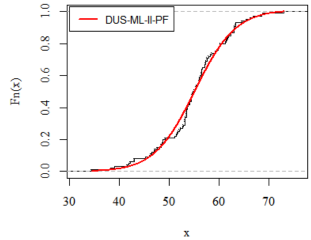


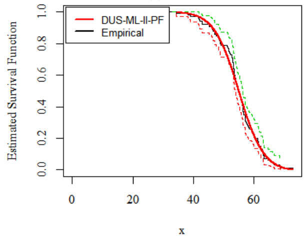
Figure 9: Fitted Plots for Fatigue Fracture of Kevlar 373/Epoxy



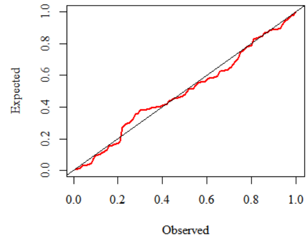
(i)



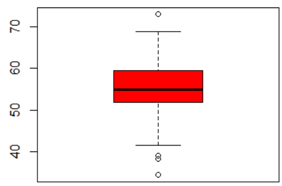
(ii)



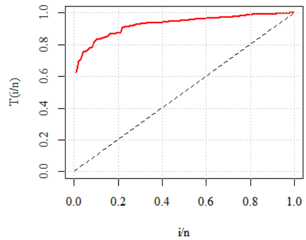
(iii)



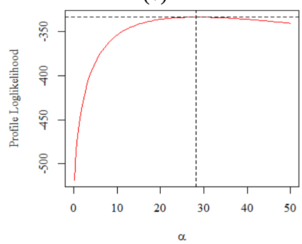
(iv)



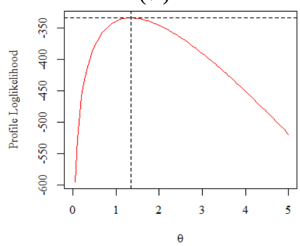
(v)



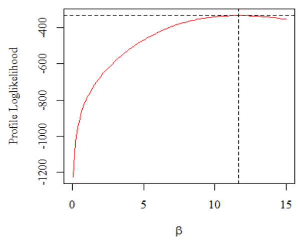
(vi)



(vii)



(viii)



(ix)

Figure 10: Fitted Plots for Body Mass of Australian Athletes

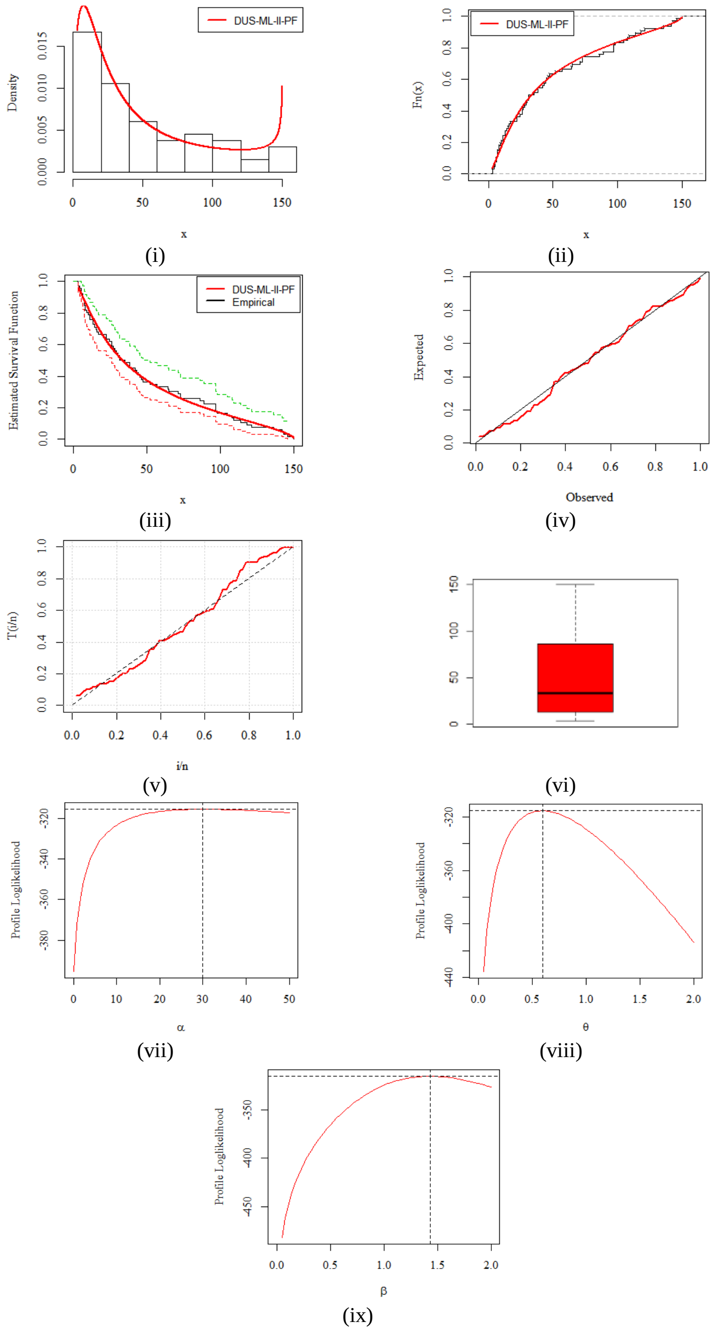


Figure 11: Fitted Plots for Daily Deaths Count Owing to COVID-19 in China

In comparison to the well-known competitors, the DUSMLII-PF distribution has the lowest CVM, AD, and KS findings with a higher p-value, as demonstrated in Tables 3 to 7. As a result, DUSMLII-PF distribution is selected as a model that is more appropriate for the subject data sets. Note that, fitted (i) pdf (ii) cdf (iii) Kaplan-Meier (iv) P-P (v) box (vi) total time on test transform, and (vii) - (ix) profile log-likelihood (α, θ, β) plots are presented in Figures 7 to 11. These plots show a better fit of the DUSMLII-PF distribution on five-lifetime data sets.

5. CONCLUSION

In this paper, a new class named the DUSMLII–G class of distributions was developed and studied. Several mathematical and reliability aspects of the new family, as well as a specific member (DUSMLII–PF) distribution, were thoroughly investigated and described. Then DUSMLII–PF distribution explored possible shapes of pdf and hrf that could be increasing, decreasing, symmetrical, or upside-down bathtub. Three well-known methods of estimation, named MLEs, LSEs, and Ps, were used with bias and MSE criteria were allocated to select a suitable one. Based on a simulation study, MLE performance was determined to be significantly better and more consistent than LSE and P, with a lower MSE in all scenarios. For practical illustration, DUSMLII–PF distribution was discussed to analyze the events, including failure times of fifty devices, failure times of thirty electronic devices, fatigue fracture of Kevlar 373/epoxy, lean body mass of Australian athletes, and daily death count owing to COVID-19 in China. Finally, we hope that the pdf and hrf's closed-form properties will persuade academics to employ the model as a forecasting tool.

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APPENDIX A
Partial Derivatives w.r.t α, θ, β of Maximum Likelihood Estimate (MLE)

$$\frac{\partial l_y(\text{ב})\big|_{MLE}}{\partial \alpha} = \frac{n}{1+\alpha} - (\beta+1)\sum_{j=1}^{n} \frac{(y/M)^{\theta}}{1+\alpha(y/M)^{\theta}} + \beta \sum_{j=1}^{n} \frac{(y/M)^{\theta}\left[1-(y/M)^{\theta}\right]^{\beta}}{[1+\alpha(y/M)^{\theta}]^{\beta+1}},$$

$$\frac{\partial l_y(\text{ב})\big|_{MLE}}{\partial \theta} = \begin{bmatrix} \frac{n}{\theta} + \sum_{j=1}^{n} log(y/M) - (\beta-1)\sum_{i=1}^{n} \frac{(y/M)^{\theta} log(y/M)}{[1-(y/M)^{\theta}]} - \\ \alpha(\beta+1)\sum_{j=1}^{n} \frac{(y/M)^{\theta} log(y/M)}{[1+\alpha(y/M)^{\theta}]} + \\ \beta(\alpha+1)\sum_{j=1}^{n} \frac{(y/M)^{\theta} log(y/M)\left[1-(y/M)^{\theta}\right]^{\beta-1}}{[1+\alpha(y/M)^{\theta}]^{\beta+1}} \end{bmatrix},$$

$$\frac{\partial l_y(\text{ב})\big|_{MLE}}{\partial \beta} = \begin{bmatrix} \frac{n}{\beta} + \sum_{j=1}^{n} log\left[1-(y/M)^{\theta}\right] - \sum_{j=1}^{n} log\left[1+\alpha(y/M)^{\theta}\right] - \\ \sum_{j=1}^{n} \left[\frac{1-(y/M)^{\theta}}{1+\alpha(y/M)^{\theta}}\right]^{\beta} log\left[\frac{1-(y/M)^{\theta}}{1+\alpha(y/M)^{\theta}}\right] \end{bmatrix}.$$

APPENDIX B
Partial Derivatives w.r.t α, θ, β of Least Square Estimation (LSE)

$$\frac{\partial l_y(\text{ב})\big|_{LS}}{\partial \alpha} = \sum_{j}^{n} \left[\frac{e^{1-\left[\left(1-(y|_{j:n}/M)^{\theta}\right)/\left(1+\alpha(y|_{j:n}/M)^{\theta}\right)\right]^{\beta}} - 1}{e-1} - \frac{j}{n+1}\right] \varrho_1\left(y|_{j:n}\right) = 0,$$

$$\frac{\partial l_y(\text{ב})\big|_{LS}}{\partial \theta} = \sum_{j}^{n} \left[\frac{e^{1-\left[\left(1-(y|_{j:n}/M)^{\theta}\right)/\left(1+\alpha(y|_{j:n}/M)^{\theta}\right)\right]^{\beta}} - 1}{e-1} - \frac{j}{n+1}\right] \varrho_2\left(y|_{j:n}\right) = 0,$$

and

$$\frac{\partial l_y(\text{ב})\big|_{LS}}{\partial \beta} = \sum_{j}^{n} \left[\frac{e^{1-\left[\left(1-(y|_{j:n}/M)^{\theta}\right)/\left(1+\alpha(y|_{j:n}/M)^{\theta}\right)\right]^{\beta}} - 1}{e-1} - \frac{j}{n+1}\right] \varrho_3\left(y|_{j:n}\right) = 0,$$

where, $\varrho_1\left(y|_{j:n}\right), \varrho_2\left(y|_{j:n}\right)$, and $\varrho_3\left(y|_{j:n}\right)$ are derived as follows:

$$\varrho_1\left(y|_{j:n}\right) = \left[\frac{\beta\left[1-\left(y|_{j:n}/M\right)^{\theta}\right]^{\beta}\left[y|_{j:n}/M\right]^{\theta}}{\left[1+\alpha\left(y|_{j:n}/M\right)^{\theta}\right]^{\beta+1}}\right],$$

$$\varrho_2(y|_{j:n}) = \left[\frac{\beta(\alpha+1)\left[1-(y|_{j:n}/M)^{\theta}\right]^{\beta-1}\left[y|_{j:n}/M\right]^{\theta} log\left[y|_{j:n}/M\right]}{\left[1+\alpha(y|_{j:n}/M)^{\theta}\right]^{\beta+1}}\right],$$

$$\varrho_3(y|_{j:n}) = \left[\left[\frac{1-(y|_{j:n}/M)^{\theta}}{1+\alpha(y|_{j:n}/M)^{\theta}}\right]^{\beta} log\left[\frac{1+\alpha(y|_{j:n}/M)^{\theta}}{1-(y|_{j:n}/M)^{\theta}}\right]\right].$$

APPENDIX C
The following Data Set have been Used in Application Section

Data Set 1:

0.1, 0.2, 1.0, 1.0, 1.0, 1.0, 1.0, 2.0, 3.0, 6.0, 7.0, 11.0, 12.0, 18.0, 18.0, 18.0, 18.0, 18.0, 21.0, 32.0, 36.0, 40.0, 45.0, 45.0, 47.0, 50.0, 55.0, 60.0, 63.0, 63.0, 67.0, 67.0, 67.0, 67.0, 72.0, 75.0, 79.0, 82.0, 82.0, 83.0, 84.0, 84.0, 84.0, 85.0, 85.0, 85.0, 85.0, 85.0, 86.0, 86.0.

Data Set 2:

275, 13, 147, 23, 181, 30, 65, 10, 300, 173, 106, 300, 300, 212, 300, 300, 300, 2, 261, 293, 88, 247, 28, 143, 300, 23, 300, 80, 245, 266.

Data Set 3:

0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.2570, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.0960.

Data Set 4:

4.37, 3.87, 4.00, 4.03, 3.50, 4.08, 2.25, 4.70, 1.73, 4.93, 1.73, 4.62, 3.43, 4.25, 1.68, 3.92, 3.68, 3.10, 4.03, 1.77, 4.08, 1.75, 3.20, 1.85, 4.62, 1.97, 4.50, 3.92, 4.35, 2.33, 3.83, 1.88, 4.60, 1.80, 4.73, 1.77, 4.57, 1.85, 3.52, 4.00, 3.70, 3.72, 4.25, 3.58, 3.80, 3.77, 3.75, 2.50, 4.50, 4.10, 3.70, 3.80, 3.43, 4.00, 2.27, 4.40, 4.05, 4.25, 3.33, 2.00, 4.33, 2.93, 4.58, 1.90, 3.58, 3.73, 3.73, 1.82, 4.63, 3.50, 4.00, 3.67, 1.67, 4.60, 1.67, 4.00, 1.80, 4.42, 1.90, 4.63, 2.93, 3.50, 1.97, 4.28, 1.83, 4.13, 1.83, 4.65, 4.20, 3.93, 4.33, 1.83, 4.53, 2.03, 4.18, 4.43, 4.07, 4.13, 3.95, 4.10, 2.72, 4.58, 1.90, 4.50, 1.95, 4.83, 4.12.

Data Set 5:

8, 16, 15, 24, 26, 26, 38, 43, 46, 45, 57, 64, 65, 73, 73, 86, 89, 97, 108, 97, 146, 121, 143, 142, 105, 98, 136, 114, 118, 109, 97, 150, 71, 52, 29, 44, 47, 35, 42, 31, 38, 31, 30, 28, 27, 22, 17, 22, 11, 7, 13, 10, 14, 13, 11, 8, 3, 7, 6, 9, 7, 4, 6, 5, 3, 5.