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LOMAX–POWER RAYLEIGH (T-X) DISTRIBUTION, STRUCTURAL PROPERTIES AND APPLICATIONS IN BIOLOGICAL SCIENCES

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ABSTRACT

The main aim to formulate this research article is to introduce a modified distribution named as Lomax-Power Rayleigh Distribution LPRD. The Lomax distribution and Power Rayleigh Distribution are joined together using "Transformed –Transformer Family (T-X)" to get the LPRD. The structural properties of LPRD which includes survival rate hazard rate function, cumulative hazard rate, mills ratio are derived. Apart from these properties, mean, median, mode, harmonic mean and variance are obtained. Expressions for moment generating function and characteristic function are framed. The parameters of the distribution are estimated by maximum likelihood estimator method. At last Shanon's Entropy and Order statistics of LPRD are derived. The LPRD resulted better fit when compared to some other distributions by inserting two bioscience data sets.

KEY WORDS

Lomax distribution, Power Rayleigh distribution, Transformed–transformer (T-X) Family, Survival function, hazard rate function, moments.

1. INTRODUCTION

Rayleigh distribution which was introduced by Lord Rayleigh in (1880) has a wide use in many fields of research. This model was basically derived in association with a problem in acoustics. It is considered to be the sophisticated one for the formation of new models by different statistical techniques using it as a parent distribution. Various authors have worked on this model due to its numerous flexible properties to generate new models. This flexibility of distribution leads it to many modifications to get more reliable and sophisticated models. Howlader and Hossain (1995), Voda (2005), Ahmad et al. (2014), Kundu and Raqab (2005), Merovci (2013), Ahmad et al. (2017), Gazal and Hasaballah (2017), Ajami and Jahanshahi (2017), Ateeq et al. (2019) and Sofi et al. (2019). Bhat et al. (2020), has introduced a new modified version of Rayleigh distribution named as Power Rayleigh distribution and Alzaatreh et al. (2013) suggested a general method that permits for many existing continuous distributions as the generator. Mathew et al. (2020), Nagarjuna et al. (2020), ZeinEldin et al. (2020), Almetwally et al. (2020) and Ijaz et al. (2020) recently, worked on Lomax distribution by considering different problems.

Suppose $F(x)$ and $p(t)$ are the cumulative distribution function of any random variable x and probability density function of a non-negative continuous random variable t respectively. Then the cumulative distribution function of the generalized distribution defined by Alzaatreh is given by;

$$G(x) = \int_0^{-log[1-F(x)]} p(t)dt \tag{1.1}$$

The family of distribution defined in equation (1) is called "Transformed-Transformer" family or "T-X distribution family" suggested by Alzaatreh et al. (2013). The probability density function (pdf) to the cumulative distribution function (cdf) in (1) is given by;

$$g(x) = \frac{f(x)}{1-F(x)} p\{-log(1-F(x)\} \tag{1.2}$$

If a random variable 'T' follows Lomax Distribution with parameter θ and λ, we get;

$$p(t) = \theta\lambda(1+\lambda t)^{-(\theta+1)}; t > 0; \theta, \lambda > 0 \tag{1.3}$$

where θ, λ are shape and scale parameter respectively.

Using equation (2), we will get the PDF of Lomax-X distribution as under;

$$g(x) = \frac{f(x)}{1-F(x)}\left[\theta\lambda\left\{1+\lambda\left(-log\big(1-F(x)\big)\right)\right\}^{-(\theta+1)}\right] \tag{1.4}$$

In this paper we introduce the Lomax–X family where X is the random variable following Power Rayleigh Distribution.

2. LOMAX-POWER RAYLEIGH DISTRIBUTION T-X FAMILY

Suppose X is a non-negative random variable following Power Rayleigh Distribution with probability density function $f(x)$ and cumulative distribution function $F(x)$

$$f(x) = \frac{\alpha}{\beta^2} x^{2\alpha-1} e^{\frac{-x^{2\alpha}}{2\beta^2}}; \alpha > 0, \beta > 0 \tag{2.1}$$

$$F(x) = 1 - e^{\frac{-x^{2\alpha}}{2\beta^2}} \tag{2.2}$$

Substituting equation (2.1) and (2.2) in equation (1. 4) then the pdf of Lomax-Power Rayleigh Distribution (LPRD) is given by;

$$g(x) = \frac{\frac{\alpha}{\beta^2} x^{2\alpha-1} e^{e^{\frac{-x^{2\alpha}}{2\beta^2}}}}{1-\left(1-e^{e^{\frac{-x^{2\alpha}}{2\beta^2}}}\right)}\left[\theta\lambda\left\{1+\lambda\left(-log\left(1-\left(1-e^{e^{\frac{-x^{2\alpha}}{2\beta^2}}}\right)\right)\right)\right\}^{-(\theta+1)}\right]$$

$$g(x) = \frac{\alpha}{\beta^2}(\theta\lambda)x^{2\alpha-1}\left\{1+\frac{\lambda x^{2\alpha}}{2\beta^2}\right\}^{-(\theta+1)}; x > 0, \alpha, \beta, \theta, \lambda > 0 \tag{2.3}$$

where n and α are shape, β and θ are scale parameters.

The above equation is said to be Lomax-Power Rayleigh Distribution and is denoted by $LPRD(n, \alpha, \beta, \theta)$.

The corresponding cdf of $LPRD(n, \alpha, \beta, \theta)$ is

$$G(x) = 1 - \frac{(2\beta^2)^\theta}{(2\beta^2 + \lambda x^{2\alpha})^\theta}. \tag{2.4}$$

3. STRUCTURAL PROPERTIES OF LOMAX-POWER RAYLEIGH DISTRIBUTION

3.1 Survival Function

It deals with the probability of survival time t in living organisms and functional time in engineering and electronics. The survival function of LPRD is given as

$$S(x) = 1 - \mathrm{G(x)} = 1 - \left(1 - \frac{(2\beta^2)^\theta}{(2\beta^2 + \lambda x^{2\alpha})^\theta}\right) = \frac{(2\beta^2)^\theta}{(2\beta^2 + \lambda x^{2\alpha})^\theta} \tag{3.1}$$

3.2 Hazard Rate Function

The probability that a component will fail or die in a particular interval of time t. The hazard rate function of LPRD is given as

$$h(x) = \frac{g(x)}{S(x)} = \frac{\left[\frac{\alpha}{\beta^2}(\theta\lambda)x^{2\alpha-1}\left\{1 + \frac{\lambda x^{2\alpha}}{2\beta^2}\right\}^{-(\theta+1)}\right]}{\left[\frac{(2\beta^2)^\theta}{(2\beta^2 + \lambda x^{2\alpha})^\theta}\right]} = 2\alpha\lambda\theta\left(\frac{2\beta^2}{x^{2\alpha}} + \lambda\right)^{-1} \tag{3.2}$$

3.3 Cumulative Hazard Rate

The cumulative hazard rate function of LPRD is given by

$$CHR = -logS(x) = -log\left[\frac{(2\beta^2)^\theta}{(2\beta^2 + \lambda x^{2\alpha})^\theta}\right] = \theta log\lambda x^{2\alpha} \tag{3.3}$$

3.4 Mills Ratio

The Mills ratio of LPRD is

$$MR = \frac{G(x)}{1 - G(x)} = \frac{\left[1 - \frac{(2\beta^2)^\theta}{(2\beta^2 + \lambda x^{2\alpha})^\theta}\right]}{\left[\frac{(2\beta^2)^\theta}{(2\beta^2 + \lambda x^{2\alpha})^\theta}\right]} = \left[\frac{(2\beta^2 + \lambda x^{2\alpha})^\theta}{(2\beta^2)^\theta} - 1\right] \tag{3.4}$$

Graphical representation of PDF, CDF, Hazard Rate Function of LPRD for different values of parameters

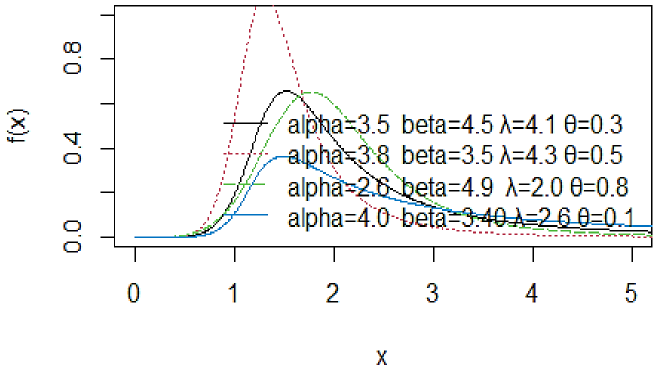


Figure 1: Density Plots of LPRD (t-x) for Different Values of Shape and Scale Parameters

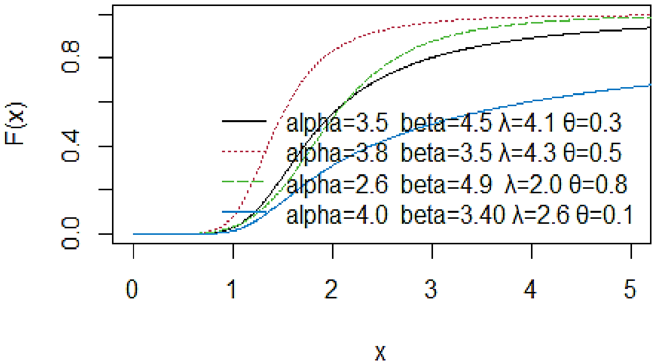


Figure 2: Distribution Function Plots of (LPRD (t-x)) for Different Values of Shape and Scale Parameters

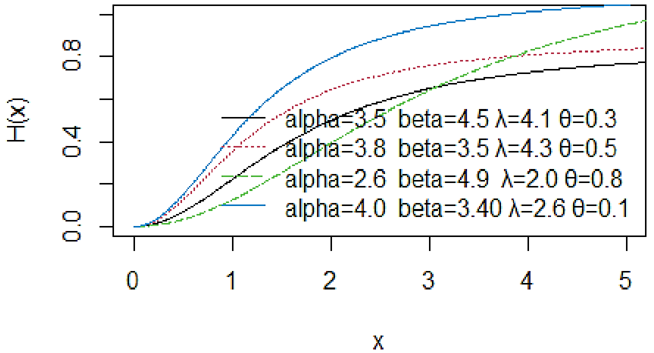


Figure 3: Survival Rate Function Plots of (LPRD (t-x)) for Different Values of Shape and Scale Parameters

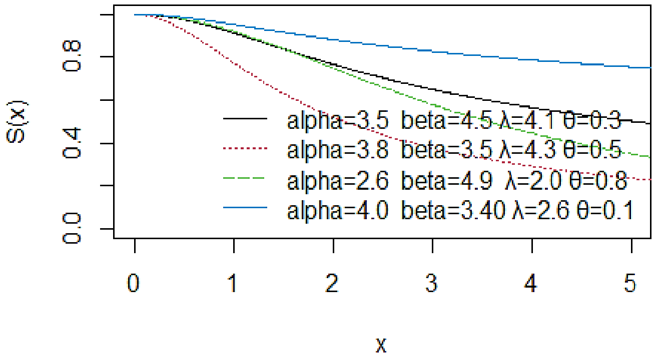


Figure 4: Hazard Rate Function Plots of (LPRD (t-x)) for Different Values of Shape and Scale Parameters

3.5 Moments

Moments are very important statistical measures of a probability model. In this section, raw moments of LPRD are derived. If ‘x’ is a random variable following LPRD then

$$\mu_r' = E(x) = \int_0^\infty x^r g(x)dx$$

$$= \int_0^\infty x^r \frac{\alpha}{\beta^2}(\theta\lambda)x^{2\alpha-1}\left\{1+\frac{\lambda x^{2\alpha}}{2\beta^2}\right\}^{-(\theta+1)} dx$$

Solving the above equation by making the substitution of $\left\{\left(\frac{\lambda x^{2\alpha}}{2\beta^2}\right)=m\right\}$, we get;

$$\mu_r' = \left(\frac{2}{\lambda}\right)^{\frac{r}{2\alpha}} \beta^{\frac{r}{\alpha}}\theta\left[\frac{\Gamma\left(\theta-\frac{\alpha r}{2}\right)\Gamma\left(1+\frac{\alpha r}{2}\right)}{\Gamma(1+\theta)}\right] \tag{3.5.1}$$

Putting $r = 1,2,3,4$ we get the first four moments of LPRD.

$$\mu_1' = \left(\frac{2}{\lambda}\right)^{\frac{1}{2\alpha}} \beta^{\frac{1}{\alpha}}\theta\left[\frac{\Gamma\left(\theta-\frac{\alpha}{2}\right)\Gamma\left(1+\frac{\alpha}{2}\right)}{\Gamma(1+\theta)}\right] \tag{3.5.2}$$

$$\mu_2' = \left(\frac{2}{\lambda}\right)^{\frac{1}{\alpha}} \beta^{\frac{2}{\alpha}}\theta\left[\frac{\Gamma(\theta-\alpha)\Gamma(1+\alpha)}{\Gamma(1+\theta)}\right] \tag{3.5.3}$$

$$\mu_3' = \left(\frac{2}{\lambda}\right)^{\frac{3}{2\alpha}} \beta^{\frac{3}{\alpha}}\theta\left[\frac{\Gamma\left(\theta-\frac{3\alpha}{2}\right)\Gamma\left(1+\frac{3\alpha}{2}\right)}{\Gamma(1+\theta)}\right] \tag{3.5.4}$$

$$\mu_4' = \left(\frac{2}{\lambda}\right)^{\frac{2}{\alpha}} \beta^{\frac{4}{\alpha}}\theta\left[\frac{\Gamma(\theta-2\alpha)\Gamma(1+2\alpha)}{\Gamma(1+\theta)}\right] \tag{3.5.5}$$

μ_1' Represents the mean of LPRD and μ_2 represents the variance of LPRD. Therefore

$$\mu_2 = \mu_2{}' - (\mu_1{}')^2$$

$$= \left(\frac{2}{\lambda}\right)^{\frac{1}{\alpha}} \beta^{\frac{2}{\alpha}}\theta\left[\frac{\Gamma(\theta-\alpha)\Gamma(1+\alpha)}{\Gamma(1+\theta)}\right] - \left\{\left(\frac{2}{\lambda}\right)^{\frac{1}{2\alpha}} \beta^{\frac{1}{\alpha}}\theta\left[\frac{\Gamma\left(\theta-\frac{\alpha}{2}\right)\Gamma\left(1+\frac{\alpha}{2}\right)}{\Gamma(1+\theta)}\right]\right\}^2$$

$$= \left(\frac{2}{\lambda}\right)^{\frac{1}{\alpha}} \beta^{\frac{2}{\alpha}} \frac{\theta}{\left(\Gamma(1+\theta)\right)^2}\left[\Gamma(\theta-\alpha)\Gamma(1+\alpha)\Gamma(1+\theta) - \theta\left(\Gamma\left(\theta-\frac{\alpha}{2}\right)\right)^2\left(\Gamma\left(1+\frac{\alpha}{2}\right)\right)^2\right]. \tag{3.5.6}$$

3.6 Harmonic Mean

The harmonic mean of LPRD is given by

$$\frac{1}{H} = E\left(\frac{1}{x}\right)$$

$$= \int_0^\infty \frac{1}{x} g(x)dx$$

$$= \int_0^\infty \frac{1}{x}\frac{\alpha}{\beta^2}(\theta\lambda)x^{2\alpha-1}\left\{1+\frac{\lambda x^{2\alpha}}{2\beta^2}\right\}^{-(\theta+1)} dx$$

$$= \frac{\theta}{\beta^{\frac{1}{\alpha}}(2\lambda)^{\frac{1}{2\alpha}}}\left[\frac{\Gamma\left(1-\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha}{2}+\theta\right)}{\Gamma(1+\theta)}\right]. \tag{3.6}$$

3.7 Mode and Median of LPRD

The mode of LPRD is derived as under

$$\log g(x) = log\left[\lambda x^{2\alpha-1}\left(\frac{\alpha\theta}{\beta^2}\right)\left(1+\frac{\lambda x^{2\alpha}}{2\beta^2}\right)^{-(\theta+1)}\right] \tag{3.7.1}$$

Substituting value of g(x) from equation (2.3) in equation (3.7), we get;

$$= log\left(\frac{\lambda\alpha\theta}{\beta^2}\right) + (2\alpha-1)logx - (\theta+1)log\left(1+\frac{\lambda x^{2\alpha}}{2\beta^2}\right) \tag{3.7.2}$$

Differentiating the above equation w.r.t x, then equating to zero to get the mode of LPRD

$$\frac{\partial Log\ L}{\partial x} = \frac{(2\alpha-1)}{x} - \frac{2\lambda(\theta+1)x^{2\alpha-1}}{(2\beta^2+\lambda x^{2\alpha})} \tag{3.7.3}$$

$$\frac{(2\alpha-1)}{2\lambda(\theta+1)} - \frac{x^{2\alpha}}{(2\beta^2+\lambda x^{2\alpha})} = 0$$

$$M_o = x = \left(\frac{2\beta^2(2\alpha-1)}{2\lambda\theta-2\alpha+3\lambda}\right)^{1/2\alpha} \tag{3.7.4}$$

The median of LPRD is defined as under;

$$M_d = \frac{1}{3}M_o + \frac{2}{3}\mu_1' \tag{3.7.5}$$

Using equations (3.5.2) and (3.7.4) in equation (3.7.5), we get the mode of LPRD as

$$M_d = \frac{1}{3}\left(\frac{2\beta^2(2\alpha-1)}{2\lambda\theta-2\alpha+3\lambda}\right)^{1/2\alpha} + \frac{2}{3}\left(\frac{2}{\lambda}\right)^{\frac{1}{2\alpha}}\beta^{\frac{1}{\alpha}}\theta\left[\frac{\Gamma\left(\theta-\frac{\alpha}{2}\right)\Gamma\left(1+\frac{\alpha}{2}\right)}{\Gamma(1+\theta)}\right]. \tag{3.7.6}$$

4. MOMENT GENERATING FUNCTION AND CHARACTERISTIC FUNCTION

Here moment generating function and characteristic function of LPRD are derived.

4.1 Moment Generating Function

Suppose "x" is a random variable following LPRD, then MGF of 'x' is defined as

$$\begin{aligned} M_x(t) &= Ee^{(tx)} \\ &= \int_0^\infty e^{(tx)} g(x)\mathrm{dx} \\ &= \int_0^\infty e^{(tx)}\left[\frac{\alpha}{\beta^2}(\theta\lambda)x^{2\alpha-1}\left\{1+\frac{\lambda x^{2\alpha}}{2\beta^2}\right\}^{-(\theta+1)}\right]\mathrm{dx} \\ &= \int_0^\infty \left\{1+tx+\frac{(tx)^2}{2!}+\frac{(tx)^3}{3!}---\right\}\left[\frac{\alpha}{\beta^2}(\theta\lambda)x^{2\alpha-1}\left\{1+\frac{\lambda x^{2\alpha}}{2\beta^2}\right\}^{-(\theta+1)}\right]dx \\ &= \sum_{r=0}^{n}\frac{(t)^r}{r!}\mu_1' \end{aligned}$$

$$M_x(t) = \sum_{r=0}^{n}\frac{(t)^r}{r!}\left(\frac{2}{\lambda}\right)^{\frac{1}{2\alpha}}\beta^{\frac{1}{\alpha}}\theta\left[\frac{\Gamma\left(\theta-\frac{\alpha}{2}\right)\Gamma\left(1+\frac{\alpha}{2}\right)}{\Gamma(1+\theta)}\right]. \tag{4.1}$$

4.2 Characteristic Function

Suppose "x" is a random variable following LPRD, then CF of 'x' is defined as

$$\phi_x(t) = Ee^{(itx)} = \sum_{r=0}^{n}\frac{(it)^r}{r!}\mu_1'. \tag{4.2}$$

5. MAXIMUM LIKELIHOOD ESTIMATION

To estimate the parameters of a statistical distribution, maximum likelihood estimator is the most trendy method. Let $x_1, x_2, x_3, x_4, \dots\dots\dots\dots, x_n$ follows LPRD then the maximum likelihood estimator L of LPRD is defined as;

$$L = \prod_{i=1}^{n} g(x)$$
$$= \prod_{i=1}^{n} \left[\frac{\alpha}{\beta^2} (\theta\lambda) x^{2\alpha-1} \left\{ 1 + \frac{\lambda x^{2\alpha}}{2\beta^2} \right\}^{-(\theta+1)} \right]$$
$$L = \left(\frac{\alpha\theta\lambda}{\beta^2} \right)^n \prod_{i=1}^{n} {x_i}^{2\alpha-1} \left(1 + \frac{\lambda {x_i}^{2\alpha}}{2\beta^2} \right)^{-(\theta+1)} \tag{5.1}$$

Applying Log on both sides of Likelihood Function (5.1) to get the Log Likelihood function of LPRD as

$$Log\ L = Log \left[\left(\frac{\alpha\theta\lambda}{\beta^2} \right)^n \prod_{i=1}^{n} {x_i}^{2\alpha-1} \left(1 + \frac{\lambda {x_i}^{2\alpha}}{2\beta^2} \right)^{-(\theta+1)} \right] \tag{5.2}$$
$$= Log \left(\frac{\alpha\theta\lambda}{\beta^2} \right)^n + Log \left[\prod_{i=1}^{n} {x_i}^{2\alpha-1} + \left(1 + \frac{\lambda {x_i}^{2\alpha}}{2\beta^2} \right)^{-(\theta+1)} \right]$$
$$= nlog(\alpha\theta\lambda) - 2nlog\beta + \sum_{i=1}^{n} Log\ {x_i}^{2\alpha-1} + \sum_{i=1}^{n} Log \left(\frac{2\beta^2 + \lambda {x_i}^{2\alpha}}{2\beta^2} \right)^{-(\theta+1)}$$
$$= nlog\alpha + nlog\ \theta + nlog\ \lambda - 2nlog$$
$$+ (2\alpha - 1) \sum_{i=1}^{n} Log x_i - (\theta + 1) \sum_{i=1}^{n} Log \left(\frac{2\beta^2 + \lambda {x_i}^{2\alpha}}{2\beta^2} \right)$$
$$= nlog\alpha + nlog\ \theta + nlog\ \lambda - 2nlog\beta$$
$$+ (2\alpha - 1) \sum_{i=1}^{n} Log x_i - (\theta + 1) \sum_{i=1}^{n} Log\ (2\beta^2 + \lambda {x_i}^{2\alpha})$$
$$+ (\theta + 1) \sum_{i=1}^{n} Log\ (2\beta^2)\ (5.3)$$

Differentiating equation (5.3) partially w.r.t α we get;

$$\frac{\partial Log\ L}{\partial \alpha} = \frac{n}{\alpha} + 2 \sum_{i=1}^{n} Log\ (x_i) - 2(\theta + 1)\lambda \sum_{i=1}^{n} Log\ (x_i) \tag{5.4}$$

Equating $\frac{\partial Log\ L}{\partial \alpha} = 0$ in equation (5.4) to get the estimate of α;

$$\frac{n}{\alpha} = 2(\theta + 1)\lambda \sum_{i=1}^{n} Log\ (x_i) - 2 \sum_{i=1}^{n} Log\ (x_i)$$

$$\frac{n}{\alpha} = 2 \sum_{i=1}^{n} Log\ (x_i) [(\theta + 1)\lambda - 1]$$

$$\hat{\alpha} = \frac{n}{2\sum_{i=1}^{n} Log\,(x_i)[(\theta+1)\lambda - 1]} \tag{5.5}$$

Differentiating equation (5.3) partially w.r.t θ we get;

$$\frac{\partial Log\ L}{\partial\theta} = \frac{n}{\theta} - 2\alpha\lambda\sum_{i=1}^{n} Log\,(x_i) \tag{5.6}$$

Equating $\frac{\partial Log\ L}{\partial\theta} = 0$ in equation (5.6) to get the estimate of θ;

$$0 = \frac{n}{\theta} - 2\alpha\lambda\sum_{i=1}^{n} Log\,(x_i)$$

$$\frac{n}{\theta} = 2\alpha\lambda\sum_{i=1}^{n} Log\,(x_i)$$

$$\hat{\theta} = \frac{n}{2\alpha\lambda\sum_{i=1}^{n} Log\,(x_i)} \tag{5.7}$$

Differentiating equation (5.3) partially w.r.t λ we get;

$$\frac{\partial Log\ L}{\partial\lambda} = \frac{n}{\lambda} - 2\alpha(\theta+1)\sum_{i=1}^{n} Log\,(x_i) \tag{5.8}$$

Equating $\frac{\partial Log\ L}{\partial\lambda} = 0$in equation (5.8) to get the estimate of λ;

$$0 = \frac{n}{\lambda} - 2\alpha(\theta+1)\sum_{i=1}^{n} Log\,(x_i)$$

$$\frac{n}{\lambda} = 2\alpha(\theta+1)\sum_{i=1}^{n} Log\,(x_i)$$

$$\hat{\lambda} = \frac{n}{2\alpha(\theta+1)\sum_{i=1}^{n} Log\,(x_i)}. \tag{5.9}$$

6. SHANON'S ENTROPY

If $x_1, x_2, x_3, x_4, \ldots \ldots \ldots \ldots, x_n$ follows LPRD then Shanon's Entropy of LPRD is defined as;

$$H(x) = -E[Log g(x)] = -E\left[Log\left\{\frac{\alpha}{\beta^2}(\theta\lambda)x^{2\alpha-1}\left(1+\frac{\lambda x^{2\alpha}}{2\beta^2}\right)^{-(\theta+1)}\right\}\right]$$

$$= -E\left(Log\frac{\alpha\theta\lambda}{\beta^2}\right) - (2\alpha-1)E(Log x) - (\theta+1)E\left[Log\left(1+\frac{\lambda x^{2\alpha}}{2\beta^2}\right)^{-(\theta+1)}\right] \tag{6.1}$$

now

$$E(Logx) = \int_0^\infty (log\ x)\, g(x)dx$$
$$= \int_0^\infty (log\ x)\left[\frac{\alpha}{\beta^2}(\theta\lambda)x^{2\alpha-1}\left\{1+\frac{\lambda x^{2\alpha}}{2\beta^2}\right\}^{-(\theta+1)}\right]dx$$
$$= \frac{\beta^2}{\alpha\lambda(\theta-1)} \tag{6.2}$$

$$E\left(Log\left(1+\frac{\lambda x^{2\alpha}}{2\beta^2}\right)\right) = \int_0^\infty Log\left(1+\frac{\lambda x^{2\alpha}}{2\beta^2}\right)\left[\frac{\alpha}{\beta^2}(\theta\lambda)x^{2\alpha-1}\left(1+\frac{\lambda x^{2\alpha}}{2\beta^2}\right)^{-(\theta+1)}\right]dx$$

Making substitution and solving the above integral we get;

$$E\left(Log\left(1+\frac{\lambda x^{2\alpha}}{2\beta^2}\right)\right) = 0 \tag{6.3}$$

Substituting the values from equation (6.2) and (6.3) in equation (6.1), we get the Shanon's entropy as

$$H(x) = -(2\alpha-1)\frac{\beta^2}{\alpha\lambda(\theta-1)}. \tag{6.4}$$

7. ORDER STATISTICS

Let $x_1, x_2, x_3, x_4, \dots \dots \dots \dots, x_n$ follows LPRD then k^{th} of LPRD is defined as;

$$f_{x(k)}(x) = \frac{n!}{(k-1)!\,(n-k)!}[G(x)]^{k-1}[1-G(x)]^{n-k}g(x) \tag{7.1}$$

Substituting the value of $g(x)$ and $G(x)$ from (2.3) and (2.4) in (7.1) we get;

$$f_{x(k)}(x) = \frac{n!}{(k-1)!\,(n-k)!}\left[1-\frac{(2\beta^2)^\theta}{(2\beta^2+\lambda x^{2\alpha})^\theta}\right]^{k-1}\left[1 - \left\{1-\frac{(2\beta^2)^\theta}{(2\beta^2+\lambda x^{2\alpha})^\theta}\right\}\right]^{n-k}\left[\frac{\alpha}{\beta^2}(\theta\lambda)x^{2\alpha-1}\left\{1+\frac{\lambda x^{2\alpha}}{2\beta^2}\right\}^{-(\theta+1)}\right] \tag{7.2}$$

Put $k = 1$ in equation (7.2) we get the 1st order statistics of LPRD as

$$f_{x(1)}(x) = n\left[\frac{(2\beta^2)^\theta}{(2\beta^2+\lambda x^{2\alpha})^\theta}\right]^{n-1} \tag{7.3}$$

Put $k = n$ in equation (7.2) we get the nth order statistics of LPRD as

$$f_{x(n)}(x) = \frac{n!}{(n-1)!\,(n-n)!}\left[1 - \frac{(2\beta^2)^\theta}{(2\beta^2+\lambda x^{2\alpha})^\theta}\right]^{n-1}\left[1 - \left\{1 - \frac{(2\beta^2)^\theta}{(2\beta^2+\lambda x^{2\alpha})^\theta}\right\}\right]^{n-n}\left[\frac{\alpha}{\beta^2}(\theta\lambda)x^{2\alpha-1}\left\{1+\frac{\lambda x^{2\alpha}}{2\beta^2}\right\}^{-(\theta+1)}\right]$$

$$f_{x(n)}(x) = n\left[1 - \frac{(2\beta^2)^\theta}{(2\beta^2+\lambda x^{2\alpha})^\theta}\right]^{n-1}\left[\frac{\alpha}{\beta^2}(\theta\lambda)x^{2\alpha-1}\left\{1+\frac{\lambda x^{2\alpha}}{2\beta^2}\right\}^{-(\theta+1)}\right]. \quad (7.4)$$

DATA ANALYSIS

This section is devoted to demonstrate the importance, flexibility and appropriateness of the PRD by means of two real data sets. For illustrating the significance and the potentiality of the proposed probability model, we compare the goodness-of-fit of the proposed model with the lifetime models.

For collation purposes the various criterions of goodness-of-fit such as AIC, BIC, AICC and HQIC. The statistic with smaller value along with large p-value is considered to be the best fit. For analysis purposes, the numerical results are obtained using R software.

Data Set 1:

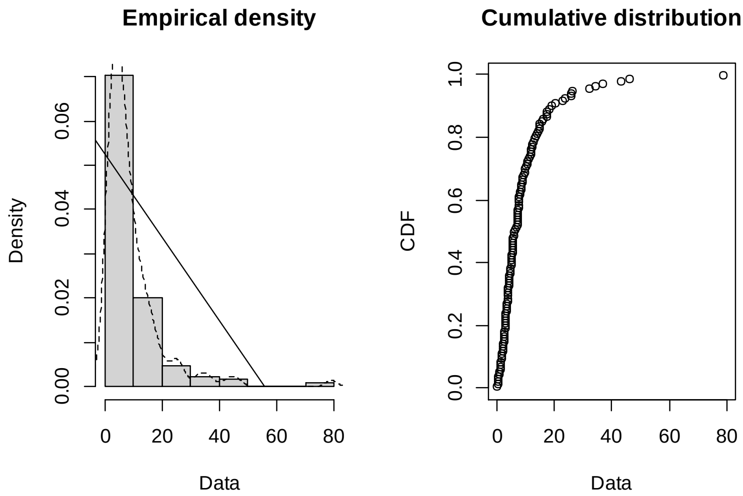
The data set represents the remission times (in months) of a random sample of 128 bladder cancer patients. The observations are follows

0.08, 2.09, 2.73, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.22, 3.52, 4.98, 6.99, 9.02, 13.29, 0.40, 2.26,3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 15.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.93, 8.65, 12.63, 22.69

The summary of the data is given in Table 1.

Table 1
Summary of Data

	Min	Mean	Median	Variance	1st Qu.	3rd Qu.	Max
DATA 1	0.080	9.311	6.050	112.178	3.295	11.678	79.050



Plot of empirical Cdf and LPR (t-x) distribution Cdf for the first data set

Estimated sd	Estimated Skewness	Estimated kurtosis
10.59141	3.359174	19.24389

Correlation Matrix:

	α	**B**	***λ***	***θ***
α	1.000000000	-0.02302501	-0.007090258	-0.81598127
β	-0.023025009	1.00000000	0.998673827	0.05407217
λ	-0.007090258	0.99867383	1.000000000	0.01444231
θ	-0.815981267	0.05407217	0.014442307	1.00000000

Cullen and Frey Graph

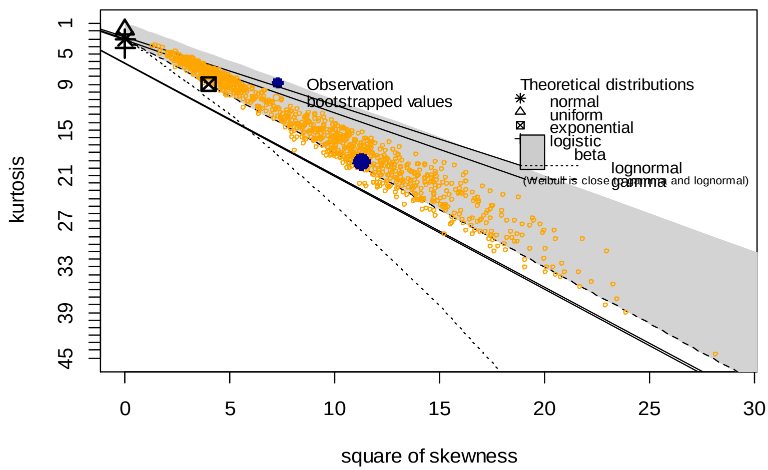


Table 2
Goodness-of-Fit Statistics for Data Set 1st

Distribution	$-log\ l$	AIC	BIC	AICC	HQIC	$K-S$	p-value
RD	451.45	957.62	960.44	957.65	958.77	0.39	0.396
WRD	452.11	959.77	965.42	959.87	962.06	0.40	0.287
AD	413.47	828.94	831.76	829.14	836.38	0.16	0.025
WAD	489.35	980.71	983.53	980.91	988.15	0.48	0.349
IAD	499.23	1007.63	1010.45	1007.83	1015.07	0.49	0.256
MARD	400.63	803.41	811.88	803.61	806.84	0.06	0.004
LPRD(t-x)	**399.79**	**800.05**	**811.33**	**800.26**	**801.50**	**0.04**	**0.859**

Table 3
The Estimation of Parameters for the First Data Set

Distribution	Estimated Parameters			
	α	**β**	**λ**	**θ**
RD	9.9490195	-	-	-
WRD	-	0.05007203	-	-0.1244486
AD	0.107404257	-	-	-
WAD	0.214803804	-	-	-
IAD	2.4395558	-	-	-
MARD	24.42251460	0.11388888	0.14964964	-
LPRD(t-x)	**3.879059e-01**	**1.440249e+03**	**1.287530e+01**	**2.028223e+04**

Data Set II:

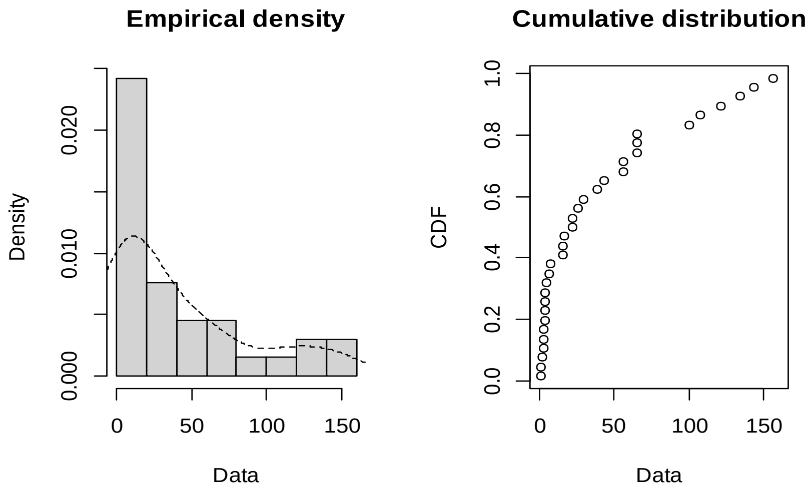
This data represents the survival times, in weeks of 33 patients suffering from acute myelogenous leukemia. The data is as follows:

65, 156, 100, 134, 16, 108, 121, 4, 39, 143, 56, 26, 22, 1, 1, 5, 65, 56, 65, 17, 7, 16, 22, 3, 4, 2, 3, 8, 4, 3, 30, 4, 43

The summary of the data is given in Table 4.

Table 4
Summary of Data

	Min	Mean	Median	Variance	1st Qu.	3rd Qu.	Max
DATA 2	1.00	40.88	22.00	2181.17	4.00	65.00	156.00



Plot of empirical Cdf and LPR $(t-x)$ distribution Cdf for the second data set

Estimated sd	Estimated Skewness	Estimated kurtosis
46.70302	1.220773	3.349298

Correlation Matrix:

	α	**β**	**λ**	**θ**
α	1.00000000	0.1446169	0.1215145	0.08912248
β	0.14461694	1.0000000	0.9562662	0.92716450
λ	0.12151452	0.9562662	1.0000000	0.77906678
θ	0.08912248	0.9271645	0.7790668	1.00000000

Cullen and Frey Graph

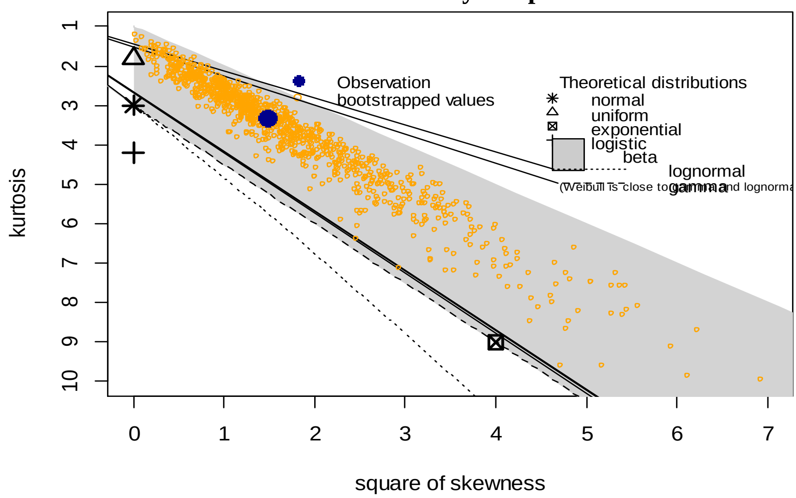


Table 5
Goodness-of-Fit Statistics for Data Set 2nd

Distribution	$-log\ l$	AIC	BIC	AICC	HQIC	$K-S$	p-value
RD	227.17	379.27	380.76	379.40	379.77	0.38	0.768
WRD	229.23	381.34	384.33	381.74	382.34	0.39	0.648
AD	220.63	349.554	354.044	350.382	351.065	0.33	0.007
WAD	217.79	437.589	439.086	438.417	443.100	0.29	0.821
IAD	225.81	366.123	367.619	366.950	371.633	0.37	0.068
MARD	171.77	345.543	347.039	346.370	351.054	0.19	0.057
LPRD(t-x)	**154.12**	**315.198**	**321.184**	**316.001**	**314.685**	**0.07**	**0.899**

Table 6
The Estimation of Parameters for the Second Data Set

Distribution	Estimated Parameters			
	α	**β**	λ	θ
RD	43.509490	-	-	-
WRD	-	0.001000	-	43.498617
AD	0.024468886	-	-	-
WAD	0.048926949	-	-	-
IAD	6.0070564	-	-	-
MARD	22.355941413	0.001000000	0.24451324	-
LPRD(t-x)	**0.71597061**	**1.22656707**	**0.09069990**	**2.00484542**

CONCLUSION

The T-X generator is used in this piece of research to join the two distributions namely Lomax and Power Rayleigh Distribution, which formed Lomax–Power Rayleigh (T-X) distribution (LPRD). The structural properties, Generating Functions and Shanon's entropy of the said distribution is derived. The Probability density function, Cumulative distribution function, reliability function and hazard rate function are represented by graphical plots. Finally a real life data set is fitted in the new distribution where the distribution performed very well when compared to parallel distributions.

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