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THE KUMARASWAMY EXPONENTIATED FRÉCHET DISTRIBUTION

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ABSTRACT

A new five-parameter model called the Kumaraswamy exponentiated Fréchet (Kw-EFr) distribution is proposed and studied. The new model generalizes many well known distributions in the literature. It is illustrated that 27 different distribution are embedded in the Kw-EFr distribution. Various structural properties including explicit expressions for ordinary and incomplete moments, quantile, generating function and Rényi and q-entropies are derived. The maximum likelihood method is used to estimate the model parameters and the observed information matrix is derived. A real data set is used to compare the new model with other competing models.

KEYWORDS

Exponentiated Fréchet; Generating Function; Maximum Likelihood Estimation; Order Statistics; Rényi Entropy.

1. INTRODUCTION

For last few decades there has been a growing interest in developing generalized class of distributions by inducting one or more additional parameter(s) to the standard probability distribution. More attention has been given studying the tail behavior of distributions by adding shape parameters. Because of their popularity in modeling real life phenomenon arising from diverse area of real life situation extreme value distributions have been generalized using these tools by several authors.

The Fréchet distribution is a special case of the generalized extreme value distribution. The Fréchet distribution has applications ranging from accelerated life testing to earthquakes, floods, horse racing, rainfall, wind speeds, sea waves, among others. For more information about the Fréchet distribution and its applications see Kotz and Nadarajah (2000), Barreto-Souza et al. (2011) and Krishna et al. (2013). Recently, some generalizations of the Fréchet distribution are introduced. For example, Nadarajah and Gupta (2004) introduced the beta Fréchet, Krishna et al. (2013) proposed the

Marshall-Olkin Fréchet, Mahmoud and Mandouh (2013) introduced transmuted Fréchet, Mead and Abd-Eltawab (2014) introduced the Kumaraswamy Fréchet, Afify et al. (2015) defined the transmuted Marshall-Olkin Fréchet and Afify et al. (2016a) proposed the Kumaraswamy Marshall-Olkin Fréchet distributions. Nadarajah and Kotz (2003) proposed the exponentiated Fréchet (EFr) distribution with cumulative distribution function (cdf) (for $x > 0$) is given by

$$F(x) = 1 - \left[1 - e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right]^{\alpha}, \tag{1}$$

where $\sigma > 0$ is a scale parameter and $\lambda > 0$and $\alpha > 0$ are two shape parameters.

The corresponding probability density function (pdf)is given by

$$f(x) = \alpha\lambda\sigma^{\lambda}x^{-(1+\lambda)}e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\left[1 - e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right]^{\alpha-1}. \tag{2}$$

The aim of this paper is to define and study a new model called the Kumaraswamy exponentiated Fréchet (Kw-EFr) distribution. The main feature of this model is that two additional parameters will be introduced in (1) to give greater flexibility in the form of the generated distribution. Using the Kumaraswamy-G (Kw-G) family introduced by Cordeiro and de Castro (2011), we construct the new five-parameter Kw-EFr model. We give a comprehensive description of some mathematical properties of the new distribution with the hope that it will attract wider applications in reliability, engineering and other areas of research. The class of Kw-G of distributions is stems from the following general construction due to Cordeiro and de Castro: if G denotes the baseline cumulative function of a random variable, then a generalized class of distributions can be defined by

$$F(x) = 1 - [1 - G(x)^{a}]^{b}, \tag{3}$$

where $a > 0$ and $b > 0$ are two additional shape parameters. Correspondingly, the pdf of Kw-G family is given by

$$f(x) = abg(x)G(x)^{a-1}[1 - G(x)^{a}]^{b-1}, \tag{4}$$

where $g(x) = dG(x)/dx$ and a and b are two additional positive shape parameters. Clearly when $a = b = 1$ we obtain the baseline distribution. An attractive feature of this distribution is that the two parameters a and b can afford greater control over the weights in both tails and in its centre.

The rest of the paper is organized as follows: In Section 2, we derive the expression the subject distribution and provide useful mixture representations for its pdf. In Section 3, we provide some mathematical properties of the Kw-EFr distribution. The maximum likelihood estimates (MLEs) of the unknown parameters are given in Section 4. Section 5 discusses simulation results to assess the performance of the proposed maximum likelihood estimation procedure. An application to a real data set is performed in Section 6 to show the potentiality of the proposed model. Finally, some concluding remarks are given in Section 7.

2. THE KW-EF MODEL

Consider the $G(x)$ in Equation (3) to be the cdf of the EFr distribution given in (1) so that the cdf of Kw-EFr distribution is given by

$$F(x) = 1 - \left\{1 - \left\{1 - \left[1 - e^{-(\frac{\sigma}{x})^{\lambda}}\right]^{\alpha}\right\}^{a}\right\}^{b},$$

where, $\sigma > 0$ is a scale parameter and $\lambda, \alpha, a, b > 0$ are shape parameters.

The pdf of the Kw-EFr distribution is given by

$$f(x) = ab\alpha\lambda\sigma^{\lambda}x^{-(1+\lambda)}e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\left\{1 - \left[1 - e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right]^{\alpha}\right\}^{a-1}$$
$$\times\left[1 - e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right]^{\alpha-1}\left\{1 - \left[1 - \left(1 - e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right)^{\alpha}\right]^{a}\right\}^{b-1}. \quad (5)$$

The hazard rate function (hrf) of X reduces to

$$h(x) = \frac{ab\alpha\lambda\sigma^{\lambda}x^{-(1+\lambda)}e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\left[1-e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right]^{\alpha-1}\left[1-\left(1-e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right)^{\alpha}\right]^{a-1}}{1-\left[1-\left(1-e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right)^{\alpha}\right]^{a}}.$$

Using the series expansion

$$(1 - z)^{b-1} = \sum_{j=0}^{\infty}\frac{(-1)^{j}\Gamma(b)}{j!\,\Gamma(b-j)}z^{j}, |z| < 1, b > 0,$$

the pdf of the Kw-EFr distribution in (5) can be expressed in the mixture form as

$$f(x) = ab\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}\Gamma(b)\Gamma(aj+a)}{i!j!\Gamma(b-j)\Gamma(aj+a-i)}\alpha\lambda\sigma^{\lambda}x^{-(1+\lambda)}e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\left[1 - e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right]^{\alpha(i+1)-1}. \quad (6)$$

Equation (6) can be further expressed in terms of the Fr densities as

$$f(x) = ab\alpha\lambda\sigma^{\lambda}\sum_{j,i,k=0}^{\infty}\frac{(-1)^{j+i+k}\Gamma(b)\Gamma(aj+a)\Gamma(\alpha i+\alpha)}{j!i!k!\Gamma(b-j)\Gamma(aj+a-i)\Gamma(\alpha i+\alpha-k)}x^{-(1+\lambda)}e^{-(k+1)\left(\frac{\sigma}{x}\right)^{\lambda}}.$$

Or equivalently

$$f(x) = \sum_{k=0}^{\infty} \upsilon_k h_{k+1}(x), \quad (7)$$

where $h_{k+1}(x)$is the Fréchet pdf of with scale parameter $\sigma(k+1)^{1/\lambda}$ and shape parameter λ and

$$\upsilon_k = \sum_{j,i=0}^{\infty}\frac{(-1)^{j+i+k}ab\alpha\Gamma(b)\Gamma(aj+a)\Gamma(\alpha i+\alpha)}{j!i!k!\Gamma(b-j)\Gamma(aj+a-i)\Gamma(\alpha i+\alpha-k)(k+1)}.$$

Equation (7) reveals that the Kw-EFr density can be expressed as an infinite linear combination of Fr densities. Thus, some of its mathematical properties can be obtained directly from those properties of the Fr distribution.

The Kw-EFr distribution is very flexible model that approaches to different distributions when its parameters are changed. The Kw-EFr distribution includes as special cases 27 well known and new probability distributions which are provided in Table 1.

Table 1
Sub Models of Kw-EFr$(\boldsymbol{\sigma}, \boldsymbol{\lambda}, \boldsymbol{\alpha}, \boldsymbol{a}, \boldsymbol{b})$ Distribution

Model	$\boldsymbol{\sigma}$	$\boldsymbol{\lambda}$	$\boldsymbol{\alpha}$	$\boldsymbol{a}$	$\boldsymbol{b}$	Author
Kw-Fr	σ	λ	1	a	b	Mead and Abd-Eltawab (2014)
Kw-IE	σ	1	1	a	b	–
Kw-IR	σ	2	1	a	b	–
Kw-EIE	σ	1	α	a	b	New
Kw-EIR	σ	2	α	a	b	New
Kw-EGuII	$p = \sigma^\lambda$	λ	α	a	b	New
Kw-EGIW	$qc^{\frac{1}{\lambda}}$	λ	α	a	b	New
Kw-GuII	$p = \sigma^\lambda$	λ	1	a	b	New
Kw-GIW	$qc^{\frac{1}{\lambda}}$	λ	1	a	b	New
Kw-IW	$c^{\frac{1}{\lambda}}$	λ	1	a	b	Shahbaz et al. (2012)
GEFr	σ	λ	α	1	b	New
EGFr	σ	λ	α	a	1	New
GEIE	σ	1	α	1	b	New
GEIR	σ	2	α	1	b	New
GEGIW	$qc^{\frac{1}{\lambda}}$	λ	α	1	b	New
EGIW	$qc^{\frac{1}{\lambda}}$	λ	1	1	b	New
GIW	$qc^{\frac{1}{\lambda}}$	λ	1	1	1	de Gusmao et al. (2011)
IW	σ	λ	1	1	1	Keller and Kamath (1982)
EGGuII	$p = \sigma^\lambda$	λ	1	a	1	New
EGuII	$p = \sigma^\lambda$	λ	1	1	b	New
GuII	$p = \sigma^\lambda$	λ	1	1	1	Gumbel (1958)
EFr	σ	λ	α	1	1	Nadarajah and Kotz (2003)
EIE	σ	1	1	1	b	–
EIR	σ	2	1	1	b	–
Fr	σ	λ	1	1	1	Fréchet (1924)
IE	σ	1	1	1	1	Keller and Kamath (1982)
IR	σ	2	1	1	1	Trayer (1964)

G=Generalized, GuII=Gumbel type-2, I=inverse, E=exponentiated, E=exponential, R=Rayliegh and W=Weibull

Figure 1 and Figure 2, respectively, display possible shape of the pdf and hrf of Kw-EFr distribution for selected values of the parameters.

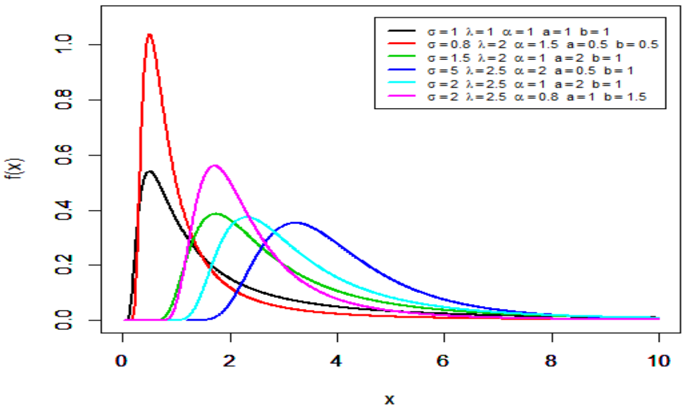

Figure 1: Probability Density Function of Kw-EFr Distribution

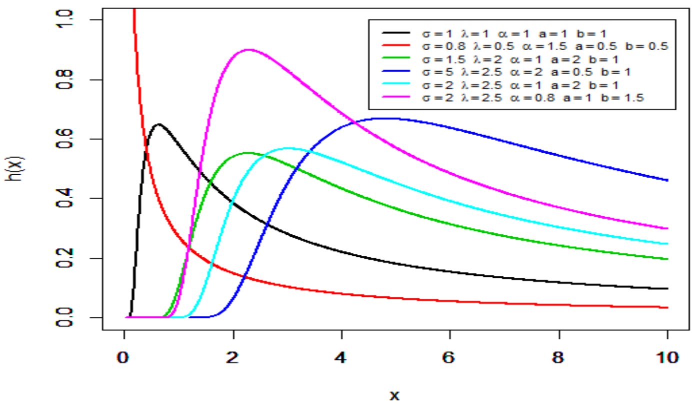

Figure 2: Hazard Rate Function of Kw-Fr Distribution

2. MATHEMATICAL PROPERTIES

In this section, some statistical properties of the Kw-EFr distribution including quantile functions (qf), ordinary and incomplete moments, moment generating functions (mgf), Rényi and q-entropies, order statistics and moments of the residual and reversed residual lifes are derived.

3.1 Quantile Function

The qf of a distribution is the real solution of $F(x_q) = q$ for $0 \leq q \leq 1$. For Kw-EFr distribution the quantiles are given by

$$x_q = \sigma \left\{ -\ln \left(1 - \sqrt[\alpha]{1 - \sqrt[a]{1 - \sqrt[b]{1 - q}}} \right) \right\}^{-1/\lambda}. \tag{8}$$

By substituting $q = 0.5$ in Equation (8) we can get the median of the Kw-EFr distribution.

3.2 Moments

Some of the most important features and characteristics of a distribution can be studied through moments (e.g. tendency, dispersion, skewness and kurtosis). The rth ordinary moment of X can be written from (7) as

$$\mu_r' = E(X^r) = \sum_{k=0}^{\infty} \upsilon_k \int_0^{\infty} x^r h_{k+1}(x)dx.$$

Then, for $r < \lambda$, we obtain

$$\mu_r' = E(X^r) = \sum_{k=0}^{\infty} \upsilon_k \sigma^r (k+1)^{\frac{r}{\lambda}-1} \Gamma\left(1-\frac{r}{\lambda}\right). \tag{9}$$

Setting $r = 1$ in (9), we have the mean of X.

The effects of parameters a and b on mean, variance, skewness and kurtosis for given values of λ, σ and α are displayed in Figure 3 and 4, respectively.

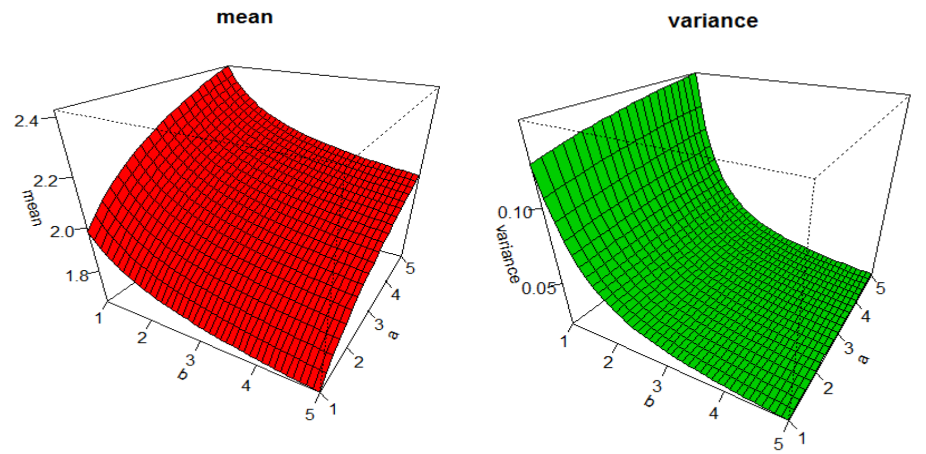

Figure 3: Plots of Mean and Variance

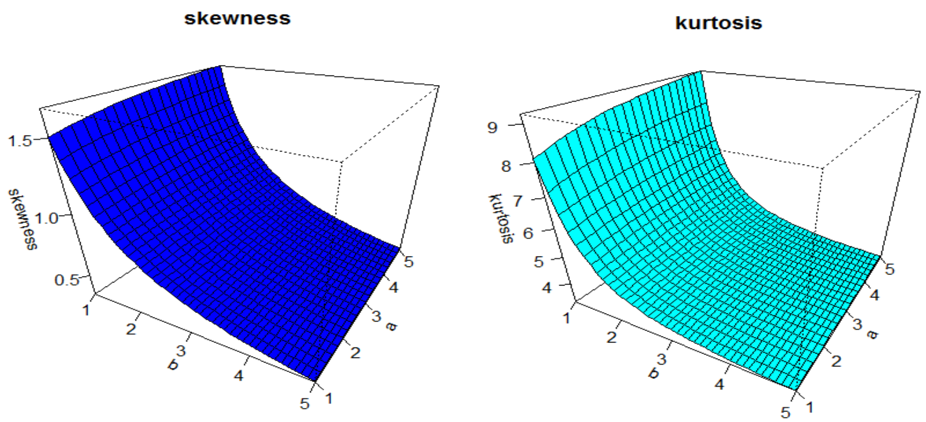

Figure 4: Plots of Skewness and Kurtosis

In Table 2 we provide the numerical measure of the median, mean, variance, skewness and kurtosis of the Kw-EFr distribution for selected values of the parameters to illustrate their effect on these measures.

Table 2
Median, Mean, Variance, Skewness, Kurtosis for selected Values of the Parameters

σ	λ	α	a	b	Median	Mean	Variance	Skewness	Kurtosis
1	5	2	1	1	0.959	0.991	0.031	1.502	8.103
2	5	2	1	1	1.919	1.982	0.124	1.502	8.103
3	5	2	1	1	2.879	2.973	0.280	1.502	8.103
5	5	2	1	1	4.799	4.956	0.777	1.502	8.103
8	5	2	1	1	7.678	7.929	1.990	1.502	8.103
2	5	2	1	1	1.919	1.982	0.124	1.502	8.103
2	6	2	1	1	1.933	1.981	0.084	1.342	6.974
2	7	2	1	1	1.942	1.981	0.060	1.237	6.327
2	8	2	1	1	1.949	1.982	0.046	1.163	5.909
2	10	2	1	1	1.959	1.984	0.029	1.064	5.406
2	5	1	1	1	2.152	2.328	0.535	3.535	48.09
2	5	2	1	1	1.919	1.982	0.124	1.502	8.103
2	5	3	1	1	1.826	1.862	0.068	1.037	5.349
2	5	5	1	1	1.733	1.752	0.036	0.667	3.976
2	5	8	1	1	1.667	1.678	0.023	0.432	3.429
2	5	2	1	1	1.919	1.982	0.124	1.502	8.103
2	5	2	2	1	2.102	2.169	0.132	1.574	8.614
2	5	2	3	1	2.211	2.280	0.138	1.615	8.905
2	5	2	5	1	2.351	2.423	0.146	1.665	9.246
2	5	2	8	1	2.482	2.558	0.156	1.705	9.528
2	5	2	1	1	1.919	1.982	0.124	1.502	8.103
2	5	2	1	2	1.771	1.796	0.047	0.809	4.426
2	5	2	1	3	1.706	1.721	0.030	0.567	3.715
2	5	2	1	5	1.639	1.646	0.019	0.342	3.280
2	5	2	1	8	1.589	1.592	0.013	0.185	3.095

3.3 Moment Generating Function

The mgf of the Kw-EFr distribution, denoted by $M_X(t) = E(e^{tX})$, (for $r < \lambda$), is given by

$$M_X(t) = \sum_{r,k=0}^{\infty} \frac{(t\sigma)^r}{r!} \upsilon_k (k+1)^{r/\lambda - 1} \Gamma\left(1 - \frac{r}{\lambda}\right).$$

We can express the mgf of Kw-EFr distribution in terms of Wright generalized hypergeometric function as below:

Note that the pdf and cdf of the Fréchet distribution (for $x > 0$) are, respectively, given by

$$g(x) = \lambda \sigma^{\lambda} x^{-(\lambda+1)} e^{-\left(\frac{\sigma}{x}\right)^{\lambda}} \text{ and } G(x) = e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}.$$

First, we provide the generating function of the Fréchet model as discussed by Afify et al. (2016b).

Setting $y = x^{-1}$, we can write this mgf as

$$M(t) = \lambda \sigma^{\lambda} \int_0^{\infty} \exp\left(\frac{t}{y}\right) y^{\lambda-1} \exp\left[-(\sigma y)^{\lambda}\right] dy.$$

By expanding $\exp\left(\frac{t}{y}\right)$ and calculating the integral, we have

$$M(t) = \lambda \sigma^{\lambda} \int_0^{\infty} \sum_{m=0}^{\infty} \frac{t^m}{m!} y^{\lambda-m-1} \exp\left[-(\sigma y)^{\lambda}\right] dy = \sum_{m=0}^{\infty} \frac{\sigma^m t^m}{m!} \Gamma\left(\frac{\lambda-m}{\lambda}\right),$$

where the gamma function is well-defined for any non-integer λ.

Consider the Wright generalized hypergeometric function (Srivastava and Karlsson, 1985, p. 21) defined by

$${}_p\Psi_q\left[\begin{matrix}(\sigma_1, A_1), \ldots, (\sigma_p, A_p) \\ (\lambda_1, B_1), \ldots, (\lambda_q, B_q)\end{matrix}; x\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\sigma_j + A_j n)}{\prod_{j=1}^{q} \Gamma(\lambda_j + \lambda_j n)} \frac{x^n}{n!}.$$

Then, we can write $M(t)$ as

$$M(t) = {}_1\Psi_0\left[\begin{matrix}(1, -\lambda^{-1}) \\ -\end{matrix}; \sigma t\right]. \tag{10}$$

Combining expressions (7) and (10), we obtain the mgf of the Kw-EFr distribution as

$$M_X(t) = \sum_{k=0}^{\infty} \upsilon_k \, {}_1\Psi_0\left[\begin{matrix}(1, -\lambda^{-1}) \\ -\end{matrix}; \sigma (k+1)^{1/\lambda} t\right].$$

3.4 Incomplete Moments

The main application of the first incomplete moment refers to the Bonferroni and Lorenz curves. These curves are very useful in economics, reliability, demography, insurance and medicine. The sth incomplete moments, denoted by $\varphi_s(t)$, of a random variable X is given by

$$\varphi_s(t) = \int_0^t x^s f(x) dx.$$

Using Equation (7) and the lower incomplete gamma function, *if* $s < \lambda$, we obtain

$$\varphi_s(t) = \sum_{k=0}^{\infty} v_k \sigma^s (k+1)^{s/\lambda - 1} \Gamma\left(1 - \frac{s}{\lambda}, (k+1)\left(\frac{\sigma}{t}\right)^{\lambda}\right). \tag{11}$$

3.5 Rényi and q-Entropies

Entropy refers to the amount of uncertainty associated with a random variable ($r.v.$). The Rényi entropy has numerous applications in information theoretic learning, statistics (e.g. classification, distribution identification problems, and statistical inference), computer science (e.g. average case analysis for random databases, pattern recognition, and image matching) and econometrics (Källberga et al., 2014). The Rényi entropy of a $r.v.\ X$ represents a measure of variation of the uncertainty. The Rényi entropy is defined by

$$I_\delta(X) = (1-\delta)^{-1} \log \int_{-\infty}^{\infty} f^\delta(x)dx, \delta > 0 \text{ and } \delta \neq 1.$$

Therefore, the Rényi entropy of the $r.v.\ X$ is given by

$$I_\delta(X) = (1-\delta)^{-1} \log \sum_{k=0}^{\infty} s_k (ab\alpha\lambda)^\delta \sigma^{\delta\lambda} \int_0^{\infty} x^{-\delta(1+\lambda)} e^{-(k+1)\left(\frac{\sigma}{x}\right)^{\lambda}} dx.$$

But

$$\int_0^{\infty} x^{-\delta(1+\lambda)} e^{-(k+1)\left(\frac{\sigma}{x}\right)^{\lambda}} dx = \frac{1}{\lambda} \sigma^{1-\delta(1+\lambda)} (k+1)^{(1-\delta(1+\lambda))/\beta} \Gamma\left(\frac{\delta(1+\lambda)-1}{\lambda}\right).$$

Therefore,

$$I_\delta(X) = (1-\delta)^{-1} \log\left\{\Gamma\left(\frac{\delta(1+\lambda)-1}{\lambda}\right) \sum_{k=0}^{\infty} s_k (k+1)^{(1-\delta(1+\lambda))/\beta}\right\},$$

where

$$s_k = \sum_{j,i=0}^{\infty} \frac{(-1)^{j+i+k} (ab\alpha)^\delta \lambda^{\delta-1} \Gamma(\delta b - \delta + 1) \Gamma(aj + (a-1)\delta + 1) \Gamma(\alpha i + (\alpha-1)\delta + 1)}{j!\, i!\, k!\, \sigma^{\delta-1} \Gamma(\delta b - \delta - j + 1) \Gamma(aj + (a-1)\delta - i + 1) \Gamma(\alpha i + (\alpha-1)\delta - k + 1)}.$$

The q-entropy, denoted by $H_q(X)$, is defined by

$$H_q(X) = \frac{1}{q-1} \log\left\{1 - \int_{-\infty}^{\infty} f^q(x)dx\right\}, q > 0 \text{and} q \neq 1.$$

$$H_q(X) = \frac{1}{q-1} \log\left\{1 - \Gamma\left(\frac{q(1+\lambda)-1}{\lambda}\right) \sum_{k=0}^{\infty} s_k^* (k+1)^{(1-q(1+\lambda))/\beta}\right\},$$

where

$$s_k^* = \sum_{j,i=0}^{\infty} \frac{(-1)^{j+i+k} (ab\alpha)^q \lambda^{q-1} \Gamma(qb - q + 1) \Gamma(aj + (a-1)q + 1) \Gamma(\alpha i + (\alpha-1)q + 1)}{j!\, i!\, k!\, \sigma^{q-1} \Gamma(qb - q - j + 1) \Gamma(aj + (a-1)q - i + 1) \Gamma(\alpha i + (\alpha-1)q - k + 1)}.$$

3.6 Order Statistics

If $X_1, X_2, \ldots, X_n$ is a random sample of size n from a continuous population with cdf $F(x)$ and pdf $f(x)$, and $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the corresponding order statistics. Then the pdf of $X_{(j)}$is given by

$$f_{j:n}(x) = \frac{n!}{(j-1)!(n-j)!} f(x)\big(F(x)\big)^{j-1}\big(1-F(x)\big)^{n-j}, j = 1,\ldots,n.$$

The joint pdf of $X_{(i:n)}$ and $X_{(j:n)}$, $1 \le i \le j \le n$, is given by

$$f_{i:j:n}(x,y) = \frac{n!}{(i-1)!\,(j-i-1)!\,(n-j)!} f(x)f(y) \times \big(F(x)\big)^{i-1}\big(F(y)-F(x)\big)^{j-i-1}\big(1-F(x)\big)^{n-j},$$

for $0 \le x \le y < \infty$.

The pdf of the jth order statistics for a Kw-EFr distributionis given by

$$f_{j:n}(x) = \frac{n!}{(j-1)!(n-j)!} ab\alpha\lambda\sigma^{\lambda}x^{-(1+\lambda)}e^{-\left(\frac{\sigma}{x}\right)^{\lambda}} l_x^{\alpha-1}[1-(1-l_x^{\alpha})^{a}]^{b-1} \times (1-l_x^{\alpha})^{a-1}\{1-[1-(1-l_x^{\alpha})^{a}]^{b}\}^{j-1}\{[1-(1-l_x^{\alpha})^{a}]^{b}\}^{n-j},$$

where $l_x^{\alpha} = [1-e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}]^{\alpha}$.

The joint pdf of $X_{(i:n)}$ and $X_{(j:n)}$, $1 \le i \le j \le n$, for a Kw-EFr distribution is given by

$$f_{i:j:n}(x,y) = \frac{n!\big(ab\alpha\lambda\sigma^{\lambda}\big)^{2}\big(l_x l_y\big)^{\alpha-1}}{(i-1)!(j-i-1)!(n-j)!(xy)^{1+\lambda}} e^{-\left(\frac{\sigma}{x}\right)^{\lambda}-\left(\frac{\sigma}{y}\right)^{\lambda}}\big[(1-l_x^{\alpha})\big(1-l_y^{\alpha}\big)\big]^{a-1} \times \Big\{[1-(1-l_x^{\alpha})^{a}]\big[1-\big(1-l_y^{\alpha}\big)^{a}\big]\Big\}^{b-1}\{1-[1-(1-l_x^{\alpha})^{a}]^{b}\}^{i-1} \times \Big\{\big[1-\big(1-l_y^{\alpha}\big)^{a}\big]^{b}-[1-(1-l_x^{\alpha})^{a}]^{b}\Big\}^{j-i-1}\{[1-(1-l_x^{\alpha})^{a}]^{b}\}^{n-j},$$

where $l_y^{\alpha} = [1-e^{-\left(\frac{\sigma}{y}\right)^{\lambda}}]^{\alpha}$.

Let $X_1, X_2, \ldots, X_n$ are independently identically distributed ordered random variables from the Kw-EFr distribution having median order X $_{m+1}$ pdf is given by

$$f_{m+1:n}(x) = \frac{(2m+1)!}{m!m!} f(x)[F(x)]^{m}[1-F(x)]^{m}$$
$$= \frac{(2m+1)!}{m!m!} ab\alpha\lambda\sigma^{\lambda}x^{-(1+\lambda)}e^{-\left(\frac{\sigma}{x}\right)^{\lambda}} l_x^{\alpha-1}(1-l_x^{\alpha})^{a-1} \times [1-(1-l_x^{\alpha})^{a}]^{b-1}\{[1-[1-(1-l_x^{\alpha})^{a}]^{b}][[1-(1-l_x^{\alpha})^{a}]^{b}]\}^{m}.$$

3.7 Moments of the Residual and Reversed Residual Lives

Several functions are defined related to the residual life. The failure rate function, mean residual life function and the left censored mean function, also called vitality function. In reliability analysis it is well known that these three functions uniquely determine $F(x)$ (see Gupta (1975), Kotz and Shanbhag (1980) and Zoroa et al. (1990)). Other interesting concept is the partial moments, defined by

$$g_n(t) = \int_t^{\infty} (x-t)^n dF(x), for\ n = 1,2,\ldots$$

Let X be a $r.v.$, usually representing the life length for a certain unit at age t (where this unit can have multiple interpretations), then the $r.v. X_t = X - t | X > t$, represents the remaining lifetime beyond that age. Moreover, the nth moments of residual life, denoted by

$$m_n(t) = E((X-t)^n | X > t), n = 1,2,3,\ldots,$$

uniquely determine $F(x)$ (Navarro et al., 1998). The nth moments of the residual life is given by

$$m_n(t) = \frac{1}{1-F(t)} \int_t^{\infty} (x-t)^n dF(x).$$

Therefore, the nth moments residual life given that $r < \lambda$ is given by

$$m_n(t) = \frac{1}{R(t)} \sum_{r=0}^{n} \frac{(-1)^{n-r}\Gamma(n+1)}{r!\Gamma(n-r+1)} \sigma^r t^{n-r} \\ \times \sum_{k=0}^{\infty} \upsilon_k (k+1)^{r/\lambda-1} \Gamma\left(1-\frac{r}{\lambda}, (k+1)\left(\frac{\sigma}{t}\right)^{\lambda}\right), \quad (10)$$

where $\Gamma(a,t) = \int_t^{\infty} y^{a-1} e^{-y} dy$ is the the upper incomplete gamma function.

The mean residual life function (MRL) of the Kw-EFr distribution can be obtained by setting $n = 1$ in (10). The MRL has many applications in survival analysis in biomedical sciences, life insurance, maintenance and product quality control, economics and social studies, demography and product technology (Lai and Xie, 2006).

The nth moments of the reversed residual life, denoted by $M_n(t)$, is defined as $M_n(t) = E((t-X)^n | X \leq t)$, $n = 1,2,3,\ldots,$. The nth moments of the residual life is given by

$$M_n(t) = \frac{1}{F(t)} \int_0^{t} (t-x)^n dF(x).$$

Therefore, we can write(for $r < \lambda$)

$$M_n(t) = \frac{1}{F(t)} \sum_{r=0}^{n} \frac{(-1)^{r}\Gamma(n+1)}{r!\Gamma(n-r+1)} \sigma^r t^{n-r} \\ \times \sum_{k=0}^{\infty} \upsilon_k (k+1)^{r/\lambda-1} \Gamma\left(1-\frac{r}{\lambda}, (k+1)\left(\frac{\sigma}{t}\right)^{\lambda}\right). \quad (11)$$

The mean waiting time (MWT) also called mean reversed residual life function of the Kw-EFR distribution can be obtained by setting $n = 1$ in Equation (11).

4. PARAMETER ESTIMATION

In this section, we consider the estimation of the parameters of Kw-EFr $(\boldsymbol{\sigma}, \boldsymbol{\lambda}, \boldsymbol{\alpha}, \boldsymbol{a}, \boldsymbol{b})$ model. Let $X_1, X_2, \ldots, X_n$ be a random sample from Kw-EFr distribution with unknown parameter vector $\theta = (\sigma, \lambda, \alpha, a, b)^T$. Therefore, the log-likelihood function ℓ is given by

$$\ell = n(\lambda \ln\sigma + \ln\lambda + \ln\alpha + \ln a + \ln b) - (1+\lambda)\sum_{i=1}^{n} \ln x_i - \sum_{i=1}^{n} \left(\frac{\sigma}{x_i}\right)^{\lambda} \\ + (\alpha - 1)\sum_{i=1}^{n} \ln(s_i) + (a-1)\sum_{i=1}^{n} \ln(z_i) + (b-1)\sum_{i=1}^{n} \ln(1 - z_i^{a}),$$

where $s_i = \left[1 - e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right]$, $z_i = (1 - s_i^{\alpha})$.

The above equation can be maximized either directly by using numerical techniques in R (optim function), SAS (PROC NLMIXED), Ox program (sub-routine MaxBFGS) or by solving the nonlinear likelihood equations obtained by differentiating ℓ.

Now differentiating ℓ with respect to the parameters $\sigma, \lambda, \alpha, a$ and b we get

$$\frac{\partial \ell}{\partial \sigma} = \frac{\lambda}{\sigma}\left(n - \sum_{i=1}^{n} (\frac{\sigma}{x_i})^{\lambda}\right) + (\alpha - 1)\sum_{i=1}^{n} \frac{r_i}{s_i} + \alpha(1-a)\sum_{i=1}^{n} \frac{r_i s_i^{\alpha-1}}{z_i}$$
$$+\alpha a(b-1)\sum_{i=1}^{n} \frac{r_i s_i^{\alpha-1} z_i^{a-1}}{1-z_i^{a}},$$

$$\frac{\partial \ell}{\partial \lambda} = n\left(\frac{1}{\lambda} + \ln\sigma\right) - \sum_{i=1}^{n} p_i e^{\left(\frac{\sigma}{x_i}\right)^{\lambda}} + (\alpha - 1)\sum_{i=1}^{n} \frac{p_i}{s_i} - \sum_{i=1}^{n} \ln x_i$$
$$+\alpha(1-a)\sum_{i=1}^{n} \frac{p_i s_i^{\alpha-1}}{z_i} + \alpha a(b-1)\sum_{i=1}^{n} \frac{p_i s_i^{\alpha-1} z_i^{a-1}}{1-z_i^{a}},$$

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \ln s_i + (1-a)\sum_{i=1}^{n} \frac{s_i^{\alpha}\ln s_i}{z_i} + a(b-1)\sum_{i=1}^{n} \frac{s_i^{\alpha} z_i^{a-1}\ln s_i}{1-z_i^{a}},$$

$$\frac{\partial \ell}{\partial a} = \frac{n}{a} + \sum_{i=1}^{n} \ln z_i + (1-b)\sum_{i=1}^{n} \frac{z_i^{a}\ln z_i}{1-z_i^{a}}$$

and

$$\frac{\partial \ell}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n} \ln(1-z_i^{a}),$$

where $p_i = (\frac{\sigma}{x_i})^{\lambda} e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}} \ln(\frac{\sigma}{x_i})$ and $r_i = \left(\frac{\lambda}{\sigma}\right)(\frac{\sigma}{x_i})^{\lambda} e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}$.

Setting the above nonlinear system of equations equal to zero and solving these equations simultaneously yields the MLE $\hat{\theta} = (\hat{\sigma}, \hat{\lambda}, \hat{\alpha}, \hat{a}, \hat{b})^T$. These equations cannot be solved analytically and statistical software can be used to solve them numerically by means of iterative techniques such as the Newton-Raphson algorithm.

5. SIMULATION STUDY

In this section, we provide the simulation results to assess the performance of the proposed maximum likelihood estimation procedure. An ideal technique for simulating from (0.5) is the inversion method.

One would simulate X by

$$x = \sigma\left\{-\ln\left(1 - \left(1 - \left(1 - (1-\xi)^{\frac{1}{b}}\right)^{\frac{1}{a}}\right)^{\frac{1}{\alpha}}\right)\right\}^{-1/\lambda}.$$

where $\xi \sim U(0,1)$ is a uniform random number. For different combination of $\sigma, \lambda, \alpha, a$ and b samples of sizes $n = 100, 200, 300, 500$ and 1000 are generated from the Kw-EFr distribution. We repeated the simulation $k = 100$ times and calculated the mean and the root mean square errors (RMSEs). The empirical results are given in Table 3.

Table 3
Empirical Means and the RMSEs of the Kw-EFr Distribution for $\sigma = 2.0, \lambda = 3.0, \alpha = 1.5, a = 0.5, b = 2.0$

n	$\hat{\sigma}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{a}$	$\hat{b}$
100	2.093 (0.471)	3.283 (1.111)	2.426 (2.574)	0.584 (0.535)	2.440 (2.141)
200	2.076 (0.387)	3.264 (0.874)	2.061 (1.497)	0.492 (0.305)	2.465 (2.062)
300	2.056 (0.387)	3.141 (0.684)	2.229 (2.029)	0.542 (0.354)	02.214 (1.703)
500	2.017 (0.281)	3.212 (0.623)	1.952 (1.215)	0.489 (0.285)	2.071 (1.289)
1000	2.002 (0.247)	3.200 (0.502)	1.917 (0.885)	0.474 (0.206)	1.922 (1.186)

It is evident that the estimates are quite stable and are close to the true value of the parameters for these sample sizes. Additionally, as the sample size increases the RMSEs, provided in the parentheses, decreases as expected.

6. APPLICATION

In this section, we provide an application to real data to illustrate the flexibility of the Kw-EFr model. The goodness-of-fit statistics for this model is compared with other competitive models and the MLEs of the model parameters are provided. We will make the use of a real data set (Smith and Naylor, 1987) consists of the following 63 observations of the strengths of 1.5 cm glass fibres, originally obtained by workers at the UK National Physical Laboratory.

We compare the fits of the Kw-EFr distribution with other competitive models, namely: the EFr, beta Fréchet (BFr) (Nadarajah and Gupta, 2004), gamma extended Fréchet (GEFr) (da Silva et al., 2013), transmuted Fréchet (TFr) (Mahmoud and Mandouh, 2013), Marshall-Olkin Fréchet (MOFr) (Krishna et al., 2013) and Fréchet (Fr) distributions with corresponding densities (for $x > 0$):

- BFr: $f(x) = \frac{\lambda\sigma^\lambda}{B(a,b)}\, x^{-(\lambda+1)}\, e^{-a\left(\frac{\sigma}{x}\right)^\lambda} \left\{1 - e^{-\left(\frac{\sigma}{x}\right)^\lambda}\right\}^{b-1}$,
- GEFr: $f(x) = \frac{\alpha\lambda\sigma^\lambda}{\Gamma(b)}\, x^{-(\lambda+1)}\, e^{-\left(\frac{\sigma}{x}\right)^\lambda} \left\{1 - e^{-\left(\frac{\sigma}{x}\right)^\lambda}\right\}^{\alpha-1} \left\{-\log\left[1 - e^{-\left(\frac{\sigma}{x}\right)^\lambda}\right]^{\alpha}\right\}^{b-1}$,
- TFr: $f(x) = \lambda\sigma^\lambda x^{-(\lambda+1)}\, e^{-\left(\frac{\sigma}{x}\right)^\lambda} \left\{1 + \delta - 2\delta\, e^{-\left(\frac{\sigma}{x}\right)^\lambda}\right\}$,
- MOFr: $f(x) = \alpha\lambda\sigma^\lambda\, x^{-(\lambda+1)}\, e^{-\left(\frac{\sigma}{x}\right)^\lambda} \left\{\alpha + (1-\alpha)\, e^{-\left(\frac{\sigma}{x}\right)^\lambda}\right\}^{-2}$.

The parameters of the above densities are all positive real numbers except for the TFr distribution for which $|\delta| \leq 1$.

In order to compare the fitted models, we consider some goodness-of-fit measures including the Akaike information criterion (AIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion ($HQIC$), consistent Akaike information criterion ($CAIC$) and $-2\hat{\ell}$, where $\hat{\ell}$ is the maximized log-likelihood. Furthermore, we use the Anderson-Darling (A^*) and the Cramér-von Mises (W^*) statistics in order to compare the fits of the new model with other nested and non-nested models.

Table 4 lists the values of $-2\hat{\ell}$, AIC, $CAIC$, $HQIC$, BIC, W^* and A^* whereas the values the MLEs, standard errors (SEs) of the parameters and 95% confidence intervals are given in Table 5. These numerical results are obtained using the MATH-CAD program. From Table 4, it is noted that the Kw-EFr distribution has the lowest values for the goodness-of-fit statistics among all fitted models. The plots comparing the Kw-EFr distribution with other competing distribution is given in Figure 5. These plots also indicate that the Kw-EFr distribution fits the subject data well.

Table 4
Goodness-of-Fit Statistics for Strengths of $\mathbf{1.5}$ cm Glass Fibre Data

Model	$-2\hat{\ell}$	AIC	BIC	$HQIC$	$CAIC$	W^*	A^*
Kw-EFr	**36.8**	**46.8**	**57.5**	**51.0**	**47.9**	**0.40028**	**2.14305**
EFr	44.3	50.5	56.7	52.8	50.7	0.54186	2.89991
BFr	60.6	68.6	77.2	72.0	69.3	0.76879	4.20206
GEFr	61.6	69.6	78.1	72.9	70.3	0.78121	4.27204
Fr	93.7	97.7	102	99.4	97.9	1.16252	6.40749
TFr	94.1	100.1	106.5	102.6	100.5	1.17022	6.45074
MOFr	95.7	101.7	108.2	104.2	102.1	1.19943	6.61660

Table 5
MLEs, SEs and 95% Confidence Intervals for Strengths of 1.5 cm Glass Fibre Data

Model	Parameters	Estimates	SEs	95% Confidence Interval
Kw-EFr	α	11.1364	0.749	(9.668,12.604)
	λ	0.1595	0.018	(0.124, 0.195)
	σ	35.619	23.409	(0.00, 81.501)
	a	75.4361	37.205	(2.51, 148.35)
	b	1246.4061	1075	(0.00, 3353.41)
EFr	λ	0.999	0.136	(0.732,1.266)
	σ	7.816	2.945	(2.044,13.588)
	α	132.827	116.63	(0.00,361.422)
BFr	λ	0.6466	0.163	(0.327,0.966)
	σ	2.0518	0.986	(0.119,3.984)
	a	15.0756	12.057	(0.00,38.707)
	b	36.9397	22.649	(0.00,81.332)
GEFr	λ	0.7421	0.197	(0.356,1.128)
	σ	1.6625	0.952	(0.00,3.528)
	a	32.112	17.397	(0.00,66.210)
	b	13.2688	9.967	(0.00,32.804)
Fr	λ	2.888	0.234	(2.429,3.347)
	σ	1.264	0.059	(1.148,1.379)
TFr	λ	2.7898	0.165	(2.466,3.113)
	σ	1.3068	0.034	(1.240,1.373)
	δ	0.1298	0.208	(-0.278, 0.537)
MOFr	λ	2.3876	0.253	(1.892,2.883)
	σ	1.5441	0.226	(1.101,1.987)
	α	0.4816	0.252	(0.00, 0.976)

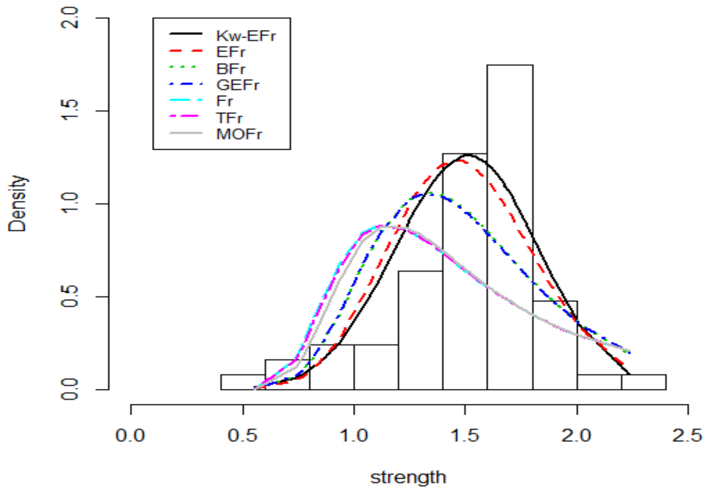

Figure 5: Fitted pdf of Kw-EFr and other Distribution for the Glass Fibre Data

7. CONCLUDING REMARKS

In this study we propose a new model, the so-called the Kw-EFr distribution which extends the EFr distribution in the analysis of data with real support. An obvious reason for generalizing a standard distribution is because the generalized form provides larger flexibility in modeling real data. We derive expressions for the ordinary and incomplete moments, quantile and generating functions, Rényi and q-entropy. We discuss maximum likelihood estimation for estimating parameters. We have presented an example to illustrate the application of the subject distribution to model real data. The Kw-EFr provides a better fit than several other nested and non-nested models for the subject data.

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