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ESTIMATION AND PREDICTION FOR NADARAJAH-HAGHIGHI DISTRIBUTION BASED ON RECORD VALUES

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ABSTRACT

This paper discusses maximum likelihood and Bayes estimation of the two unknown parameters of Nadarajah and Haghighi distribution based on record values. It assumed that in Bayes case, the unknown parameters of Nadarajah and Haghighi distribution have gamma prior densities. Explicit forms of Bayes estimators cannot be obtained. Lindley approximation is exploited to obtain point estimators for the unknown parameters. The Bayesian and non-Bayesian predictions of both point and interval predictions of the future record values are also discussed. A simulation study is used to the comparison between the Bayesian and non-Bayesian methods. Analysis of a real dataset is presented for illustrative purposes.

KEYWORDS

Exponential extension distribution, record values, prediction, Bayes estimation, squared error loss function, Lindley approximation, maximum likelihood estimation

1. INTRODUCTION

In a sequence of events, the event value that exceeds all previous values is of particular importance in the scientific and applied fields and so their values are recorded. In sporting events, for example, focus attention is usually on recording results that exceed their predecessor, as the hydrologists usually tend to monitor the higher values of the floods. Also, the meteorologists usually concern with upper and lower record temperatures. For more details on the concept of record values and their application see, for example, Ahsanullah (2004) and Arnold et al. (1998). The statistical treatment of the record values has been introduced for the first time by Chandler (1952). Since many studies on record values and their associated statistical inference have been done for some distributions by several authors such as Selim (2012) studied Bayesian estimation of Chen distribution based on record values. Hussian and Amin (2014) discussed the Bayesian and non-Bayesian estimations and prediction of record values from the Kumaraswamy inverse Rayleigh distribution. Asgharzadeh et al. (2016) derived the maximum likelihood (ML) and Bayes estimators for the two unknown parameters of the logistic distribution based on record data.

Nadarajah and Haghighi (2011) recently introduced a new generalization of the one parameter exponential distribution by introducing a shape parameter of its cumulative distribution function (cdf) to become as follow

$$\mathrm{F(x)} = 1 - \exp\{1 - (1 + \lambda \mathrm{x})^{\alpha}\}, \mathrm{x} > 0, \lambda, \beta > 0. \quad (1.1)$$

and the probability density function (pdf) is

$$\mathrm{f(x)} = \alpha\lambda(1 + \lambda \mathrm{x})^{\alpha-1}\exp\{1 - (1 + \lambda \mathrm{x})^{\alpha}\}, \mathrm{x} > 0, \lambda, \beta > 0 \quad (1.2)$$

where $\lambda > 0$ and $\alpha > 0$ are scale and shape parameters, respectively. The Nadarajah and Haghighi distribution will be denoted by (NH) distribution. The NH distribution is introduced as an alternative to the gamma, Weibull and exponentiated exponential distributions in lifetime studies. This distribution has a little number of studies regard to classical and Bayesian estimation. Among these studies, Singh et al. (2015) discussed the classical and Bayesian estimations for NH model under progressive type-II censored data. The ML and Bayes estimators of the unknown parameters of NH distribution under progressive type-II censored data with binomial removals have been also obtained by Singh et al. (2014). MirMostafaee et al. (2016) derived recurrence relations for moments of record values from NH distribution, and they also derived the BLUEs of the unknown two parameters of NH distribution.

The objective of this paper is twofold; to study the Bayesian and non-Bayesian estimation of the unknown parameters of the NH distribution based on record data and to study the Bayesian and non-Bayesian prediction of the future record values based on record data of the NH distribution. The rest of the paper is organized as follows; the maximum likelihood and Bayes estimations are discussed in Section 2. Bayesian and non-Bayesian predictions are discussed in Section 3. The estimation and prediction procedures are applied to real data set and simulation data in Section 4, 5 respectively. Finally, conclusions appear in Section 6.

2. ESTIMATION

In this section, we study the classical and Bayesian estimation of the two unknown parameters of Nadarajah and Haghighi distribution based on a sample of record values.

2.1 Maximum Likelihood Estimation

Let $\mathrm{X}_{\mathrm{U}(1)} = \mathrm{x}_1, \mathrm{X}_{\mathrm{U}(2)} = \mathrm{x}_2, \ldots, \mathrm{X}_{\mathrm{U(m)}} = \mathrm{x}_{\mathrm{m}}$ are the first m observed upper record values from NH distribution with cdf (1.2) and pdf (1.1). Then, the likelihood function of the m upper records is given by (Ahsanullah (2004))

$$L(\alpha, \lambda \mid \underline{x}) = (\alpha\lambda)^m exp\{1 - (1 + \lambda x_m)^{\alpha}\} \prod_{i=1}^{m} (1 + \lambda x_i)^{\alpha-1} \quad (2.1)$$

Taking the logarithm of the likelihood function (2.1), we get

$$\begin{aligned} lnL(\alpha, \lambda \mid \underline{x}) &= m\ ln(\alpha\lambda) + 1 - (1 + \lambda x_m)^{\alpha} \\ &+ (\alpha - 1)\sum_{i=1}^{m} ln(1 + \lambda x_i) \end{aligned} \quad (2.2)$$

Then, the MLEs of the parameters α and λ are a solution of the following likelihood equations

$$\frac{m}{\alpha} - (1 + \lambda x_m)^{\alpha} \, ln(1 + \lambda x_m) + \sum_{i=1}^{m} ln(1 + \lambda x_i) = 0 \tag{2.3}$$

$$\frac{m}{\lambda} - \alpha x_m (1 + \lambda x_m)^{\alpha-1} + (\alpha - 1) \sum_{i=1}^{m} \frac{x_i}{(1 + \lambda x_i)} = 0 \tag{2.4}$$

The previous equations (2.3), (2.4) cannot be solved analytically for α and λ. Therefore, we suggest using the iterative methods to find the numerical solutions of these equations.

The asymptotic variance–covariance matrix of the MLE for the parameters α and λ can be approximated as follows

$$\hat{I}(\alpha, \lambda) = \begin{bmatrix} -L_{\alpha\alpha} & -L_{\alpha\lambda} \\ -L_{\lambda\alpha} & -L_{\lambda\lambda} \end{bmatrix}^{-1}_{\hat{\alpha},\hat{\lambda}} = \begin{bmatrix} \hat{\sigma}_{\alpha\alpha} & \hat{\sigma}_{\alpha\lambda} \\ \hat{\sigma}_{\lambda\alpha} & \hat{\sigma}_{\lambda\lambda} \end{bmatrix} \tag{2.5}$$

where

$$L_{\alpha\alpha} = \frac{\partial^2 L}{\partial \alpha^2} = -\frac{m}{\alpha^2} - \ln^2(1 + \lambda x_m)\,(1 + \lambda x_m)^{\alpha} \tag{2.6}$$

$$L_{\lambda\lambda} = \frac{\partial^2 L}{\partial \lambda^2} = -\frac{m}{\lambda^2} - \alpha(\alpha - 1){x_m}^2 (1 + \lambda x_m)^{\alpha-2} - (\alpha - 1) \sum_{i=1}^{m} \frac{{x_i}^2}{(1 + \lambda x_i)^2} \tag{2.7}$$

$$L_{\alpha\lambda} = L_{\lambda\alpha} = \frac{\partial^2 L}{\partial \alpha \partial \lambda} = -x_m (1 + \lambda x_m)^{\alpha-1} [\alpha \ln(1 + \lambda x_m) + 1] + \sum_{i=1}^{m} \frac{x_i}{(1 + \lambda x_i)} \tag{2.8}$$

The asymptotic normality of the MLEs can be used to compute approximate $100(1 - \tau)\%$ confidence intervals for the parameters α and λ, as follow

$$\hat{\alpha} \pm z_{\tau/2} \sqrt{\hat{\sigma}^2_{\hat{\alpha}}} \quad \text{and} \quad \hat{\lambda} \pm z_{\tau/2} \sqrt{\hat{\sigma}^2_{\hat{\lambda}}}$$

where $z_{\tau/2}$ is an upper $\tau/2$ % of the standard normal distribution.

2.2 Bayes Estimation

Assuming that the unknown parameters α and λ are independent and follow gamma distribution i. e. $\alpha \sim \mathrm{gamma}(a, b)$ and $\lambda \sim \mathrm{gamma}(c, d)$. Thus, the joint prior distribution for α and λ is

$$\pi(\alpha, \lambda) \propto \alpha^{a-1} \lambda^{c-1} e^{-b\alpha - d\lambda}, \alpha, \lambda > 0 \tag{2.9}$$

where a, b, c and d are the hyper parameters that are assumed to be nonnegative and known. Combining the joint prior (2.9) with the likelihood function (2.1) and applying Bayes' theorem, we get the joint posterior function of α and λ as follows

$$\pi(\alpha,\lambda|\underline{x}) = \frac{L(\alpha,\lambda \mid \underline{x})\pi(\alpha,\lambda)}{\int_0^\infty \int_0^\infty L(\alpha,\lambda \mid \underline{x})\pi(\alpha,\lambda)\, d\alpha d\lambda}$$
$$= \frac{1}{K}\alpha^{a+m-1}\lambda^{c+m-1}e^{-b\alpha-d\lambda-(1+\lambda x_m)^\alpha}\prod_{i=1}^{m}(1+\lambda x_i)^{\alpha-1} \quad (2.10)$$

where

$$K = \int_0^\infty \int_0^\infty \alpha^{a+m-1}\lambda^{c+m-1}e^{-b\alpha-d\lambda-(1+\lambda x_m)^\alpha}\prod_{i=1}^{m}(1+\lambda x_i)^{\alpha-1}\, d\alpha d\lambda \quad (2.11)$$

The Bayes estimators of α and λ under the squared error loss function (SELF) are the posterior mean as follow

$$\hat{\alpha} = E(\alpha|\underline{x}) = \frac{1}{K}\int_0^\infty \int_0^\infty \alpha^{a+m}\lambda^{c+m-1}e^{-b\alpha-d\lambda-(1+\lambda x_m)^\alpha} \prod_{i=1}^{m}(1+\lambda x_i)^{\alpha-1}\, d\alpha d\lambda \quad (2.12)$$

and

$$\hat{\lambda} = E(\lambda|\underline{x}) = \frac{1}{K}\int_0^\infty \int_0^\infty \alpha^{a+m-1}\lambda^{c+m}e^{-b\alpha-d\lambda-(1+\lambda x_m)^\alpha} \prod_{i=1}^{m}(1+\lambda x_i)^{\alpha-1}\, d\alpha d\lambda \quad (2.13)$$

It may be noted here that, the integral ratios in (2.12) and (2.13) cannot be expressed in simple closed forms. Therefore, we suggest using the Lindley's approximation method to obtain the Bayes estimators of α and λ. Lindley (1980) introduced a method to approximate the ratio of integrals as in (2.10). This approximation has been used to achieve the Bayes estimation based on record values by many authors; see, among others, Ahmadi et al. (2009) and Badr (2015). Let we have a ratio of integrals of the following form

$$E(u(\theta)|x) = \frac{\int u(\theta)e^{\mathcal{L}(\theta)+\rho(\theta)}d\theta}{\int e^{\mathcal{L}(\theta)+\rho(\theta)}d\theta} \quad (2.14)$$

where $\theta = (\theta_1, \theta_2, \dots, \theta_n)$ and $\mathcal{L}(\theta)$ is the logarithm of the likelihood function, $\rho(\theta) = log(\pi(\theta))$, $\pi(\theta)$ is the joint prior distribution of θ and $u(\theta)$ is a function of θ. This ratio of integrals can be asymptotically approximated using Lindley's approach as follows

$$E(u(\theta)|x) = \left[u(\theta) + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\sigma_{ij}[u_{ij} + 2u_i\rho_j] + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{m}\mathcal{L}_{ijk}\sigma_{ij}\sigma_{kl}u_i\right]_{\hat{\theta}} \quad (2.15)$$

where $u_{ij}=\frac{\partial^2 u}{\partial\theta_i\partial\theta_j}$, $u_i=\frac{\partial u}{\partial\theta_i}$, $\mathcal{L}_{ijk}=\frac{\partial^3\mathcal{L}}{\partial\theta_i\partial\theta_j\partial\theta_k}$, $\rho_j=\frac{\partial\rho}{\partial\theta_j}$, $\sigma_{ij}=\frac{-1}{\mathcal{L}_{ij}}$

Accordingly, the Lindley's approximation of (2.12) and (2.13) are

$$\hat{\alpha}_{LB}=\hat{\alpha}+\hat{\rho}_1\hat{\sigma}_{11}+\frac{1}{2}\hat{\sigma}_{11}\left(\hat{\mathcal{L}}_{122}\hat{\sigma}_{22}+\hat{\mathcal{L}}_{111}\hat{\sigma}_{11}\right) \tag{2.16}$$

and

$$\hat{\lambda}_{LB}=\hat{\lambda}+\hat{\rho}_2\hat{\sigma}_{22}+\frac{1}{2}\hat{\sigma}_{22}\left(\hat{\mathcal{L}}_{112}\hat{\sigma}_{11}+\hat{\mathcal{L}}_{222}\hat{\sigma}_{22}\right) \tag{2.17}$$

That can be rewritten as follow

$$\lambda_{LB}=\left\{\begin{array}{c}\lambda+\dfrac{\left(\frac{c-1}{\lambda}-d\right)}{\frac{m}{\lambda^2}+\alpha(\alpha-1)x_m^2(1+\lambda x_m)^{\alpha-2}+(\alpha-1)\sum_{i=1}^m\frac{x_i^2}{(1+\lambda x_i)^2}}\\+\dfrac{\frac{2m}{\lambda^3}-\alpha(\alpha-1)(\alpha-2)x_m^3(1+\lambda x_m)^{\alpha-3}+(\alpha-1)\sum_{i=1}^m\frac{2x_i^3}{(1+\lambda x_i)^3}}{2\left[\frac{m}{\lambda^2}+\alpha(\alpha-1)x_m^2(1+\lambda x_m)^{\alpha-2}+(\alpha-1)\sum_{i=1}^m\frac{x_i^2}{(1+\lambda x_i)^2}\right]^2}\\+\dfrac{-x_m\ln(1+\lambda x_m)(1+\lambda x_m)^{\alpha-1}(\alpha\ln(1+\lambda x_m)+2)}{2\left[\frac{m}{\lambda^2}+\alpha(\alpha-1)x_m^2(1+\lambda x_m)^{\alpha-2}+(\alpha-1)\sum_{i=1}^m\frac{x_i^2}{(1+\lambda x_i)^2}\right]\left[\frac{m}{\alpha^2}+\ln^2(1+\lambda x_m)(1+\lambda x_m)^{\alpha}\right]}\end{array}\right\}_{\alpha=\hat{\alpha},\lambda=\hat{\lambda}} \tag{2.18}$$

and

$$\alpha_{LB}=\left\{\begin{array}{c}\alpha+\dfrac{\left(\frac{a-1}{\alpha}-b\right)}{\frac{m}{\alpha^2}+\ln^2(1+\lambda x_m)(1+\lambda x_m)^{\alpha}}+\dfrac{\frac{2m}{\alpha^2}-\ln^3(1+\lambda x_m)(1+\lambda x_m)^{\alpha}}{2\left[\frac{m}{\alpha^2}+\ln^2(1+\lambda x_m)(1+\lambda x_m)^{\alpha}\right]^2}\\+\dfrac{-\alpha(\alpha-1)x_m^2(1+\lambda x_m)^{\alpha-2}\left[\frac{1}{(\alpha-1)}+\frac{1}{\alpha}+\ln(1+\lambda x_m)\right]-\sum_{i=1}^m\frac{x_i^2}{(1+\lambda x_i)^2}}{2\left[\frac{m}{\lambda^2}+\alpha(\alpha-1)x_m^2(1+\lambda x_m)^{\alpha-2}+(\alpha-1)\sum_{i=1}^m\frac{x_i^2}{(1+\lambda x_i)^2}\right]\left[\frac{m}{\alpha^2}+\ln^2(1+\lambda x_m)(1+\lambda x_m)^{\alpha}\right]}\end{array}\right\}_{\alpha=\hat{\alpha},\lambda} \tag{2.19}$$

where $\hat{\alpha}$ and $\hat{\lambda}$ are MLEs of α and λ, respectively.

3. THE PREDICTION

In this section, we study the classical and Bayesian predictions of unknown future record values based on a sample of observed record values from NH distribution.

3.1 Non-Bayesian Prediction

Let $X_{U(1)}=x_1, X_{U(2)}=x_2, \dots, X_{U(m)}=x_m$ are the first m observed upper record values taking from $NH(\alpha,\lambda)$ distribution, where α and λ are unknown parameters. Based on this sample of the record values, we intend to predict the future s^{th} upper record value $X_{U(s)}$, $m<s$. The joint predictive likelihood function of $X_{U(s)}=x_s$, is given by Basak and Balakrishnan (2003) as follows

$$L(x_s;\alpha,\lambda,x)=\frac{[\ln\bar{F}(x_m;\alpha,\lambda)-\ln\bar{F}(x_s;\alpha,\lambda)]^{s-m-1}}{\Gamma(s-m)}\prod_{i=1}^{m}\frac{f(x_i;\alpha,\lambda)}{\bar{F}(x_i;\alpha,\lambda)}f(x_s;\alpha,\lambda) \quad (3.1)$$

Then, the predictive likelihood function for the $NH(\alpha,\lambda)$ distribution is

$$L(x_s;\alpha,\lambda,x)\propto(\alpha\lambda)^{m+1}(1+\lambda x_s)^{\alpha-1}exp\{1-(1+\lambda x_m)^{\alpha}\} \times[(1+\lambda x_s)^{\alpha}-(1+\lambda x_m)^{\alpha}]^{s-m-1} \times\prod_{i=1}^{m}(1+\lambda x_i)^{\alpha-1} \quad (3.2)$$

Taking the natural logarithm of the predictive likelihood function (3.2) we get

$$lnL(x_s;\alpha,\lambda,x)=(m+1)\,ln(\alpha\lambda)+(\alpha-1)\,ln(1+\lambda x_s) -(1+\lambda x_m)^{\alpha} +(s-m-1)\,ln[(1+\lambda x_s)^{\alpha}-(1+\lambda x_m)^{\alpha}]+1 +(\alpha-1)\sum_{i=1}^{m}ln(1+\lambda x_i) \quad (3.3)$$

Differentiating the equation (3.3) with respect to λ and x_s, and by equating to zero, we obtain the following likelihood equations

$$\frac{m+1}{\alpha}+ln(1+\lambda x_s)-ln(1+\lambda x_s)\,(1+\lambda x_s)^{\alpha}+\sum_{i=1}^{m}ln(1+\lambda x_i) +(s-m-1)\frac{ln(1+\lambda x_s)\,(1+\lambda x_s)^{\alpha}-ln(1+\lambda x_m)\,(1+\lambda x_m)^{\alpha}}{(1+\lambda x_s)^{\alpha}-(1+\lambda x_m)^{\alpha}} =0 \quad (3.4)$$

$$\frac{m+1}{\lambda}+\frac{(\alpha-1)x_s}{(1+\lambda x_s)}-\alpha x_s(1+\lambda x_s)^{\alpha-1}+\sum_{i=1}^{m}\frac{(\alpha-1)x_i}{(1+\lambda x_i)} +(s-m -1)\frac{\alpha x_s(1+\lambda x_s)^{\alpha-1}-\alpha x_m(1+\lambda x_m)^{\alpha-1}}{(1+\lambda x_s)^{\alpha}-(1+\lambda x_m)^{\alpha}}=0 \quad (3.5)$$

$$\frac{(\alpha-1)\lambda}{(1+\lambda x_s)}-\alpha\lambda(1+\lambda x_s)^{\alpha-1}+\frac{(s-m-1)\alpha\lambda(1+\lambda x_s)^{\alpha-1}}{(1+\lambda x_s)^{\alpha}-(1+\lambda x_m)^{\alpha}}=0 \quad (3.6)$$

The three likelihood equations (3.4), (3.5) and (3.6) can be solved simultaneously using numerical solution to yield the predictive maximum likelihood estimators (PMLE) $\hat{\alpha}^{*}$ and $\hat{\lambda}^{*}$ of the parameters α and λ respectively, and MLP $\hat{x}_s$ of the s^{th} upper record value.

3.2 Highest Conditional Prediction Interval

To make prediction interval for the s^{th} upper record value x_s, where $1\le m<s$. Arnold et al. (1998) presented the conditional pdf of $X_{U(s)}$ given $X_{U(m)}$ as follows

$$f(x_s|x_m;\alpha,\lambda)$$

$$= \frac{[\ln \bar{F}(x_m;\alpha,\lambda) - \ln \bar{F}(x_s;\alpha,\lambda)]^{s-m-1}}{\Gamma(s-m)} \frac{f(x_s;\alpha,\lambda)}{\bar{F}(x_m;\alpha,\lambda)}, x_m < x_s < \infty \quad (3.7)$$

For $NH(\alpha,\lambda)$ distribution with cdf and pdf defined in (1.1) and (1.2), the conditional pdf of $X_{U(s)}$ given $X_{U(m)}$ can be approximated by replacing the unknown parameters α and λ by their maximum likelihood estimates $\hat{\alpha}$ and $\hat{\lambda}$ to become

$$\hat{f}(x_s|x_m;\alpha,\lambda) = \frac{\hat{\lambda}\hat{\alpha}\left(1+\hat{\lambda}x_s\right)^{\hat{\alpha}-1} exp\left\{-\left(1+\hat{\lambda}x_s\right)^{\hat{\alpha}}\right\}}{\Gamma(s-m) exp\left\{-\left(1+\hat{\lambda}x_m\right)^{\hat{\alpha}}\right\}} \left[\left(1+\hat{\lambda}x_s\right)^{\hat{\alpha}} - \left(1+\hat{\lambda}x_m\right)^{\hat{\alpha}}\right]^{s-m-1} \quad (3.8)$$

Then, the $100(1-\tau)\%$ highest conditional density (HCD) prediction limits for $X_{U(s)}$ are given by

$$L_{HCD} = (1+\upsilon_1)x_m \; and \; U_{HCD} = (1+\upsilon_2)x_m \quad (3.9)$$

where υ_1 and υ_2 are the simultaneous solution of the following equations:

$$\int_{(1+\upsilon_1)x_m}^{(1+\upsilon_2)x_m} \frac{\hat{\lambda}\hat{\alpha}\left(1+\hat{\lambda}x_s\right)^{\hat{\alpha}-1} exp\left\{-\left(1+\hat{\lambda}x_s\right)^{\hat{\alpha}}\right\}}{\Gamma(n-m) exp\left\{-\left(1+\hat{\lambda}x_m\right)^{\hat{\alpha}}\right\}} \left[\left(1+\hat{\lambda}x_s\right)^{\hat{\alpha}} - \left(1+\hat{\lambda}x_m\right)^{\hat{\alpha}}\right]^{s-m-1} dx_s = 1-\tau \quad (3.10)$$

and

$$\hat{f}\left((1+\upsilon_1)x_m|x_m\right) = \hat{f}\left((1+\upsilon_2)x_m|x_m\right) \quad (3.11)$$

we can simplify the eq. (3.11) as follows

$$\left[\frac{1+\hat{\lambda}(1+\upsilon_1)x_m}{1+\hat{\lambda}(1+\upsilon_2)x_m}\right]^{\alpha-1} \left[\frac{\left(1+\hat{\lambda}(1+\upsilon_1)x_m\right)^{\hat{\alpha}} - \left(1+\hat{\lambda}x_m\right)^{\hat{\alpha}}}{\left(1+\hat{\lambda}(1+\upsilon_2)x_m\right)^{\hat{\alpha}} - \left(1+\hat{\lambda}x_m\right)^{\hat{\alpha}}}\right]^{s-m-1} \times \left[\frac{e^{-\left[1+\hat{\lambda}(1+\upsilon_1)x_m\right]^{\hat{\alpha}}}}{e^{-\left[1+\hat{\lambda}(1+\upsilon_2)x_m\right]^{\hat{\alpha}}}}\right] = 1 \quad (3.12)$$

Using the numerical solution of the equations (3.10) and (3.12) yield the values υ_1 and υ_2, and then the prediction limits (L_{HCD}, U_{HCD}) are obtained from equations in (3.9).

3.3 Bayesian prediction method

The Bayesian predictive density function of $X_{U(s)}$ for given the past (m) records, is

$$q\left(x_s|\underline{x}\right) = \int_{\theta} f(x_s|x_m;\theta)\pi\left(\theta|\underline{x}\right)d\theta \quad (3.13)$$

where $f(x_s|x_m;\theta)$ is the conditional density function as provided in (3.7), and $\pi\left(\theta|\underline{x}\right)$ the posterior density function. Thus, the predictive density function of x_s given the observed past (m) records $\underline{x}$ for $NH(\alpha,\lambda)$ distribution is

$$q(x_s|\underline{x}) = \int_\alpha \int_\lambda \frac{\lambda^{c+m}\alpha^{a+m}(1+\lambda x_s)^{\alpha-1}}{j_0\Gamma(s-m)} e^{-(b\alpha+d\lambda+(1+\lambda x_s)^{\alpha})} [(1 + \lambda x_s)^{\alpha} - (1+\lambda x_m)^{\alpha}]^{s-m-1} \prod_{i=1}^{m} (1+\lambda x_i)^{\alpha-1}\, d\alpha d\lambda \quad (3.14)$$

The Bayes point prediction of the s^{th} upper record value based on the squared error loss function is given by

$$\hat{x}_s = E(x_s|\underline{x}) = \int_{x_m}^{\infty}\int_0^{\infty}\int_0^{\infty} \frac{\lambda^{c+m}\alpha^{a+m}x_s(1+\lambda x_s)^{\alpha-1}}{K\Gamma(s-m)} e^{-(b\alpha+d\lambda+(1+\lambda x_s)^{\alpha})} [(1+\lambda x_s)^{\alpha} - (1+\lambda x_m)^{\alpha}]^{s-m-1} \prod_{i=1}^{m} (1+\lambda x_i)^{\alpha-1}\, d\alpha d\lambda dx_s \quad (3.15)$$

To make a prediction interval for $X_{U(s)}$based on upper record values, we need to derive Bayesian prediction bounds for $X_{U(s)}$ by evaluating $P(X_{U(s)} \geq \delta|\underline{x})$, where δ is a positive value, as follows

$$P(X_{U(s)} \geq \delta|\underline{x}) = \int_{\delta}^{\infty} q(x_s|\underline{x})dx_s \quad (3.16)$$

The Bayesian predictive bounds of a two-sided interval with cover τ, for the future upper record value $X_{U(s)}$, is such that $P[L_B < X_{U(s)} < U_B] = \tau$, where L_B and U_B are the lower and upper Bayesian predictive bounds, which can be obtained by solving the following two equations:

$$P(X_{U(s)} > L_B|\underline{x}) = \frac{(1+\tau)}{2} \quad (3.17)$$

and

$$P(X_{U(s)} > U_B|\underline{x}) = \frac{(1-\tau)}{2} \quad (3.18)$$

where $P(X_{U(s)} > L_B|\underline{x})$ and $P(X_{U(s)} > U_B|\underline{x})$ are given by (3.16) after replacing δ by L_B and U_B, respectively. It is not possible to obtain the solutions analytically. Therefore, the numerical integration procedures are required to solve the above two equations to obtain L_B and U_B.

4. APPLICATION TO REAL DATA

To illustrate the practical usefulness of the proposed procedures in this paper, we consider the following real data set which represent the total annual rainfall (in inches) during the month of January from 1880 to 1916 recorded at Los Angeles Civic Center (see the website of Los Angeles Almanac: www.laalmanac.com/weather/we08aa.htm). These data are, *1.33, 1.43, 1.01, 1.62, 3.15, 1.05, 7.72, 0.2, 6.03, 0.25, 7.83, 0.25, 0.88, 6.29, 0.94, 5.84, 3.23, 3.7, 1.26, 2.64, 1.17, 2.49, 1.62, 2.1, 0.14, 2.57, 3.85, 7.02, 5.04, 7.27, 1.53, 6.7, 0.07, 2.01, 10.35, 5.42, 13.3.*

To check the validity of NH model to fit this data, the Kolmogorov-Smirnov (K-S) goodness of fit test is used based on MLEs ($\hat{\alpha} = 1.233$ and $\hat{\lambda} = 0.217$). The result of Kolmogorov-Smirnov test is $K - S = 0.0977$ with $p - value = 0.872$. Thus, the NH model provides a good fit to this data. This can be also concluded through the straight line pattern of Quantile-Quantile (Q-Q) plot of MLEs in Fig. 1. Now, the following eight upper record values are extracted from the previous data set: 1.33, 1.43, 1.62, 3.15, 7.72, 7.83, **10.35, 13.3**.

In order to estimate the unknown parameters α and λ, the first six records (m=6) are considered as the observed upper record values, while the two remains record values will be predictable via ML and Bayes methods. Using the previous six upper record values, the ML estimates of α and θ are $\hat{\alpha}_{ML} = 0.875$ and $\hat{\lambda}_{ML} = 1.052$. The Lindley approximation of the Bayes estimates of α and λ under the SE loss function for the hyper-parameters ($a = 1, b = 7, c = 3, d = 1$) are $\hat{\alpha}_{BS} = 0.901$ and $\hat{\lambda}_{BS} = 0.255$. To assess the performance of these estimators, the empirical and fitted cdf is plotted using maximum likelihood and Bayes estimates in Figure 2. These plots have shown that the Bayes estimators provide a better fit than the maximum likelihood estimators.

In the previous upper record sample, only the first six record values are considered as observed records and the last two records as unseen. Then the first six upper record values are used to predict the future 8th upper record value of rainfall. The maximum likelihood point prediction for the 8th upper record value is (8.739) and the highest conditional interval with 95% confidence level is (7.899, 11.287). The Bayesian point prediction for the 8th upper record value is (12.182) and the 95% prediction interval is (12.587, 13.338).

It is clear that the Bayesian predictions of the 8th record are much better than both maximum likelihood and highest conditional interval predictions. Also, it is notable that the 95% Bayesian prediction interval for the 8th upper record contains the true value of the 8th upper record.

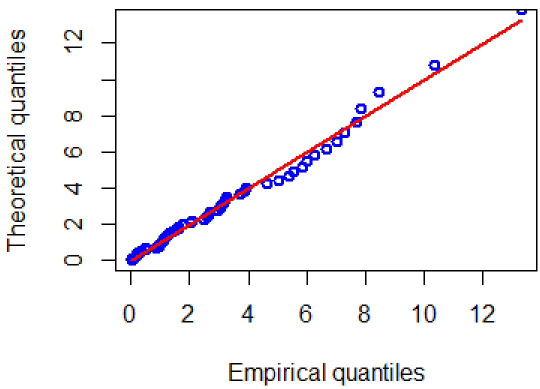

Fig. 1: Quantile-Quantile(Q-Q) Plot for Rainfall Data using MLEs are $\hat{\alpha} = 1.233$ and $\hat{\lambda} = 0.217$.

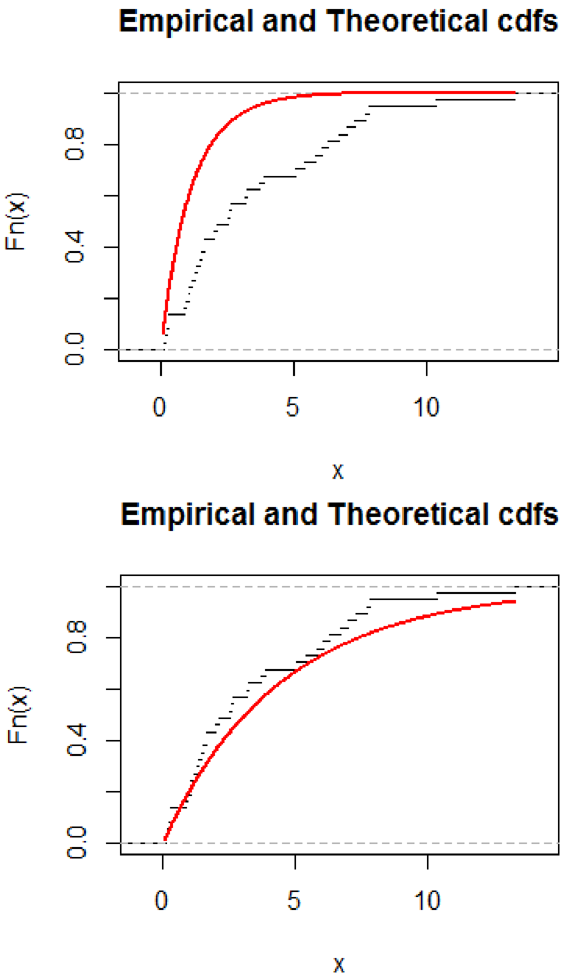

Fig. 2: Empirical and Fitted cdf for Rainfall Data using MLEs (Upper Panel); Empirical and Fitted cdf for Rainfall Data using Bayes Estimates (Bottom Panel).

5. NUMERICAL EXAMPLE

The discussed procedures in this paper were implemented using the MathCAD (2001) program. The simulation data from NH(α, λ) distribution for each combination of $\alpha = 0.5, 1, 1.5$ and $\lambda = 0.5, 1.5$ are generated using the transformation $x = \frac{(1-ln(1-u))^{1/\alpha}-1}{\lambda}, 0 \leq u \leq 1,$ where u is uniform random variable. Subsequently, the first 12^{th} upper record values are listed in Table 1. Finally, the percentage errors (PE) are computed to assess the performance of the estimators by formula, $EP = \frac{|estimate\ value - exact\ value|}{|exact\ value|} 100\%.$

The Bayesian estimation and prediction are obtained under the squared error (SE) loss function using informative and non-informative priors, when $(a, b, c, d > 0)$ and $(a = b = c = d = 0.0001)$, respectively. The results of the ML and Bayes estimates for the parameters α and λ along with the corresponding percentage errors (PE) are shown in Tables 2 and 3. Also, the results of the Bayesian and non-Bayesian predictions for the future upper record value both point and interval predictions along the corresponding percentage errors are shown in Tables 4 and 5.

6. RESULTS AND DISCUSSION

From Tables 2 and 3 we observed that; while the PE of the Bayes estimates with non-informative priors for the shape parameter α are smaller than PE of maximum likelihood estimates, the PE of maximum likelihood estimates for the scale parameter λ are smaller than PE of Bayes estimates with non-informative prior. However, the PE of Bayes estimates under informative prior for both parameters are smaller compared to the others. Moreover, the performances of all estimators are improved when the sample size increases. As may be seen from Tables 4 and 5 that, the Bayes point prediction for the future upper record value under informative as well as non-informative have smaller percentage error than that for maximum likelihood predicted values. The width of the Bayesian prediction interval is shorter as compared to highest conditional prediction interval. Also, according to the percentage errors of predicted values, the performances of all predictors are improved when the sample size increases. Lastly, the Bayesian method to both of estimating the parameters and prediction of future record values are superior to maximum likelihood method for NH distribution. More work is needed in this direction.

Table 1
Samples of Upper Record Values for Different Parameter Values

λ	α	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}	y_{11}	y_{12}
0.5	0.5	5.227	12.268	24.633	35.497	48.677	58.365	85.999	105.747	149.696	180.206	241.282	264.31
	1	0.964	2.541	3.212	7.451	10.684	11.273	13.383	14.803	16.893	18.891	21.573	22.159
	1.5	0.678	1.105	1.155	3.224	4.009	4.533	4.968	6.125	6.392	6.956	7.044	7.537
1.5	0.5	0.797	2.77	5.334	25.173	26.145	28.697	38.774	46.389	62.658	77.075	91.951	96.607
	1	0.45	1.061	1.370	1.587	1.797	2.027	2.975	3.338	3.758	4.307	4.658	5.233
	1.5	0.244	0.523	0.705	1.136	1.582	1.782	2.079	2.173	2.451	2.677	2.972	3.196

Table 2

ML and Bayes Estimates for α and λ and the Corresponding Percentage Errors (in the Parentheses) when $\lambda = 0.5$, and Hyper Parameters ($a = 2, b = 4, c = 2, d = 5$)

α	M	MLEs		Non-informative Bayes		Informative Bayes	
		$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\alpha}_{LB}$	$\hat{\lambda}_{LB}$	$\hat{\alpha}_{LB}$	$\hat{\lambda}_{LB}$
0.5	8	0.739 (47.866%)	0.175 (65.057%)	0.709 (41.751%)	0.161 (67.856)	0.695 (38.991%)	0.206 (58.778%)
	9	0.624 (24.819%)	0.258 (48.398%)	0.605 (20.91%)	0.243 (51.489%)	0.6 (19.901%)	0.303 (39.464%)
	10	0.615 (22.957%)	0.266 (46.794%)	0.598 (19.662%)	0.252 (49.636%)	0.594 (18.883%)	0.308 (38.405%)
1	8	1.166 (16.575%)	0.377 (24.648%)	1.106 (10.569%)	0.343 (31.462%)	1.045 (4.521%)	0.361 (27.707%)
	9	1.113 (11.279%)	0.409 (18.159%)	1.064 (6.442%)	0.376 (24.733%)	1.02 (2.038%)	0.39 (21.902%)
	10	1.093 (9.327%)	0.421 (15.729%)	1.052 (5.213%)	0.391 (21.817%)	1.017 (1.667%)	0.403 (19.494%)
1.5	8	1.128 (24.784%)	0.981 (96.299%)	1.071 (28.61%)	0.893 (78.546%)	1.141 (23.933%)	0.591 (18.293%)
	9	1.39 (7.32%)	0.662 (32.446%)	1.325 (11.659%)	0.610 (21.916%)	1.393 (7.133%)	0.548 (9.570%)
	10	1.421 (5.263%)	0.632 (26.432%)	1.363 (9.157%)	0.587 (17.452%)	1.424 (5.067%)	0.542 (8.414%)

Table 3

ML and Bayes estimates for α and λ and the Corresponding Percentage Errors (in the Parentheses) when $\lambda = 1.5$, and Hyper Parameters ($a = 2, b = 4, c = 3, d = 1$)

α	M	MLEs		Non-informative Bayes		Informative Bayes	
		$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\alpha}_{LB}$	$\hat{\lambda}_{LB}$	$\hat{\alpha}_{LB}$	$\hat{\lambda}_{LB}$
0.5	8	0.596 (19.166%)	0.852 (43.216%)	0.576 (15.111%)	0.798 (46.79%)	0.571 (14.235%)	1.32 (12.006%)
	9	0.545 (8.924%)	1.089 (27.367%)	0.53 (6.028%)	1.04 (30.685%)	0.529 (5.718%)	1.671 (11.369%)
	10	0.536 (7.261%)	1.132 (24.545%)	0.524 (4.833%)	1.087 (27.554%)	0.523 (4.622%)	1.69 (12.672%)
1	8	1.463 (46.296%)	1.049 (30.05%)	1.383 (38.253%)	0.955 (36.302%)	1.274 (27.37%)	1.111 (25.948%)
	9	1.388 (38.767%)	1.136 (24.287%)	1.323 (32.30%)	1.045 (30.301%)	1.244 (24.397%)	1.196 (20.289%)
	10	1.242 (24.219%)	1.371 (8.592%)	1.193 (19.332%)	1.273 (15.165%)	1.143 (14.317%)	1.436 (4.283%)
1.5	8	2.017 (34.435%)	0.905 (39.641%)	1.897 (26.46%)	0.827 (44.844%)	1.658 (10.512%)	0.931 (37.917%)
	9	1.68 (11.987%)	1.197 (20.179%)	1.597 (6.491%)	1.104 (26.382%)	1.469 (2.051%)	1.227 (18.174%)
	10	1.634 (8.908%)	1.246 (16.914%)	1.564 (4.273%)	1.159 (22.702%)	1.463 (2.483%)	1.273 (15.114%)

Table 4
The Bayesian and Non-Bayesian Predictions for the Future sth Upper Record Value and the Corresponding Percentage Errors (in the Parentheses), when $\lambda = 0.5$

α	m, s	Non-Bayesian Predictions		Non-Informative Bayes		Informative Bayes	
		$\hat{x}_s$	L_{HCD}, U_{HCD}	$\hat{\hat{x}}_s$	L_B, U_B	$\hat{\hat{x}}_s$	L_B, U_B
0.5	8,10	117.989 (34.525)	105.747, 192.094	148.421 (17.638)	133.744, 137.378	155.11 (13.93)	139.693, 143.921
	9,11	168.641 (30.106)	149.773, 214.724	200.512 (16.897)	191.21, 196.352	204.21 (15.39)	197.6, 203.361
	10,12	201.723 (23.68)	181.156, 259.374	229.726 (13.085)	163.02, 228.833	230.29 (12.87)	162.34, 230.845
1	8, 10	15.867 (16.009)	14.803, 22.458	20.512 (8.581)	13.628, 18.286	20.223 (7.051)	13.628, 18.286
	9, 11	18.118 (16.015)	16.893, 24.972	20.783 (3.662)	15.692, 20.378	22.231 (3.050)	15.552, 20.821
	10, 12	20.205 (8.818)	18.891, 27.15	22.825 (3.006)	17.678, 22.351	22.637 (2.157)	17.544, 22.751
1.5	8,10	6.614 (4.924)	6.125, 9.378	7.842 (12.739)	5.661, 7.457	7.428 (6.786)	5.739, 7.171
	9,11	6.784 (3.687)	6.392, 8.942	7.878 (11.838)	5.974, 7.559	7.561 (7.337)	6.036, 7.34
	10,12	7.353 (2.446)	6.956, 9.411	8.359 (10.907)	6.549, 8.072	8.087 (7.295)	6.605, 7.88

Table 5
The Bayesian and Non-Bayesian Predictions for the Future sth Upper Record Value and the Corresponding Percentage Errors (in the Parentheses) when $\lambda = 1.5$

α	m, s	Non-Bayesian Predictions		Non-Informative Bayes		Informative Bayes	
		$\hat{x}_s$	L_{HCD}, U_{HCD}	$\hat{\hat{x}}_s$	L_B, U_B	$\hat{\hat{x}}_s$	L_B, U_B
0.5	8, 10	52.485 (31.904)	46.28, 67.08	64.443 (16.39)	59.054, 60.662	67.801 (12.032)	62.473, 64.547
	9,11	71.303 (22.455)	63.44, 89.11	81.324 (11.56)	78.170, 80.143	82.362 (10.429)	80.918, 83.286
	10,12	87.213 (9.724)	78.04, 110.36	92.06 (4.707)	93.303, 95.396	90.597 (6.221)	95.166, 97.586
1	8,10	3.574 (17.021)	3.338, 4.773	4.200 (2.478)	3.937, 4.005	4.208 (2.292)	3.942, 4.011
	9,11	4.010 (13.921)	3.758, 5.251	4.590 (1.469)	4.346, 4.412	4.597 (1.299)	4.352, 4.418
	10, 12	4.595 (12.192)	4.307, 5.272	5.155 (1.491)	4.914, 4.981	5.163 (1.34)	5.030, 5.112
1.5	8,10	2.286 (17.676)	2.173, 2.941	2.754 (2.876)	2.020, 2.594	2.707 (1.124)	2.016, 2.602
	9,11	2.588 (12.922)	2.451, 3.307	2.957 (0.516)	2.299, 2.862	2.965 (0.226)	2.296, 2.870
	10, 12	2.820 (11.768)	2.677, 3.532	3.157 (1.213)	2.529, 3.071	3.165 (0.984)	2.527, 3.077

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REFERENCES

1. Ahmadi, J., Jozani, M.J., Marchand, É. and Parsian, A. (2009). Bayes estimation based on k-record data from a general class of distributions under balanced type loss functions. *Journal of Statistical Planning and Inference*, 139(3), 1180-1189.
2. Ahsanullah, M. (2004). *Record values--theory and applications*: University Press of America.
3. Arnold, B., Balakrishnan, N. and Nagaraja, H. (1998). *Records*, John Wiley and Sons. New York.
4. Asgharzadeh, A., Valiollahi, R. and Abdi, M. (2016). Point and interval estimation for the logistic distribution based on record data. *SORT-Statistics and Operations Research Transactions*, 1(1), 89-112.
5. Badr, M. (2015). Mixture of exponentiated Frechet distribution based on upper record values. *Journal of American Science*, 11(7), 54-64.
6. Basak, P. and Balakrishnan, N. (2003). Maximum likelihood prediction of future record statistic. In *Mathematical and Statistical Methods in Reliability: World Scientific*, 159-175.
7. Chandler, K. (1952). The distribution and frequency of record values. *Journal of the Royal Statistical Society. Series B (Methodological)*, 220-228.
8. Hussian, M. and Amin, E.A. (2014). Estimation and prediction for the Kumaraswamy-inverse Rayleigh distribution based on records. *International Journal of Advanced Statistics and Probability*, 2(1), 21-27.
9. Lindley, D.V. (1980). Approximate bayesian methods. *Trabajos de estadística y de investigación operative*, 31(1), 223-245.
10. MirMostafaee, S.K., Asgharzadeh, A. and Fallah, A. (2016). Record values from NH distribution and associated inference. *Metron*, 74(1), 37-59.
11. Nadarajah, S. and Haghighi, F. (2011). An extension of the exponential distribution. *Statistics*, 45(6), 543-558.
12. Selim, M.A. (2012). Bayesian estimations from the two-parameter bathtub-shaped lifetime distribution based on record values. *Pakistan Journal of Statistics and Operation Research*, 8(2), 155-165.
13. Singh, S., Singh, U., Kumar, M. and Vishwakarma, P. (2014). Classical and Bayesian inference for an extension of the exponential distribution under progressive type-ii censored data with binomial removals. *Journal of Statistics Applications and Probability Letters* 1(3), 75-86.
14. Singh, U., Singh, S.K. and Yadav, A.S. (2015). Bayesian Estimation for Extension of Exponential Distribution Under Progressive Type-II Censored Data Using Markov Chain Monte Carlo Method. *Journal of Statistics Applications & Probability*, 4(2), 275-283.