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THE EXPONENTIATED INVERSE WEIBULL GEOMETRIC DISTRIBUTION

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ABSTRACT

In recent years, there are many efforts to develop new statistical distribution with more flexibility which can be fitted well to complex data. In this paper we develop a compounded distribution of two-parameter inverse Weibull with geometric random variables, so-called the exponentiated inverse Weibull geometric (EIWG) distribution. We study the properties of the EIWG distribution such as moments, moment generating function, mean deviations, Bonferroni and Lorenz curves, entropies and order statistics, and obtain the maximum likelihood estimations of the parameter of the distribution using EM algorithm and Bayesian estimations using Gibbs sampler. Finally, the EIWG distribution is applied to model a simulated and real failure data, with appropriate model comparison with the competing models.

KEY WORDS

Bayesian estimation, Exponentiated inverse Weibull geometric (EIWG) distribution, Gibbs sampler, Inverse Weibull distribution, Maximum likelihood estimation (MLE), Metropolis-Hastings algorithm.

1. INTRODUCTION

The inverse Weibull (IW) distribution is suitable for modelling lifetime data to explain, say the degradation phenomena of mechanical components (Murthy et al., 2004). Erto and Rapone (1984) argued that the physical failure process could lead to IW distribution. They showed that IW distribution provided a goodness of fit to several data given in the literature, such as the times to the breakdown of an insulating fluid subject to the action of a constant tension (Nelson, 1982). Khan et al. (2008) mentioned that the IW distribution could be used to a variety of failure characteristics such as infant mortality, useful lifetime and wear-out periods. Kundu and Howlader (2010) considered the Bayesian approach and prediction of the IW distribution for the type-II censored data. Khan (2009) proposed the probability density function (pdf) and cumulative distribution function (cdf) of inverse generalized exponential distribution and investigated its properties and estimation.

In recent years, many statisticians and reliability engineers try to develop new statistical distributions with more flexibility which can be fitted well to complex data. Often they consider a compounded distribution of one of well-known lifetime distributions and one of discrete distributions. Let the positive random variable Y be the lifetime of a system, defined as $Y = \min_{1\le i\le Z}(Y_i)$ or $Y = \max_{1\le i\le Z}(Y_i)$, where the distribution of Y_i belongs to one of well-known lifetime distributions, and the random variable Z has one of discrete distributions. Then the unconditional marginal distribution of Y is obtained as a new statistical lifetime distribution. Adamidis and Loukas (1998) proposed two-parameter exponential geometric distribution which exhibited the decreasing failure rate property. Barreto-Souza et al. (2011) proposed one generalization of the generalized exponential distribution, referred to as Weibull geometric (WG) distribution, for modeling monotone and unimodal failure rates. Also, Chung and Kang (2014) proposed the exponentiated Weibull geometric (EWG) distribution and studied its properties and estimations.

In this article, we introduce a new exponentiated inverse Weibull geometric (EIWG) distribution that generalizes the IWG and IW distributions in section 2. We further study the properties of the EIWG distribution in section 3. We consider the maximum likelihood and Bayesian estimations in section 4. Simulation data and real data are applied to the EIWG distribution in section 5 and section 6, respectively. Finally, concluding remarks are given in section 7.

2. THE EIWG DISTRIBUTION

Suppose that $\{Y\}_{i=1}^{Z}$ are independent and identically distributed (iid) random variables from the exponentiated inverse Weibull distribution $\mathrm{EIW}(\alpha,\gamma)$ with a shape parameter $\alpha > 0$ and a scale parameter $\gamma > 0$ where z is an integer. Then the corresponding pdf and cdf are given by

$$g(y|\alpha,\gamma) = \alpha\gamma y^{-(\alpha+1)}\exp(-\gamma y^{-\alpha}) \tag{1}$$

and

$$G(y|\alpha,\gamma) = \exp(-\gamma y^{-\alpha}), \tag{2}$$

respectively, for $y > 0$. Clearly, for $\gamma = 1$, it represents the standard inverse Weibull distribution (IW) and for $\alpha = 1$, it represents the exponentiated standard inverse exponential (EIE) distribution. Therefore, the EIW distribution includes the IW distribution as well as the EIE distribution as submodels. Suppose

$$Y = min_{1\le i\le Z}(Y_i), \tag{3}$$

where Y_i's are distributed independently from $\mathrm{EIW}(\alpha,\gamma)$ with pdf as given in (1) for $i = 1,\cdots,Z$. Then the conditional pdf of Y given $Z = z$ is obtained as

$$g_Y(y|z,\alpha,\gamma) = z[1 - G(y|\alpha,\gamma)]^{z-1} g(y|\alpha,\gamma),$$

where $g(y|\alpha,\gamma)$ and $G(y|\alpha,\gamma)$ are expressed in (1) and (2), respectively. Here, we assume that Z is a geometric random variable with its probability function

$$p(z|q) = (1-q)q^{z-1},$$

where $z = 1,2,\cdots$ and $0 < q < 1$. Therefore, the unconditional marginal pdf of Y is given by

$$f_Y(y|q,\alpha,\gamma) = \sum_{z=1}^{\infty} g_Y(y|z,\alpha,\gamma)p(z|q) = \frac{\alpha\gamma(1-q)y^{-(\alpha+1)}\exp(-\gamma y^{-\alpha})}{[1-q(1-\exp(-\gamma y^{-\alpha}))]^2}, y > 0. \tag{4}$$

Here, the proposed pdf of Y in (4) has three-parameters q, α and γ which is called the exponentiated inverse Weibull geometric distribution, denoted by $\mathrm{EIWG}(q, \alpha, \gamma)$. Our proposed exponentiated inverse Weibull geometric distribution brings much more flexibility in modeling various types of hazard functions in the context of reliability and survival analysis. If $\gamma = 1$, the EIWG distribution is called the inverse Weibull geometric (IWG) distribution. Thus the IWG distribution is a submodel among the EIWG distribution. When q approaches to zero, the EIWG distribution goes to the EIW distribution in (1). If $\alpha = 1$, the EIWG distribution is called the exponentiated inverse exponential geometric (EIEG) distribution. Also, when q approaches to zero with $\alpha = 1$, the EIWG distribution goes to the inverse exponential distribution. Therefore, the EIWG distribution is much more flexible than the IWG, EIEG, IW and EIE distributions. Figure 1 shows the EIWG distribution with some parameters α and γ when $q = 0.1, 0.3, 0.5, 0.7, 0.9$.

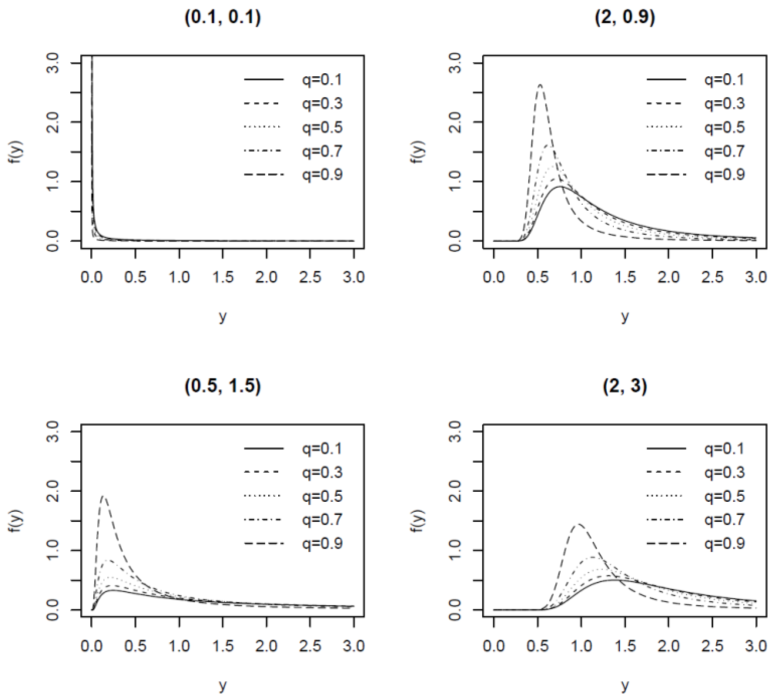

Figure 1: The pdf of the EIWG distribution with some parameters α and γ when $q = 0.1, 0.3, 0.5, 0.7, 0.9$.

Now we develop some important properties of EIWG distribution.

Theorem 1.
The EIWG distribution in (4) is unimodal.

Proof. $\partial f_Y(y|q, \alpha, \gamma)/\partial y = 0$ implies

$$(1-\mathrm{q}+\mathrm{q}e^{-u})\mathrm{u}-(1-\mathrm{q}+\mathrm{q}e^{-u})\frac{\alpha+1}{\alpha}-2que^{-u}=0, \quad (5)$$

where $f_Y(y|q,\alpha,\gamma)$ is in (4), and $u=\gamma y^{-\alpha}$. Let $h(u)=u-\frac{q}{1-q}e^{-u}(u+\frac{\alpha+1}{\alpha})$, then the equation (5) can be expressed as $h(u)=\frac{\alpha+1}{\alpha}$, $\lim_{u\to 0}h(u)<0$ and $\lim_{u\to\infty}h(u)=\infty$. Also $h(u)$ is increasing in $u>0$ since $h'(u)>0$. Since $\frac{\alpha+1}{\alpha}>0$, $h(u)=\frac{\alpha+1}{\alpha}$ has one solution at $u>0$, i.e. $\exists!\ y>0$ such that $u=\gamma y^{-\alpha}>0$. The proof is completed.

Recall that for $|w|<1$ and $k>0$,

$$(1-w)^{-k}=\sum_{j=0}^{\infty}\frac{\Gamma(k+j)}{\Gamma(k)j!}w^j. \quad (6)$$

Expanding $[1-q(1-\exp(-\gamma y^{\alpha}))]^{-2}$ as in (6) and using binomial expansion, we can rewrite the equation (4) as

$$\begin{aligned} f_Y(y|q,\alpha,\gamma) &= \alpha\gamma(1-q)y^{-(\alpha+1)}e^{-\gamma y^{-\alpha}}\sum_{j=0}^{\infty}\frac{\Gamma(2+j)}{\Gamma(2)j!}(q(1-e^{-\gamma y^{-\alpha}}))^j \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{\Gamma(j+2)(-1)^k(1-q)q^j}{k!(j-k)!}\alpha\gamma y^{-(\alpha+1)}e^{-(k+1)\gamma y^{-\alpha}} \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{\Gamma(j+2)(-1)^k(1-q)q^j}{(k+1)!(j-k)!}f(y|\alpha,\gamma_k), \end{aligned} \quad (7)$$

where $f(y|\alpha,\gamma_k)$ is the IW density with parameters α and γ_k, where $\gamma_k=(k+1)\gamma$. Therefore, the EIWG distribution can be expressed as an infinite mixture of IW distribution, and consequently several mathematical properties (e.g. cdf, moments, percentiles, moment generating function, factorial moments, etc.) of the EIWG distribution from the corresponding properties of the IW distribution can be obtained using (7).

3. PROPERTIES OF THE EIWG DISTRIBUTION

Let Y be a random variable distributed as $\mathrm{EIWG}(q,\alpha,\gamma)$ in (4), then the cdf is given by

$$F_Y(y|q,\alpha,\gamma)=\frac{\exp(-\gamma y^{-\alpha})}{1-q(1-\exp(-\gamma y^{-\alpha}))}, y>0. \quad (8)$$

Thus the reliability (or survival) function and the corresponding hazard function are given by

$$S_Y(y|q,\alpha,\gamma)=\frac{(1-q)(1-\exp(-\gamma y^{-\alpha}))}{1-q(1-\exp(-\gamma y^{-\alpha}))} \quad (9)$$

and

$$H_Y(y|q,\alpha,\gamma)=\frac{\alpha\gamma y^{-(\alpha+1)}\exp(-\gamma y^{-\alpha})}{(1-q(1-\exp(-\gamma y^{-\alpha})))(1-\exp(-\gamma y^{-\alpha}))} \quad (10)$$

respectively. Clearly, when q approaches to zero, the hazard function (10) of the EIWG distribution goes to the hazard function of the IW distribution. And, when q approaches to zero with $\alpha=1$, the hazard function (10) of the EIWG distribution goes to the hazard function of the inverse gamma $(1,\gamma)$ distribution. Figure 2 shows hazard functions of the EIWG distribution with some parameters α and γ when $q=0.1, 0.3, 0.5, 0.7, 0.9$. Figure 2 demonstrates tremendous flexibility of hazard functions.

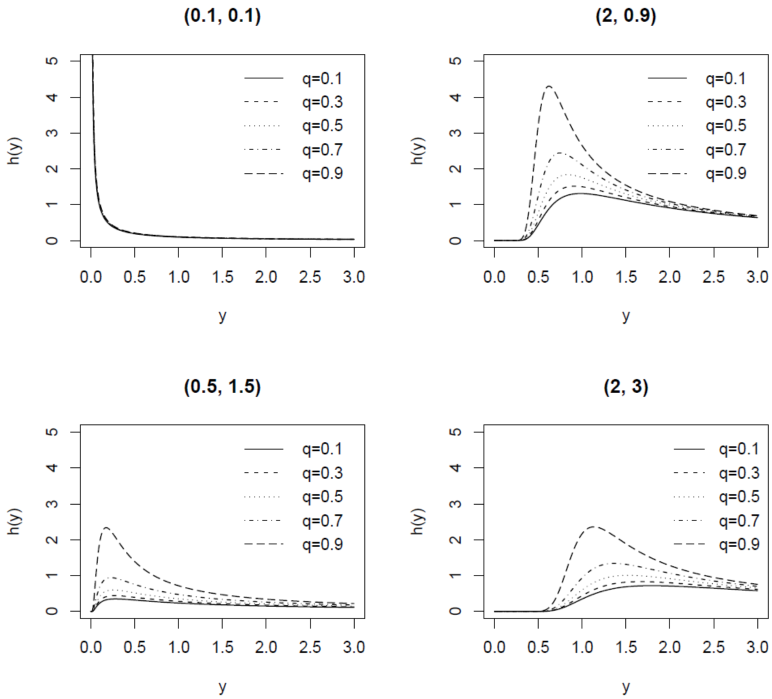

Figure 2: The Hazard Functions of EIWG Distribution with Some Parameters α and γ when q = 0.1, 0.3, 0.5, 0.7, 0.9.

3.1 Quantiles and Moments

The quantile ω (y_ω) of the EIWG distribution follows from (8) as

$$y_\omega = \{\frac{1}{\gamma}\log\frac{1-\omega q}{\omega(1-q)}\}^{-1/\alpha},$$

where the median is $y_{0.5} = \{\frac{1}{\gamma}\log\frac{1-0.5q}{0.5(1-q)}\}^{-1/\alpha}$. And the sth moment of the EIWG distribution is given by, for $s < \alpha$,

$$\begin{aligned} E[Y^s] &= \int_0^\infty y^s f_Y(y|q,\alpha,\gamma)dy \\ &= \sum_{j=0}^\infty \sum_{k=0}^j \frac{\Gamma(j+2)(-1)^k(1-q)q^j\alpha\gamma}{\Gamma(2)k!(j-k)!}\int_0^\infty y^{s-(\alpha+1)}e^{-(k+1)\gamma y^{-\alpha}}\,dy \\ &= \Gamma(1-\frac{s}{\alpha})\sum_{j=0}^\infty \sum_{k=0}^j \frac{\Gamma(j+2)(-1)^k(1-q)q^j}{(k+1)!(j-k)!}(k+1)^{s/\alpha}\gamma^{s/\alpha} \\ &= \frac{\pi}{\sin(\pi\frac{s}{\alpha})\Gamma(\frac{s}{\alpha})}\sum_{j=0}^\infty \sum_{k=0}^j \frac{\Gamma(j+2)(-1)^k(1-q)q^j}{(k+1)!(j-k)!}(k+1)^{s/\alpha}\gamma^{s/\alpha}, \end{aligned} \quad (11)$$

where $\Gamma\left(1-\frac{s}{\alpha}\right)\Gamma\left(\frac{s}{\alpha}\right) = \pi/\sin(\pi\frac{s}{\alpha})$ is Euler's reflection formula and $E[Y^s]$ does not exist for $s \geq \alpha$. Also, the moment generating function of the random variable Y follows from (11) as

$$M_Y[t] = \int_0^\infty e^{ty} f_Y(y|q,\alpha,\gamma)dy$$
$$= \sum_{s=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j} \frac{\pi t^s \Gamma(j+2)(-1)^k (1-q) q^j (k+1)^{s/\alpha} \gamma^{s/\alpha}}{\sin\left(\pi \frac{s}{\alpha}\right)\Gamma\left(\frac{s}{\alpha}\right) s!(k+1)!(j-k)!}, \tag{12}$$

where $\frac{d^s}{dt^s}|_{t=0} M_Y[t] = M_Y^{(s)}[0] = \mathrm{E}(\mathrm{Y}^s)$.

3.2 Mean Deviations

The amount of scatter in a population is evidently measured to some extent by the totality of deviations from the mean and median. These are known as the mean deviation about the mean and the mean deviation about the median as defined respectively by

$$\delta_1(\mathrm{Y}) = \int_0^\infty |y-\mu| f(y)dy \tag{13}$$

and

$$\delta_2(\mathrm{Y}) = \int_0^\infty |y-M| f(y)dy \tag{14}$$

where $\mu = \mathrm{E(Y)}$ and $\mathrm{M} = \mathrm{median(Y)}$. Then the mean deviations of the EIWG distribution for $\delta_1(\mathrm{Y})$ and $\delta_2(\mathrm{Y})$ are as follows :

$$\delta_1(\mathrm{Y}) = \int_0^\infty |y-\mu| f(y)dy$$
$$= \int_0^\mu \mu f(y)dy - \int_0^\mu y f(y)dy + \int_\mu^\infty y f(y)dy - \int_\mu^\infty \mu f(y)dy$$
$$= 2\mu \mathrm{F}(\mu) - 2\mathrm{J}(\mu) \tag{15}$$

and

$$\delta_2(\mathrm{Y}) = \int_0^\infty |y-M| f(y)dy$$
$$= \int_0^M M f(y)dy - \int_0^M y f(y)dy + \int_M^\infty y f(y)dy - \int_M^\infty M f(y)dy$$
$$= \mu - 2\mathrm{J(M)} \tag{16}$$

where $\mathrm{F}(\cdot)$ is the cumulative distribution function and $\mathrm{J(a)} = \int_0^a y f(y)dy$, for $\mathrm{a} > 0$ (see Appendix for more details).

3.3 Bonferroni and Lorenz Curves

The Bonferroni and Lorenz curves for a positive random variable Y are defined as the graphs of the ratios

$$\mathrm{B(F(y))} = \frac{E(Y|Y\le y)}{E(Y)} \tag{17}$$

and

$$\mathrm{L}\big(\mathrm{F(y)}\big) = \frac{E(Y|Y\le y)F(y)}{E(Y)} = B(F(y))F(y), \tag{18}$$

respectively. Writing $\mathrm{p} = \mathrm{F(y)}$, the Lorenz curve has the properties $\mathrm{L(p)} \le \mathrm{p}$, $\mathrm{L}(0) = 0$ and $\mathrm{L}(1) = 1$. Nadarajah (2011) mentioned that if Y represents annual income, L(p) is the proportion of total income that accrues to individuals having the 100p percentage lowest incomes. If all individuals earn the same income then $\mathrm{L(p)} = \mathrm{p}$ for all p. The area between the line $\mathrm{L(p)} = \mathrm{p}$ and Lorenz curve may be regarded as a measure of inequality of income, or more generally, of variability of Y. It follows from (18) that

$$E(Y|Y \le y) = \sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{\Gamma(j+2)(-1)^k(1-q)q^j\alpha\gamma}{F(y)k!(j-k)!}\int_0^y t^{-\alpha}e^{-(k+1)\gamma t^{-\alpha}}dt$$
$$= \sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{\Gamma(j+2)(-1)^k(1-q)q^j(k+1)^{\alpha}\gamma^{\alpha}}{F(y)(k+1)!(j-k)!}[1-\upsilon(1-\alpha^{-1},(k+1)\gamma y^{-\alpha})],$$
$$\alpha > 1. \qquad (19)$$

Hence, the Bonferroni and Lorenz curves are obtained as follows :

$$B(p) = \frac{1}{p\mu}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{\Gamma(j+2)(-1)^k(1-q)q^j(k+1)^{\alpha}\gamma^{\alpha}}{(k+1)!(j-k)!}[1-\upsilon(1-\alpha^{-1},(k+1)\gamma y^{-\alpha})] \qquad (20)$$

and

$$L(p) = B(p) \times p, \qquad (21)$$

respectively. For lifetime models, it is of interest to know what $E(Y^s|Y > y)$ is. Utilizing (9), (11) and changing variables $u = (k+1)\gamma t^{-\alpha}$, we have

$$E(Y^s|Y > y) = \frac{\int_y^{\infty} t^s f(t)dt}{1-F(y)} = \frac{\int_0^{\infty} t^s f(t)dt - \int_0^y t^s f(t)dt}{S(y)}$$
$$= \frac{\pi}{\sin(\pi\frac{s}{\alpha})\Gamma(\frac{s}{\alpha})}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{\Gamma(j+2)(-1)^k(1-q)q^j}{S(y)(k+1)!(j-k)!}(k+1)^{s/\alpha}\gamma^{s/\alpha}$$
$$-\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{\Gamma(j+2)(-1)^k(1-q)q^j\alpha\gamma}{S(y)k!(j-k)!}\int_0^y t^{s-(\alpha+1)}e^{-(k+1)\gamma t^{-\alpha}}dt$$
$$= \sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{\Gamma(j+2)(-1)^k(1-q)q^j(k+1)^{s/\alpha}\gamma^{s/\alpha}}{S(y)(k+1)!(j-k)!}$$
$$\times\left[\frac{\pi}{\sin\left(\pi\frac{s}{\alpha}\right)\Gamma\left(\frac{s}{\alpha}\right)} - \{1-\upsilon(1-s\alpha^{-1},(k+1)\gamma y^{-\alpha})\}\right], s < \alpha. \qquad (22)$$

Note that the mean residual lifetime function is $E(Y|Y > y) - y$.

3.4 Entropies

Entropy is a state function for the explanation of the flow of useless energy in the thermodynamic system. In statistics, it can be thought to be the logarithm of the numbers of microscopic states corresponding to given macroscopic states. So we consider Rneyi entropy defined by $I_R(\tau) = \frac{1}{1-\tau}\log(\int_0^{\infty} f^{\tau}(y)dy)$, where $\tau > 0$ and $\tau \neq 1$. Using the equation (6) and binomial expansion, we have

$$\int_0^{\infty} f^{\tau}(y)dy = \sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{\Gamma(j+2\tau)(-1)^k(1-q)^{\tau}q^j\alpha^{\tau}\gamma^{\tau}}{\Gamma(2\tau)k!(j-k)!}\int_0^{\infty} y^{-\tau(\alpha+1)}e^{-(\tau+k)\gamma y^{-\alpha}}dy$$
$$= \sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{\Gamma(j+2\tau)\Gamma(\tau+\frac{\tau-1}{\alpha})(-1)^k(1-q)^{\tau}q^j\alpha^{\tau-1}}{\Gamma(2\tau)k!(j-k)!(\tau+k)^{\tau+\frac{\tau-1}{\alpha}}\gamma^{\frac{\tau-1}{\alpha}}}. \qquad (23)$$

Utilizing the equation (24), the Renyi entropy is obtained as follows :

$$I_R(\tau) = \frac{1}{1-\tau}\log\left(\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{\Gamma(j+2\tau)\Gamma(\tau+\frac{\tau-1}{\alpha})(-1)^k(1-q)^{\tau}q^j\alpha^{\tau-1}}{\Gamma(2\tau)k!(j-k)!(\tau+k)^{\tau+\frac{\tau-1}{\alpha}}\gamma^{\frac{\tau-1}{\alpha}}}\right),$$

where Shannon entropy is the particular case of $I_R(\tau)$ for $\tau \to 1$.

3.5 Order Statistics and L moments

Let $Y_1, \cdots, Y_n$ be iid random variables where $Y_i \sim EIWG(q, \alpha, \gamma)$, for $i = 1, \cdots, n$. The pdf of the rth order statistic, $Y_{r:n}$ is given by

$$f_{r:n}(y|q,\alpha,\gamma) = \frac{(1-q)^{n-r+1}}{\left(1-q\left(1-e^{-\gamma y^{-\alpha}}\right)\right)^{n+1}} g_{r:n}(y|\alpha,\gamma), y > 0, \quad (24)$$

where $g_{r:n}(y|\alpha,\gamma)$ is the pdf of the rth IW order statistic, which is given by

$$g_{r:n}(y|\alpha,\gamma) = \frac{n!}{(r-1)!(n-r)!} \alpha\gamma y^{-(\alpha+1)} e^{-r\gamma y^{-\alpha}} \left(1 - e^{-\gamma y^{-\alpha}}\right)^{n-r}. \quad (25)$$

Further, we can express the EIWG distribution of $Y_{r:n}$ using equations (6) and (25) as a mixture of IW order statistic densities, as follows :

$$f_{r:n}(y|q,\alpha,\gamma) = (1-q)^{n-r+1} \sum_{k=0}^{\infty} \frac{(n-r+k)!}{(n-r)!k!} q^k g_{r:n+k}(y|\alpha,\gamma). \quad (26)$$

Also the sth moment of the rth order statistic using the equation (27) is given by

$$\begin{aligned} E[Y_{r:n}^s] &= \int_0^\infty y^s f_{r:n}(y|q,\alpha,\gamma) dy \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{n+k-j} \Gamma(1-\tfrac{s}{\alpha})(1-q)^{n-r+1} q^k \gamma^{\frac{s}{\alpha}} (r+j)^{\frac{s}{\alpha}-1} w_{(nkjr)} \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{n+k-j} \frac{\pi(1-q)^{n-r+1} q^k \gamma^{\frac{s}{\alpha}} (r+j)^{\frac{s}{\alpha}-1} w_{(nkjr)}}{\sin(\pi\frac{s}{\alpha})\Gamma(\frac{s}{\alpha})}, \end{aligned} \quad (27)$$

where $w_{(nkjr)} = \frac{(-1)^j (n+k)!(n-r+k)!}{k!j!(n-r)!(r-1)!(n-r+k-j)!}$. And the moment generating function of a random variable $Y_{r:n}$ follows from (27) as

$$\begin{aligned} M_{r:n}[t] &= \int_0^\infty y^{ty} f_{r:n}(y|q,\alpha,\gamma) dy \\ &= \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{n+k-j} \Gamma(1-\tfrac{l}{\alpha}) t^l (1-q)^{n-r+1} q^k \gamma^{\frac{1}{\alpha}} (r+j)^{\frac{l}{\alpha}-1} w_{(nkljr)} \\ &= \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{n+k-j} \frac{\pi t^l (1-q)^{n-r+1} q^k \gamma^{\frac{1}{\alpha}} (r+j)^{\frac{l}{\alpha}-1} w_{(nkljr)}}{\sin(\pi\frac{l}{\alpha})\Gamma(\frac{l}{\alpha})}, \end{aligned} \quad (28)$$

where $w_{(nkljr)} = \frac{(-1)^j (n+k)!(n-r+k)!}{k!l!j!(n-r)!(r-1)!(n-r+k-j)!}$.

L-moments are summary statistics for probability distributions and data samples (Hosking, 1990). They are analogous to ordinary moments but are computed from the linear functions of the ordered data values. The rth L-moment is defined by

$$\begin{aligned} \lambda_r &= \frac{1}{r} \sum_{i=0}^{r-1} \frac{(-1)^i (r-1)!}{i!(r-i-1)!} E[y_{r-i:r}] \\ &= \frac{1}{r} \sum_{i=0}^{r-1} \frac{(-1)^i (r-1)!}{i!(r-i-1)!} \left[\sum_{k=0}^{\infty} \sum_{j=0}^{k+i} \frac{\pi(1-q)^{i+1} q^k \gamma^{\frac{1}{\alpha}} (r-i+j)^{\frac{1}{\alpha}-1} w_{(ikjr)}}{\sin(\frac{\pi}{\alpha})\Gamma(\frac{1}{\alpha})}\right], \end{aligned} \quad (29)$$

where $w_{(ikjr)} = \frac{(-1)^j (r+k)!(i+k)!}{i!k!j!(r-i-1)!(i+k-j)!}$.

4. ESTIMATIONS

4.1 Maximum Likelihood Estimation

Let $y_1, y_2, \cdots, y_n$ denote a sequence of independent random variables from the EIWG distribution (4) with $\theta = (q, \alpha, \gamma)$, then the log likelihood function ℓ for the parameter vector θ is

$$\ell(\theta) = n\log\alpha + n\log\gamma + n\log(1-q) + (\alpha+1)\sum_{i=1}^{n}\log y_i - \sum_{i=1}^{n}\gamma y_i^{-\alpha} - 2\sum_{i=1}^{n}\log(1-q(1-e^{-\gamma y_i^{-\alpha}})). \quad (30)$$

The score functions for θ are given by

$$\frac{\partial\ell}{\partial q} = \frac{n}{q-1} + 2\sum_{i=1}^{n}\frac{\nu_i}{1-q\nu_i}, \quad (31)$$

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}(1-\gamma y_i^{-\alpha})\log y_i - 2\sum_{i=1}^{n}\frac{q\gamma y_i^{-\alpha}\log y_i(1-\nu_i)}{1-q\nu_i} \quad (32)$$

and

$$\frac{\partial\ell}{\partial\gamma} = \frac{n}{\gamma} - \sum_{i=1}^{n}y_i^{-\alpha}\left(1-\frac{2q(1-\nu_i)}{1-q\nu_i}\right), \quad (33)$$

where $\nu_i = 1 - e^{-\gamma y_i^{-\alpha}}$.

The MLE $\hat{\theta}$ of θ is obtained by solving the nonlinear likelihood equations $\frac{\partial\ell}{\partial q} = 0$, $\frac{\partial\ell}{\partial\alpha} = 0$ and $\frac{\partial\ell}{\partial\gamma} = 0$. But, these equations cannot be explicitly solved. So we use the EM algorithm (Dempster et al., 1977; McLachlan and Krishnan, 1997) to obtain $\hat{\theta}$. First, we define the complete log likelihood function $\ell_c(\theta) = \ell_c(q, \alpha, \gamma|Y, Z)$ as follows:

$$\ell_c(\theta) = n\log(\alpha\gamma(1-q)) + \sum_{i=1}^{n}\log(z_i q^{z_i-1}) - (\alpha+1)\sum_{i=1}^{n}y_i - \sum_{i=1}^{n}\gamma y_i^{-\alpha} + \sum_{i=1}^{n}(z_i-1)\log(1-e^{-\gamma y_i^{-\alpha}}). \quad (34)$$

Here, the E-step of EM algorithm is $Q(\theta;\theta_{(k)}) = E[\ell_c(\theta)|Y,\theta_{(k)}]$ which is the conditional expectation of Z given Y and $\theta_{(k)}$, where $\theta_{(k)} = (q_{(k)}, \alpha_{(k)}, \gamma_{(k)})$ is the kth current estimate of θ. Using the conditional density $P(z_i|y_i,\theta) = z_i q^{z_i-1}(1-e^{-\gamma y_i^{-\alpha}})^{z_i-1}\{1-q(1-e^{-\gamma y_i^{-\alpha}})\}^2$, for $i = 1, \cdots, n$, the conditional expectation of Z_i given Y_i and θ is given by

$$E[Z_i|Y_i,\theta_{(k)}] = \frac{1+q_{(k)}(1-e^{-\gamma_{(k)}y_i^{-\alpha_{(k)}}})}{1-q_{(k)}(1-e^{-\gamma_{(k)}y_i^{-\alpha_{(k)}}})}. \quad (35)$$

Utilizing the equations (34) and (35), the E-step is as follows :

$$\begin{aligned} Q(\theta;\theta_{(k)}) &= E[\ell_c(\theta)|Y,\theta_{(k)}] \\ &= n\log(\alpha\gamma(1-q)) - (\alpha+1)\sum_{i=1}^{n}\log y_i - \sum_{i=1}^{n}\gamma y_i^{-\alpha} \\ &\quad + \sum_{i=1}^{n}E[\log z_i|y_i,\theta_{(k)}] + \log q\sum_{i=1}^{n}E[z_i-1|y_i,\theta_{(k)}] \\ &\quad + \sum_{i=1}^{n}E[z_i-1|y_i,\theta_{(k)}]\log(1-e^{-\gamma y_i^{-\alpha}}) \end{aligned} \quad (36)$$

Next, let $\theta_{(k+1)} = argmax\,Q(\theta;\theta_{(k)})$ in (36), then $\theta_{(k+1)} = (q_{(k+1)}, \alpha_{(k+1)}, \gamma_{(k+1)})$ is obtained as follows:

$$q_{(k+1)} = 1 - \mathrm{n}\left(\sum_{i=1}^{n} w_{i(k)}\right)^{-1}, \tag{37}$$

$$\frac{n}{\alpha_{(k+1)}} - \sum_{i=1}^{n} \log y_i \left(1 - \frac{\gamma_{(k+1)} y_i^{-\alpha_{(k+1)}} \left(1 - w_{i(k)} e^{-\gamma_{(k+1)} y_i^{-\alpha_{(k+1)}}}\right)}{1 - e^{-\gamma_{(k+1)} y_i^{-\alpha_{(k+1)}}}}\right) = 0 \tag{38}$$

and

$$\frac{n}{\gamma_{(k+1)}} - \sum_{i=1}^{n} y_i^{-\alpha_{(k+1)}} \left(\frac{1 - w_{i(k)} e^{-\gamma_{(k+1)} y_i^{-\alpha_{(k+1)}}}}{1 - e^{-\gamma_{(k+1)} y_i^{-\alpha_{(k+1)}}}}\right) = 0, \tag{39}$$

where $w_{i(k)} = E[z_i | y_i, \theta_{(k)}]$.

4.2 Bayesian Estimation

We consider the Bayesian estimation of parameters q, α and γ. The prior distributions of q, α and γ would be assumed that independently

$$\pi(\mathrm{q}) = \frac{\Gamma(\mathrm{a}_0 + \mathrm{b}_0)}{\Gamma(\mathrm{a}_0)\Gamma(\mathrm{b}_0)} q^{\mathrm{a}_0 - 1}(1 - q)^{\mathrm{b}_0 - 1}, \tag{40}$$

$$\pi(\alpha) = \frac{b_1^{\mathrm{a}_1}}{\Gamma(\mathrm{a}_1)} \alpha^{\mathrm{a}_1 - 1} e^{-\mathrm{b}_1 \alpha} \tag{41}$$

and

$$\pi(\gamma) = \frac{b_2^{\mathrm{a}_2}}{\Gamma(\mathrm{a}_2)} \gamma^{\mathrm{a}_2 - 1} e^{-\mathrm{b}_2 \gamma}, \tag{42}$$

where $(\mathrm{a}_0, \mathrm{b}_0)$, $(\mathrm{a}_1, \mathrm{b}_1)$ and $(\mathrm{a}_2, \mathrm{b}_2)$ are known constants. The posterior density function for parameters is obtained as follows :

$$\pi(\mathrm{q}, \alpha, \gamma | \mathrm{y}_1, \cdots, \mathrm{y}_n) \propto \mathrm{L}(\mathrm{q}, \alpha, \gamma | \mathrm{y}_1, \mathrm{y}_2, \cdots, \mathrm{y}_n) \times \pi(\mathrm{q})\pi(\alpha)\pi(\gamma)$$

$$\propto \prod_{i=1}^{n} \frac{\alpha\gamma(1-q) y_i^{-(\alpha+1)} \mathrm{e}^{-\gamma y_i^{-\alpha}}}{(1 - q(1 - e^{-\gamma y_i^{-\alpha}}))^2} \times q^{\mathrm{a}_0 - 1}(1 - q)^{\mathrm{b}_0 - 1} \alpha^{\mathrm{a}_1 - 1} e^{-\mathrm{b}_1 \alpha} \gamma^{\mathrm{a}_2 - 1} e^{-\mathrm{b}_2 \gamma} \tag{43}$$

by Bayes theorem, where $\mathrm{L}(\mathrm{q}, \alpha, \gamma | \mathrm{y}_1, \mathrm{y}_2, \cdots, \mathrm{y}_n)$ is the likelihood function of q, α and γ given $\mathrm{y}_1, \mathrm{y}_2, \cdots, \mathrm{y}_n$.

Due to complex form of the likelihood function, the equation (43) cannot be simplified to the closed form. So we can consider the MCMC methods such as Gibbs sampler (Gelfand and Smith, 1990) and Metropolis-Hastings algorithm (Metropolis et al., 1953; Hastings, 1970). Using (3) and the augmented data (Tanner and Wong, 1987), the joint posterior function can be re-expressed as

$$\pi(\mathrm{q}, \alpha, \gamma | \mathrm{y}_1, \cdots, \mathrm{y}_n) \propto \sum_{z_i=1}^{\infty} \cdots \sum_{z_n=1}^{\infty} \phi(\mathrm{q}, \alpha, \gamma, \mathrm{z}_1, \cdots, \mathrm{z}_n | \mathrm{y}_1, \cdots, \mathrm{y}_n),$$

where

$$\begin{aligned} &\phi(\mathrm{q}, \alpha, \gamma, \mathrm{z}_1, \cdots, \mathrm{z}_n | \mathrm{y}_1, \cdots, \mathrm{y}_n) \\ &\quad = \prod_{i=1}^{n} [z_i \left(1 - e^{-\gamma y_i^{-\alpha}}\right)^{z_i - 1} \alpha\gamma y_i^{-(\alpha+1)} e^{-\gamma y_i^{-\alpha}} (1 - q) q^{z_i - 1}] \\ &\qquad \times q^{\mathrm{a}_0 - 1}(1 - q)^{\mathrm{b}_0 - 1} \alpha^{\mathrm{a}_1 - 1} e^{-\mathrm{b}_1 \alpha} \gamma^{\mathrm{a}_2 - 1} e^{-\mathrm{b}_2 \gamma} \end{aligned} \tag{44}$$

Therefore, the full conditional distributions of all parameters based on the complete posterior function in (44) are given by, for $\mathrm{i} = 1, \cdots, \mathrm{n}$,

$$\pi(\mathrm{z}_i|\mathrm{q}, \alpha, \gamma, \mathrm{z}_1, \cdots, \mathrm{z}_{i-1}, \mathrm{z}_{i+1}, \cdots, \mathrm{z}_n, \mathrm{y}_1, \cdots, \mathrm{y}_n) \propto \mathrm{z}_i(q(1-e^{-\gamma y_i^{-\alpha}}))^{\mathrm{z}_i-1}, \qquad (45)$$

$$\pi(\mathrm{q}|\alpha, \gamma, \mathrm{z}_1, \cdots, \mathrm{z}_n, \mathrm{y}_1, \cdots, \mathrm{y}_n) = \mathrm{Beta}(\textstyle\sum_{i=1}^{n} \mathrm{z}_i - n + a_0, \mathrm{n} + b_0), \qquad (46)$$

$$\pi(\alpha|q, \gamma, \mathrm{z}_1, \cdots, \mathrm{z}_n, \mathrm{y}_1, \cdots, \mathrm{y}_n) \propto \prod_{i=1}^{n} \left[\left(1-e^{-\gamma y_i^{-\alpha}}\right)^{z_i-1} e^{-\gamma y_i^{-\alpha}}\right] \times \mathrm{G}(n + a_1, \textstyle\sum_{i=1}^{n} \log y_i + \mathrm{b}_1) \qquad (47)$$

and

$$\pi(\gamma|q, \alpha, \mathrm{z}_1, \cdots, \mathrm{z}_n, \mathrm{y}_1, \cdots, \mathrm{y}_n) \propto \prod_{i=1}^{n} \left(1-e^{-\gamma y_i^{-\alpha}}\right)^{z_i-1} \times \mathrm{G}(n + a_2, \textstyle\sum_{i=1}^{n} y_i^{-\alpha} + \mathrm{b}_2), \qquad (48)$$

where $\mathrm{Beta}(\mathrm{a}, \mathrm{b})$ means the beta distribution with parameters a and b, and $\mathrm{G}(\mathrm{a}, \mathrm{b})$ denotes the gamma density with parameters a and b. For generating z_i from (46), let $\mathrm{z}_i^* = \mathrm{z}_i - 1$. Then the full conditional density of z_i given other parameters becomes the negative binomial(NB) distribution with 2 and $1 - \mathrm{q}(1 - e^{-\gamma y_i^{-\alpha}})$. It can be seen that the full conditional distribution of q in (47) is a beta density with parameters $\sum_{i=1}^{n} \mathrm{z}_i - n + a_0$ and $\mathrm{n} + b_0$, and therefore samples of q can be easily generated using any beta generating routine. However both full conditional distributions of α and γ in (48) and (49) can not be reduced analytically to well-known distrinbutions, respectively and therefore it is impossible to sample directly by a standard methods. As suggested by Chib and Greenberg (1995), a hybrid MCMC algorithm is used by combined a Metropolis-Hastings sampling with Gibbs sampler using the suitable proposal distributions. To generate samples of α and γ from (48) and (49), $\mathrm{G}(n + a_1, \sum_{i=1}^{n} \log y_i + \mathrm{b}_1)$ and $\mathrm{G}(n + a_2, \sum_{i=1}^{n} y_i^{-\alpha} + \mathrm{b}_2)$ distributions are employed as their proposal distributions, respectively. The hybrid MCMC algorithm is working as follows:

Step 1: Start with initial point $(\mathrm{q}^{(0)}, \alpha^{(0)}, \gamma^{(0)})$.

Step 2: Set $\mathrm{j} = 1$

Step 3: For $\mathrm{i} = 1, \cdots, \mathrm{n}$, generate $\mathrm{z}_i^{*(j)}$ from
$\mathrm{NB}(2.1 - \mathrm{q}^{(j-1)}(1 - e^{-\gamma^{(j-1)} y_i^{-\alpha^{(j-1)}}}))$ and set $\mathrm{z}_i^{(j)} = \mathrm{z}_i^{*(j)} + 1$.

Step 4: Generate $\mathrm{q}^{(j)}$ from $\mathrm{Beta}(\sum_{i=1}^{n} \mathrm{z}_i^{(j)} - n + a_0, \mathrm{n} + b_0)$.

Step 5: Using Metropolis-Hastings algorithm, generate $\alpha^{(j)}$ from
$\pi\left(\alpha \middle| q^{(j)}, \gamma^{(j-1)}, \mathrm{z}_1^{(j)}, \cdots, \mathrm{z}_n^{(j)}, \mathrm{y}_1, \cdots, \mathrm{y}_n\right)$ in (48) with gamma proposal distribution, $\mathrm{G}(n + a_1, \sum_{i=1}^{n} \log y_i + \mathrm{b}_1)$.

Step 6: Using Metropolis-Hastings algorithm, generate $\gamma^{(j)}$ from
$\pi\left(\gamma \middle| q^{(j)}, \alpha^{(j)}, \mathrm{z}_1^{(j)}, \cdots, \mathrm{z}_n^{(j)}, \mathrm{y}_1, \cdots, \mathrm{y}_n\right)$ in (49) with gamma proposal distribution, $\mathrm{G}(n + a_2, \sum_{i=1}^{n} y_i^{-\alpha} + \mathrm{b}_2)$.

Step 7: Set $\mathrm{j} = \mathrm{j} + 1$.

Sept 8: Repeat Steps 3-7, N times.

5. SIMULATION STUDY

First, our study in this section is conducted to compare the performances between MLE using EM algorithm and Bayesian estimation using MCMC method. To perform the simulation study, we generated the data set from EIWG distribution with parameters q, α and γ using inverse cdf method. Mean squared error (MSE) criteria is used for comparison purpose. MSEs are computed based on the 10000 repetition under the different values of q, α and γ. Next, we perform the MLE and Bayesian estimation utilizing the previous section 4.1 and 4.2, respectively. In the absence of any strong prior information, a flat prior is used for q, α and γ. That is, set $a_0 = b_0 = 1$, $a_1 = a_2 = 1$ and $b_1 = b_2 = 0$. The joint posterior density is proportional to the likelihood function. As expected, the MLE and the Bayes estimate under flat prior are very similar. The MSE results for all parameters under the different values of q, α and γ are reported in Table 1. Table 1 shows that the Bayesian estimators (BEs) are approximately closer to the pre-assigned values of parameters q, α and γ than the MLEs in terms of MSE values.

Next, we also compared the different models such as Weibull (W), exponentiated inverse Weibull (EIW), inverse Weibull geometric (IWG) and exponentiated inverse Weibull geometric (EIWG) distribution using the Akaike information criterion (AIC) and the Bayesian information criterion (BIC) given as follows :

$$\mathrm{AIC} = 2\mathrm{k} - 2\ln(\mathrm{L}) \text{ and } \mathrm{BIC} = \mathrm{k}\ln(\mathrm{n}) - 2\ln(\mathrm{L}) \quad (49)$$

respectively, where k is the number of parameters in the given model, n is the sample size and L is the maximized value of the likelihood function for the model. Given a set of competing models for the given data, the preferred model is the one with the minimum AIC (or BIC) value. For this performance, we generated a sequence of independent random variables $y_1, \cdots, y_n$ from the EIWG distribution with $q = 0.9$, $\alpha = 3.0$, $\gamma = 2.3$ and $n = 200$. Then, the MLEs of the parameters, $\log L$, AIC and BIC values for the fitted distributions are displayed in Table 2. In general, the best model can be chosen among one of the competing models with the smallest value of AIC (or BIC). So the EIWG distribution is better than the other three distributions in terms of AIC or (BIC) criteria. The histogram of simulated data Y and density plots of the fitted EIWG, IWG, EIW and W distribution are shown in Figure 3.

Table 1
The MSE Results of BE and MLE for all Parameters

True Value		$\hat{q}$ MSE		$\hat{\alpha}$ MSE		$\hat{\gamma}$ MSE	
		BE	MLE	BE	MLE	BE	MLE
$\alpha = 2$, $\gamma = 0.9$	$q = 0.3$	0.0011	0.0118	0.0148	0.0190	0.0042	0.0080
	$q = 0.5$	0.0025	0.0153	0.0209	0.0392	0.0088	0.0469
	$q = 0.7$	0.0034	0.0109	0.0098	0.0373	0.0192	0.0602
$\alpha = 0.5$, $\gamma = 1.5$	$q = 0.3$	0.0006	0.0111	0.0009	0.0014	0.0083	0.0284
	$q = 0.5$	0.0019	0.0108	0.0011	0.0011	0.0067	0.0225
	$q = 0.7$	0.0016	0.0087	0.0011	0.0032	0.0206	0.0648
$\alpha = 2$, $\gamma = 3$	$q = 0.3$	0.0007	0.0135	0.0112	0.0171	0.0320	0.0737
	$q = 0.5$	0.0012	0.0098	0.0182	0.0215	0.0352	0.0802
	$q = 0.7$	0.0013	0.0091	0.0039	0.0195	0.0322	0.1847

Table 2
A Analysis of a Simulation Data for $q = 0.9$, $\alpha = 3.0$ and $\gamma = 2.3$ in EIWG(q, α, γ)

Model	Estimates			log *L*	AIC	BIC
EIWG(q, α, γ)	$\hat{q} = 0.8709$	$\hat{\alpha} = 3.1676$	$\hat{\gamma} = 2.0763$	-16.6378	39.2756	49.1705
IWG(q, α)	$\hat{q} = 0.4527$	$\hat{\alpha} = 4.4497$		-20.3614	44.7227	51.3193
EIW(α, γ)		$\hat{\alpha} = 4.8774$	$\hat{\gamma} = 0.7338$	-24.0117	52.0234	58.6200
W(α, γ)		$\hat{\alpha} = 2.1487$	$\hat{\gamma} = 1.7130$	-139.2266	282.4532	289.0498

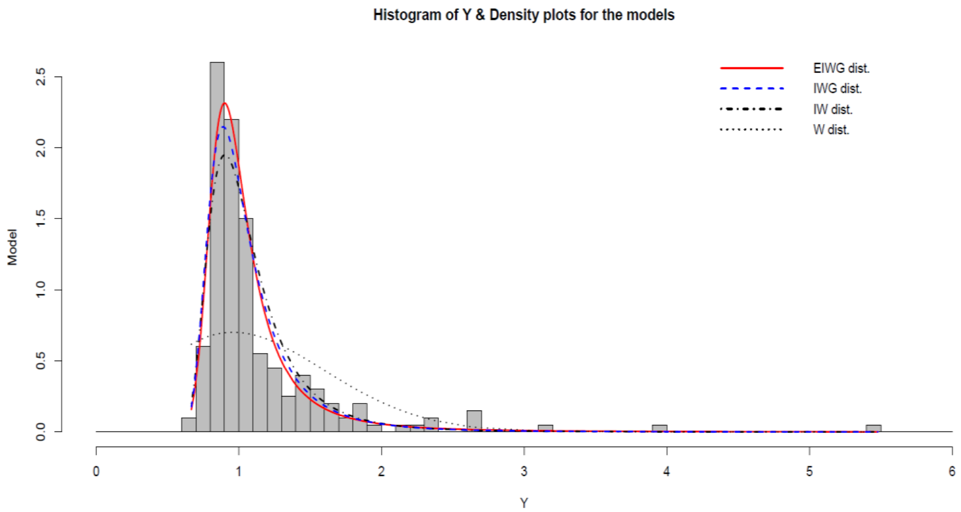

Figure 3: Histogram of Simulation data Y and Density Plots for Different Models

6. APPLICATION TO REAL DATA

In this section, we fit the EIWG models to the conjunctive soil salinity and groundwater data which were gathered in 835 regular surveys on a total of 83 fields over the period 1999-2005 in the Upstream Study Region of the Lower Arkansas River Valley (LARV) in Colorado by Morway and Gates (2012).

We compared the proposed EIWG distribution with several other two parameter lifetime distributions to these data by the method of maximum likelihood. We consider following two parameter lifetime distributions which are defined as

$$\text{IWG}(q, \alpha) : f(y|q, \alpha) = \frac{\alpha(1-q)y^{-(\alpha+1)}e^{-y^{-\alpha}}}{[1-q(1-e^{-y^{-\alpha}})]^2},$$

$$\text{EIEG}(q, \gamma) : f(y|q, \gamma) = \frac{\gamma(1-q)y^{-2}e^{-\gamma y^{-1}}}{[1-q(1-e^{-\gamma y^{-1}})]^2},$$

$$\text{EIW}(\alpha, \gamma) : f(y|\alpha, \gamma) = \alpha\gamma y^{-(\alpha+1)}e^{-\gamma y^{-\alpha}},$$

$$W(\alpha,\gamma) : f(y|\alpha,\gamma) = \alpha\gamma^{-1}y^{\alpha-1}e^{-\gamma^{-1}y^{\alpha}},$$

$$\text{Gamma (G)} : f(y|\alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}y^{\alpha-1}e^{-\beta y},$$

$$\text{Inverse Gamma (IG)} : f(y|\alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}y^{-(\alpha+1)}e^{-\frac{\beta}{y}},$$

$$\text{Log} - \text{normal (LN)} : f(y|\mu,\sigma) = \frac{1}{y\sigma\sqrt{2\pi}}e^{-\frac{(\ln y-\mu)^2}{2\sigma^2}},$$

$$\text{Inverse Gaussian (IGau)} : f(y|\mu_*,\lambda) = (\frac{\lambda}{2\pi y^3})^{\frac{1}{2}}e^{-\frac{\lambda(y-\mu_*)^2}{2\mu_*^2 y}},$$

where $0 < q < 1$, $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\mu \in R$, $\sigma > 0$, $\mu_* > 0$ and $\lambda > 0$. Here $\text{IWG}(q,\alpha)$ is exactly same as $\text{EIWG}(q,\alpha,\gamma = 1)$. The MLEs of the parameters, $\log L$, AIC and BIC for the fitted models are displayed in Table 3. Therefore, EIWG model is best among the competing models in terms of AIC (or BIC).

Table 3
A Analysis of the Conjunctive Soil Salinity and Groundwater Data

Model	Estimates			log L	AIC	BIC
$\text{EIWG}(q,\alpha,\gamma)$	$\hat{q} = 0.9026$	$\hat{\alpha} = 1.7211$	$\hat{\gamma} = 19.6583$	-1497.194	3000.388	3014.570
$\text{IWG}(q,\alpha)$	$\hat{q} = 0.0001$	$\hat{\alpha} = 1.2539$		-2456.069	4916.137	4925.592
$\text{EIEG}(q,\gamma)$	$\hat{q} = 0.9906$		$\hat{\gamma} = 15.9288$	-1502.167	3008.334	3017.789
$\text{EIW}(\alpha,\gamma)$		$\hat{\alpha} = 2.8927$	$\hat{\gamma} = 25.0701$	-1509.375	3022.750	3032.205
$W(\alpha,\gamma)$		$\hat{\alpha} = 1.9354$	$\hat{\gamma} = 19.7550$	-1753.168	3510.336	3519.791
$G(\alpha,\beta)$		$\hat{\alpha} = 4.8238$	$\hat{\beta} = 1.1716$	-1648.770	3301.539	3310.994
$IG(\alpha,\beta)$		$\hat{\alpha} = 6.3100$	$\hat{\beta} = 21.5156$	-1529.890	3063.780	3073.235
$LN(\mu,\sigma^2)$		$\hat{\mu} = 1.3080$	$\widehat{\sigma^2} = 0.1860$	-1509.729	3023.458	3032.912
$\text{IGau}(\mu,\lambda)$		$\hat{\mu} = 4.1172$	$\hat{\lambda} = 19.8429$	-1575.614	3155.228	3164.683

Next, we consider the three different types of testings based on the likelihood ratio test as follows:

Case I : $H_0 : q = 0$ (EIW model) against $H_1 : q \neq 0$ (EIWG model)

Case II : $H_0 : \alpha = 1$ (EIEG model) against $H_1 : \alpha \neq 1$ (EIWG model)

Case III : $H_0 : \gamma = 1$ (IWG model) against $H_1 : \gamma \neq 1$ (EIWG model)

Table 4
Likelihood ratio test statistics for Cases I, II and III

Case		**LRT**	**p − value**
Case I	$\mathrm{H}_0 : q = 0$ vs $\mathrm{H}_1 : q \neq 0$	24.362	8.000×10^{-7}
Case II	$\mathrm{H}_0 : \alpha = 1$ vs $\mathrm{H}_1 : \alpha \neq 1$	9.946	1.612×10^{-3}
Case III	$\mathrm{H}_0 : \gamma = 1$ vs $\mathrm{H}_1 : \gamma \neq 1$	1917.75	1.000×10^{-20}

The values of the correponding LRT and their p-values under the given three different tests are displayed in Table 4. Table 4 says that each H_0 is rejected at significant level $: \alpha = 0.005$. It follows that the EIWG distribution provides a significantly better goodness of fit than the IWG, EIEG and EIW distributions.

The histogram of conjunctive soil salinity and groundwater data and plots of the fitted EIWG, IWG, IW, W, gamma (G), inverse gamma (IG), log-normal (LN) and inverse Gaussian (IGau) densities are shown in Figure 4. As mentioned before, it indicates that the proposed EIWG distribution provides better goodness of fit than the other models. So it can be used to the analysis given a real data set, effectively.

Finally, for the Bayesian inference of a real data set, flat priors are used in the absence of any strong prior information. That is, the prior $\pi(q)$ for q is assumed to be the beta distribution with parameters $a_0 = 1$ and $b_0 = 1$. Each prior for α and γ is gamma distribution with parameters $a_1 = a_2 = 1$ and $b_1 = b_2 = 0$. Then the joint posterior density is proportional to the likelihood function. In order to obtain the Bayes estimates of q, α and γ, we apply the Metropolis-Hasting algorithms into the FCDs (46) - (49) within Gibbs sampler. Here, as mentioned in section 4.2, the suitable gamma distributions are used as the proposed distributions in hybrid MCMC method. The results of MLEs and Bayes estimators for parameters q, α and γ are presented in Table 5. Table 5 shows that the Bayes estimators are approximately closer to parameters in comparison to MLEs and Bayes estimators. And, since standard error is smaller, Bayes estimators are better than MLEs.

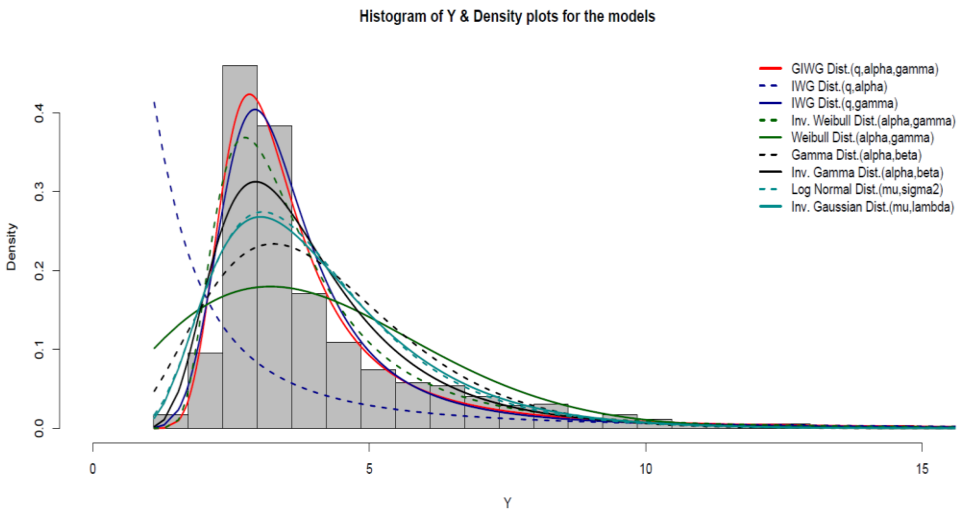

Figure 4: Histogram of a Real Data Y and Density Plots for Different Models

Table 5
MLEs and Bayes Estimators for a Real Data Set

Parameter	$\hat{q}$ mean (S.E.)	$\hat{\alpha}$ mean (S.E.)	$\hat{\gamma}$ mean (S.E.)
MLE	0.9026(0.0554)	1.7211(0.2405)	19.6584(1.8099)
Bayes estimator	0.8548(0.0480)	1.8677(0.1463)	20.4613(1.3310)

The values in parentheses denote the standard errors.

7. CONCLUSIONS

In this paper, we introduced the $\text{EIWG}(q, \alpha, \gamma)$ distribution as an extension to the $\text{EIW}(\alpha, \gamma)$ distribution. We studied some important and useful mathematical properties of the EIWG distribution that could be obtained from the corresponding properties of the EIW distribution. For a real data analysis, it can be observed that the EIWG distribution can be serving as alternative to an EIW distribution, and it is expected that it might work better than the EIW distribution. We expect that the new EIWG distribution will be useful for the practitioners, because of its very flexible hazard function which can accommodate many life time distributions.

Appendix: Simplify $J(\eta)$ in (15) and (16)

Here utilizing the equation (6) and changing variables $u = (k+1)\gamma y^{-\alpha}$, $J(\eta)$ is obtained as follows, for $\eta = \mu$ or M :

$$\begin{aligned} J(\eta) &= \int_0^{\eta} y f(y) dy \\ &= \sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{\Gamma(j+2)(-1)^k (1-q) q^j \alpha\gamma}{k!(j-k)!} \int_0^{\eta} y^{-\alpha} e^{-(k+1)\gamma y^{-\alpha}} dy \\ &= \sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{\Gamma(j+2)(-1)^k (1-q) q^j (k+1)^{\alpha} \gamma^{\alpha}}{(k+1)!(j-k)!} \left[1 - \int_0^{(k+1)\gamma\eta^{-\alpha}} u^{\frac{\alpha-1}{\alpha}-1} e^{-u} du\right] \\ &= \sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{\Gamma(j+2)(-1)^k (1-q) q^j (k+1)^{\alpha} \gamma^{\alpha}}{(k+1)!(j-k)!} \left[1 - \upsilon(1-\alpha^{-1}, (k+1)\gamma\eta^{-\alpha})\right], \alpha > 1, \end{aligned} \quad (50)$$

where $\upsilon(a, b) = \int_0^b u^{a-1} e^{-u} du$ is the lower incomplete gamma function.

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