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BOOTSTRAP PREDICTION INTERVALS FOR TIME SERIES MODELS WITH HETROSCEDASTIC ERRORS

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ABSTRACT

In this paper, we propose two bootstrap procedures to construct prediction intervals for Autoregressive Fractionally Integrated Moving Average with Conditionally Hetroscedastic errors *(ARFIMA-GARCH)* models. The first method is based on the model based bootstrap, in which the order of the model is assumed to be known. The second bootstrap method is based on the idea of approximating the *ARFIMA* part by an *AR* model. In modeling the *ARFIMA-GARCH* model, the first step is to determine the order of *ARFIMA* part and determination the order of *ARFIMA* model is a complicated task. To simplify the model building procedure, we approximate the *ARFIMA* part of the *ARFIMA-GARCH* model by an *AR(p)* model and fit an *AR-GARCH* model instead of *ARFIMA-GARCH* model. The methodology has also been applied to *ARMA-GARCH* models. To check the performance of the proposed methods, we perform simulation study.

KEYWORDS

Bootstrap; Time Series; Prediction interval; Hetroscedastic Errors.

1. INTRODUCTION

In time series analysis a variety of models such as *AR*, *ARMA*, *ARIMA* are used to model the observed series and make predictions. For prediction based on these models, it is typically assumed that the variance of the error terms is constant. The assumption of constant variance of the error term is not realistic for many financial and economic time series. These series show bursts of unusually high volatility and the assumption of a constant variance is not appropriate for such series. However, Engle (1982) found that the classical *ARIMA* model failed to achieve the desired effect of the fitting for UK inflation rate. By carefully studying the sequence of the residuals, they found that the series of the residuals faced the problem of heteroscdasticity. Recently, many researchers have shown that various financial time series exhibit heteroscdasticity. They have found positive relationship between the standard deviation and the level. That is, the sequence of fluctuation remains low with low level of the sequence and the sequence of fluctuation becomes high with increasing sequence of the level. For example, in financial time series, small returns are followed by more small returns (in case when market crashes) and large returns are followed by more large returns in the growth period. In time series analysis forecasting is an important objective. In classical interval forecasting, it is typically assumed that the innovations of the model have some known distribution. In most of the applications, this assumption is not satisfied and the prediction intervals based on it are not valid. To deal with this problem, several bootstrap procedures have been introduced for the construction of prediction intervals in time series analysis e.g. Thombs and Schucany (1990); Masarotto (1990); Grigoletto (1998) Cao et al. (1997); Alanso, A.M, et al. (2002, 2003); Pascual et al. (2004), Clements and Kim (2007), Amjad et al. (2015) among others. These bootstrap prediction intervals have the assumption of homescedasticity for the innovations of the model. In the context of time series models with hetroscedastic errors, Miguel and Olave (1999) proposed a bootstrap method to construct prediction intervals for *ARMA-ARCH* models. This method was further improved by Pascual et al. (2006) by incorporating the parameter variability and applied to construct prediction intervals for *ARMA-GARCH* models. In the current study, we propose two bootstrap procedures to construct prediction intervals for Autoregressive Fractionally Integrated Moving Average (*ARFIMA)* Models with *GARCH* errors. The first method is based on the parametric bootstrap, in which the order of the model is assumed to be known. The second bootstrap method is based on the idea of approximating the *ARFIMA* part by an *AR* model. In modeling the *ARFIMA-GARCH* model, the first step is to determine the order of *ARFIMA* part. Determination of the order of *ARFIMA* model is a complicated task. To simplify the model building procedure, we approximate the *ARFIMA* part of the *ARFIMA-GARCH* model by an *AR(p)* model and fit an *AR-GARCH* model instead of *ARFIMA-GARCH* model.

2. THE MODEL

2.1 The ARMA-*GARCH* Model

The *ARMA-GARCH* model is the combination of the linear *ARMA* model with *GARCH* errors. This is also called the conditional mean and conditional variance model. In most of the applications the *GARCH* model is not directly observed but the innovations of the linear *ARMA* process follows the *GARCH* model. The time series $X_1, X_2, ..., X_n$ follows the *ARMA(p,q)-GARCH(r,s)* model if:

$$\left.\begin{aligned}
& X_t + \sum_{i=1}^{p} \varphi_i X_{t-i} = \sum_{j=1}^{q} \theta_j \varepsilon_{t-j} + \varepsilon_t \\
& or \quad \varphi(B) X_t = \theta(B) \varepsilon_t \\
& \varepsilon_t = z_t \sigma_t \\
& \sigma_t^2 = w + \sum_{i=i}^{r} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{s} \beta_j \sigma_{t-j}^2 \\
& z_t \sim i.i.d. \quad N(0,1)
\end{aligned}\right\}.$$

The *ARMA(p,q)* model for the conditional mean is assumed to be covariance stationary and invertible i.e. the roots of $\varphi(B)$ and $\theta(B)$ lie outside the unit circle.

For the conditional variance model to be stationary $w > 0$, $\alpha_j \geq 0$, $\beta_j \geq 0$ and $\sum_{i=i}^{r} \alpha_i + \sum_{j=1}^{s} \beta_j < 1$.

2.2 The *ARFIMA-GARCH* Model

The *ARFIMA-GARCH* model is obtained by combing the *ARFIMA* model with *GARCH* innovations. The stochastic process X_t, $t \in R$ has *ARFIMA(p,d,q)-GARCH(r,s)* model if it satisfies

$$\varphi(B) X_t = \theta(B)(1-B)^{-d} \varepsilon_t ,$$

$$\varepsilon_t = \sigma_t z_t$$

with $$\sigma_t^2 = w + \sum_{i=i}^{r} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{s} \beta_j \sigma_{t-j}^2$$

$$z_t \sim i.i.d. \;\; N(0,1),$$

where $\varphi(B)$ and $\theta(B)$ are polynomials of order p and q respectively and d is the long memory parameter.

3. BOOTSTRAP PREDICTION INTERVALS FOR *ARFIMA-GARCH* MODELS

The construction of the prediction intervals through bootstrap methods is a nonparametric approach, which does not assume any parametric hypotheses on the error distribution. In the current study, we propose two bootstrap procedures to construct prediction intervals for *ARFIMA-GARCH* models. One is model based bootstrap in which we assume that the model is known and the other is a sieve type bootstrap procedure. These methods are discussed as below.

3.1 The Model Based Bootstrap Method to Construct Prediction Intervals (A1)

In the model based bootstrap method also called the parametric bootstrap method, we assume that the order of the *ARFIMA*(p,d,q)-*GARCH*(r,s) is known. The steps to construct model based bootstrap prediction intervals are outlined as follows.

1) Estimate the fractional difference parameter d for the given series. A number of methods are available to estimate d. Here the semi-parametric local Whittle estimator of d is used.

2) Take the fractional difference of the given series by using d, estimated in step-1. The filtered series thus obtained will follow *ARMA(p,q)-GARCH(r,s)* model.

3) Estimate the *ARMA(p,q)-GARCH(r,s)* model for the filtered series by quasi-maximum likelihood. The vector of estimated parameters is given by

   $$(\hat{\varphi}_1, \ldots, \hat{\varphi}_p, \hat{d}, \hat{\theta}_1, \ldots\ldots, \hat{\theta}_q, \hat{w}, \hat{\alpha}_1, \ldots\ldots, \hat{\alpha}_r, \hat{\beta}_1, \ldots\ldots, \hat{\beta}_s)$$

   where $\hat{d}$ is the estimate of the fractional difference parameter calculated in step 2.

4) Estimate the residuals $\hat{\varepsilon}_t$ from the fitted model and calculate the standardized residuals by

$$\hat{z}_t = \frac{\hat{\varepsilon}_t}{\hat{\sigma}_t} \qquad for \quad t = p+1, \ldots\ldots, n.$$

5) Draw an i.i.d sample from $\hat{G}_{\tilde{\varepsilon}_t}$ denoted by z_t^*, where $\hat{G}_{\tilde{\varepsilon}_t}$ is the empirical distribution function of the centered residuals and is defined by

$$\hat{G}_{\tilde{\varepsilon}}(x) = \frac{1}{n}\sum_{t=1}^{n} I(\tilde{z}_t \le x),$$

where $\tilde{z}_t = \hat{z}_t - \bar{\hat{z}}_t$ and $\bar{\hat{z}}_t = (n-p)^{-1}\sum_{t=p+1}^{n} \hat{z}_t$.

6) Generate the bootstrap sample by recursion

$$\hat{\varphi}(B)(X_t^* - \bar{X}) = \hat{\theta}(B)(1-B)^{-\hat{d}}\,\hat{\varepsilon}_t^*$$

$$\hat{\sigma}_t^{*2} = \hat{w} + \sum_{i=i}^{r} \hat{\alpha}_i \hat{\varepsilon}_{t-i}^{*2} + \sum_{j=1}^{s} \hat{\beta}_j \hat{\sigma}_{t-j}^{*2}$$

$$\hat{\varepsilon}_t^* = \hat{z}_t^* \hat{\sigma}_t^*.$$

Note that, we generate $n+200$ observations by the above recursion in order to stabilize the series and discard the first 200 values.

7) Estimate the model parameter $(\hat{\varphi}_1^*, \ldots, \hat{\varphi}_p^*, \hat{d}^*, \hat{\theta}_1^*, \ldots, \hat{\theta}_q^*, \hat{w}^*, \hat{\alpha}_1^*, \ldots, \hat{\alpha}_r^*, \hat{\beta}_1^*, \ldots, \hat{\beta}_s^*)$ for the bootstrap sample $(X_1^*, X_2^*, \ldots, X_n^*)$. The estimation is done on the same lines as we did in steps (1-3) for the original series.

8) Calculate the h-steps ahead forecast values by using the recursion

$$X_{t+h}^* - \bar{X} = -\sum_{j=1}^{t+h-1} a_j^* (X_{t+h-1}^* - \bar{X}) + \hat{\varepsilon}_{t+h-1}^*$$

$$\hat{\sigma}_{t+h}^{*2} = \hat{w}^* + \sum_{i=i}^{r} \hat{\alpha}_i^* \hat{\varepsilon}_{t+h-i}^{*2} + \sum_{j=1}^{s} \hat{\beta}_j^* \hat{\sigma}_{t+h-j}^{*2}$$

$$\hat{\varepsilon}_{t+h}^* = \hat{z}_{t+h}^* \hat{\sigma}_{t+h}^*.$$

The bootstrap distribution of the predicted values is obtained by repeating steps (5-8) B times.

3.2 The Sieve Bootstrap Approach to Construct Prediction Intervals (A2)

In the model based bootstrap, it was assumed that order of the *ARFIMA* part of *ARFIMA-GARCH* model is known. In practice, identifying the order of *ARFIMA* model is not very simple and leads to wrong inferences if it is not correctly identified. To deal with this complexity, we approximate it by an *AR* model as the identification of an *AR* model is very simple compared to an *ARFIMA* model. Therefore, to construct prediction intervals for *ARFIMA-GARCH* model, we fit an *AR-GARCH* model to the given series. The steps to construct prediction intervals are outlined as follow.

1) Approximate the conditional mean equation of the *ARFIMA-GARCH* model by an *AR* model and determine the order p of the *AR* model.
2) Estimate the parameters of the *AR(p)-GARCH*(r,s) model given by
$$(\hat{\varphi}_1,...,\hat{\varphi}_p,\hat{w},\hat{\alpha}_1,.....,\hat{\alpha}_r,\hat{\beta}_1,.....,\hat{\beta}_s).$$
3) From the model fitted in step 2, calculate the residuals $\hat{\varepsilon}_t$ and the standardized residuals $\hat{z}_t = \dfrac{\hat{\varepsilon}_t}{\hat{\sigma}_t} \quad for \quad t = p+1,......,n$.
4) Define the empirical distribution function $\hat{G}_{\tilde{\varepsilon}_t}$ for the centered standardized residuals as $\hat{G}_{\tilde{\varepsilon}}(x) = \dfrac{1}{n}\sum_{t=1}^{n} I(\tilde{z}_t \le x)$. The centered residuals are given $\tilde{z}_t = \hat{z}_t - \bar{\hat{z}}_t$ where $\bar{\hat{z}}_t = (n-p)^{-1}\sum_{t=p+1}^{n}\hat{z}_t$.
5) The bootstrap sample $(X_1^*, X_2^*,..., X_n^*)$ is generated by the following recursion
$$\hat{\varphi}(B)(X_t^* - \bar{X}) = \hat{\varepsilon}_t^*$$
$$\hat{\sigma}_t^{*2} = \hat{w} + \sum_{i=i}^{r}\hat{\alpha}_i\hat{\varepsilon}_{t-i}^{*2} + \sum_{j=1}^{s}\hat{\beta}_j\hat{\sigma}_{t-j}^{*2}$$
$$\hat{\varepsilon}_t^* = \hat{z}_t^*\hat{\sigma}_t^*,$$
where $\hat{\varphi}(B) = (1-\hat{\varphi}_1 - .... - \hat{\varphi}_p)$ and $X_t^* = X_t \quad for \quad t \le p.$
Using the above recursion, we generate *(n+200)* and remove the first 200 values to minimize the effect of initial values.
6) Fit the *AR(p)-GARCH*(r,s) model to the bootstrap sample $(X_1^*, X_2^*,..., X_n^*)$ generated in step 5 and estimate the model parameters given by $(\hat{\varphi}_1^*,...,\hat{\varphi}_p^*,\hat{w}^*,\hat{\alpha}_1^*,.....,\hat{\alpha}_r^*,\hat{\beta}_1^*,.....,\hat{\beta}_s^*)$.
7) Calculate the h-steps ahead forecast values by using the recursion
$$X_{t+h}^* - \bar{X} = -\sum_{j=1}^{p}\varphi_j^*(X_{t+h-1}^* - \bar{X}) + \hat{\varepsilon}_{t+h-1}^*$$
$$\hat{\sigma}_{t+h}^{*2} = \hat{w}^* + \sum_{i=i}^{r}\hat{\alpha}_i^*\hat{\varepsilon}_{t+h-i}^{*2} + \sum_{j=1}^{s}\hat{\beta}_j^*\hat{\sigma}_{t+h-j}^{*2}$$
$$\hat{\varepsilon}_{t+h}^* = \hat{z}_{t+h}^*\hat{\sigma}_{t+h}^*.$$
The bootstrap distribution of the predicted values is obtained by repeating steps 5-7, *B* times.

4. SIMULATION STUDIES

The finite sample performance of the model based bootstrap and sieve bootstrap methods to construct prediction intervals for *ARFIMA-GARCH* models has been investigated thorough simulation studies. Here, we present results for the following models.

$$M1: \; ARFIMA(0, d, 0) - GARCH(1\,,\,1)$$
$$M2: \; ARFIMA(1, d, 0) - GARCH(1\,,\,1)$$
$$M3: \; ARFIMA(0, d, 1) - GARCH(1\,,\,1)$$
$$M4: \; ARFIMA(1, d, 1) - GARCH(1\,,\,1)$$

The value of the long memory parameter *d* is set to be 0.3 for all the models. For the *ARFIMA* part the values of the autoregressive parameter φ_1 and the moving average parameter θ_1 are fixed to be 0.5 and 0.3 respectively. We also apply the sieve bootstrap approach to construct prediction intervals for *ARMA* models with conditional hetroscedastic errors. The following models are considered in our simulation study.

$$M5: \; ARMA(0.,\, -0.6) - GARCH(1\,,\,1)$$
$$M6: \; ARMA(0.5\,,\, -0.3) - GARCH(1\,,\,1)$$

The parameters of the *GARCH(1,1)* model are taken as $w = 0.05,\ \alpha_1 = 0.10,\ \beta_1 = 0.85$ and for the *ARCH(1)* model, these are set to be $w = 1,\ \alpha_1 = 0.4$. We use two sample sizes 200 and 400 and three different error distributions: the standard normal, the *t*-distribution with 5 degrees of freedom (i.e. leptokurtic one) and the exponential distribution with scale parameter equal to one (i.e. the asymmetric one). The exponential and *t*-distributions are centered and scaled to have zero mean and unit variance. We construct *h* = 1, 3, 5, 10 steps ahead forecast intervals at the nominal coverage level of 90, 95 and 99 percent, but here the results are given for 95 percent level of significance. To evaluate the performance of the prediction intervals, the empirical coverage and the length of the intervals are calculated with their corresponding standard errors.

To check the performance of the model based bootstrap and sieve bootstrap prediction intervals, we calculate the empirical coverage and length of the intervals with corresponding standard errors. The number of simulations is taken as *S=100* and the number of bootstrap resamples *B* is set to be 1000. For each combination of the model, parameters, sample size and innovations distribution, we perform the following steps.

1) Generate a realization of size *n*. Also generate *R=1000* future values of X_{T+h}. These future values are generated conditional on the past *n* values of the generated realization, the true values of the parameters and the true error distribution.

2) Calculate the bootstrap forecast interval $\left[Q^*\left(\frac{1-\beta}{2}\right), Q^*\left(\frac{1+\beta}{2}\right)\right]$ based on B=1000 bootstrap resamples, where $\left[Q^*\left(\frac{1-\beta}{2}\right)\right]$ and $\left[Q^*\left(\frac{1+\beta}{2}\right)\right]$ are the $\left(\frac{1-\beta}{2}\right)$ *th* and $\left(\frac{1+\beta}{2}\right)$ *th* percentiles of the 1000 bootstrap predicted values.

3) Using the true *R=1000* future values, we calculate the empirical coverage of the interval. The empirical coverage (β^*) is obtained as the percentage of R future values that lie in-between $\left[Q^*\left(\frac{1-\beta}{2}\right)\right]$ and $\left[Q^*\left(\frac{1+\beta}{2}\right)\right]$. The length of the interval is calculated as $L^* = Q^*\left(\frac{1+\beta}{2}\right) - Q^*\left(\frac{1-\beta}{2}\right)$..

We repeat the above steps S=100 times and obtain the empirical mean length ($\overline{L}^*$) and the empirical mean coverage $\overline{\beta}^*$ with corresponding standard errors for each of the forecast intervals as follows.

$$\overline{L}^* = S^{-1}\sum_{i=1}^{S} L_i^* \quad SE(\overline{L}^*) = (S\ \ (S-1))^{-1}\sum_{i=1}^{S}(L_i^* - \overline{L}^*)^2)^{1/2}$$

$$\overline{\beta}^* = S^{-1}\sum_{i=1}^{S} \beta_i^*\ ,\ SE(\overline{\beta}^*) = (S\ \ (S-1))^{-1}\sum_{i=1}^{S}(\beta_i^* - \overline{\beta}^*)\ ^2)^{1/2}$$

The results for model 1 to 12 are presented in Tables 1 to 6. Both methods have reasonable coverage for *n=200*, but increases with increasing the sample size to *n=400* as expected. Since constructing prediction intervals by bootstrap are non-parametric methods, therefore, different error distributions have no significant effect on the percentage coverage. This is true for all models and both sample sizes, but in most of the cases length of the prediction interval for t-distribution is a little bit wider than the normal and exponential distribution. While constructing prediction intervals for *ARFIMA-GARCH* models our simulation results reveal that the performance of the Sieve Bootstrap method becomes weaker as the long memory parameter *"d"* approaches *"0.5"* which is its limiting value.

It is very natural as the performance of AR-approximation deteriorates as the model becomes more persistent (Poskitt, 2007). The same nature of performance has also been reported by Amjad et al. (2015) while constructing prediction intervals for ARFIMA models with white noise errors.

5. CONCLUSION

This work is concerned with forecasting of time series models with conditional hetroscedastic errors through bootstrap methods. We considered the long memory *ARFIMA* and short memory *ARMA* models with conditional hetroscedastic errors. In the current study, two bootstrap methods for the construction of prediction intervals have been proposed for *ARFIMA-GARCH* model; the model based bootstrap and the sieve bootstrap. Simulation studies showed that both the methods have good coverage performance. The proposed sieve bootstrap procedure showed good performance when applied to construct prediction intervals for *ARMA*-GARCH models.

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Table 1
Simulation Results for M1

Step-ahead	Sample size	Distr.	A1		A2	
			$\overline{\beta}^*$ (se)	$\overline{L}^*$ (se)	$\overline{\beta}^*$ (se)	$\overline{L}^*$ (se)
1	200	N	92.6(0.069)	4.539(0.0087)	91.9(0.070)	4.546(0.0079)
		t(5)	92.8(0.064)	5.053(0.0101)	93.1(0.065)	5.069(0.0099)
		EXP.	93.1(0.081)	4.460(0.0130)	92.9(0.099)	4.609(0.0086)
	400	N	94.3(0.060)	4.541(0.0055)	94.2(0.074)	4.579(0.0066)
		t(5)	94.5(0.061)	4.458(0.0077)	94.4(0.073)	4.644(0.0074)
		EXP.	93.9(0.073)	4.461(0.0087)	93.8(0.093)	4.548(0.0066)
3	200	N	92.3(0.064)	4.575(0.0086)	92.1(0.060)	5.056(0.0078)
		t(5)	93.2(0.067)	4.637(0.0122)	93.1(0.060)	4.487(0.0095)
		EXP.	92.9(0.104)	4.494(0.0134)	93.0(0.109)	4.212(0.0106)
	400	N	94.8(0.056)	4.561(0.0057)	94.6(0.057)	4.121(0.0063)
		t(5)	94.5(0.062)	4.677(0.0071)	95.0(0.077)	4.737(0.0065)
		EXP.	94.8(0.075)	4.514(0.0081)	94.7(0.066)	4.803(0.0072)
5	200	N	92.1(0.085)	4.679(0.0083)	91.9(0.083)	4.823(0.0799)
		t(5)	92.4(0.094)	4.771(0.0120)	92.3(0.082)	5.073(0.0093)
		EXP.	92.6(0.098)	4.573(0.0132)	92.6(0.097)	4.981(0.0088)
	400	N	94.7(0.055)	4.624(0.0073)	94.1(0.061)	4.646(0.0066)
		t(5)	94.3(0.051)	4.728(0.0069)	94.5(0.059)	5.089(0.0063)
		EXP.	94.5(0.061)	4.995(0.0090)	94.3(0.062)	4.746(0.0074)
10	200	N	92.0(0.098)	4.995(0.0083)	92.1(0.093)	5.282(0.0099)
		t(5)	92.1(0.096)	5.071(0.120)	92.0(0.080)	4.936(0.0103)
		EXP.	92.4(0.097)	4.597(0.0135)	92.1(0.074)	5.238(0.0110)
	400	N	93.8(0.050)	4.512(0.0065)	94.0(0.064)	5.806(0.0050)
		t(5)	94.2(0.052)	5.146(0.0070)	93.9(0.052)	5.809(0.0068)
		EXP.	94.6(0.056)	5.082(0.0081)	93.8(0.044)	4.987(0.0081)

Table 2
Simulation Results for M2

Step-ahead	Sample size	Distr.	A1		A2	
			$\bar{\beta}^*$ (se)	$\bar{L}^*$ (se)	$\bar{\beta}^*$ (se)	$\bar{L}^*$ (se)
1	200	N	93.0(0.071)	4.686(0.0088)	92.9(0.070)	4.756(0.0104)
		t(5)	93.1(0.076)	5.127(0.0105)	93.3(0.068)	4.995(0.0088)
		EXP.	92.9(0.103)	4.552(0.0104)	93.1(0.110)	4.671(0.0135)
	400	N	94.9(0.051)	4.355(0.0054)	94.0(0.051)	4.446(0.0078)
		t(5)	94.6(0.057)	4.470(0.0067)	94.2(0.044)	4.668(0.0084)
		EXP.	94.7(0.062)	4.501(0.0080)	94.3(0.052)	4.474(0.0091)
3	200	N	92.9(0.053)	4.871(0.0080)	92.6(0.068)	4.865(0.0109)
		t(5)	92.8(0.055)	5.163(0.0110)	91.9(0.041)	4.990(0.0098)
		EXP.	92.4(0.090)	4.700(0.0122)	91.9(0.077)	4.839(0.0117)
	400	N	94.8(0.043)	4.348(0.0053)	94.6(0.051)	4.517(0.0081)
		t(5)	94.5(0.052)	4.506(0.0067)	94.6(0.040)	4.440(0.0085)
		EXP.	94.7(0.071)	4.611(0.0090)	94.8(0.066)	4.684(0.0095)
5	200	N	92.7(0.055)	4.996(0.0082)	92.3(0.053)	5.023(0.0108)
		t(5)	92.6(0.084)	5.607(0.0117)	92.2(0.050)	5.224(0.0095)
		EXP.	91.9(0.091)	4.901(0.0127)	91.9(0.073)	4.976(0.0113)
	400	N	94.9(0.043)	4.394(0.0057)	94.4(0.049)	4.459(0.0086)
		t(5)	94.5(0.052)	4.736(0.0071)	94.1(0.041)	4.628(0.0085)
		EXP.	94.4(0.067)	4.706(0.0090)	94.7(0.050)	4.845(0.0093)
10	200	N	93.7(0.061)	5.104(0.0083)	93.5(0.058)	5.150(0.0103)
		t(5)	93.7(0.067)	5.162(0.0105)	93.3(0.044)	5.210(0.0095)
		EXP.	92.9(0.087)	4.966(0.0136)	93.0(0.070)	5.053(0.0119)
	400	N	94.7(0.037)	4.944(0.0058)	94.5(0.042)	4.957(0.0085)
		t(5)	95.0(0.050)	4.858(0.0072)	94.4(0.032)	5.012(0.0085)
		EXP.	94.4(0.089)	4.848(0.0095)	94.9(0.062)	4.969(0.0096)

Table 3
Simulation Results for M3

Step-ahead	Sample size	Distr.	A1		A2	
			$\overline{\beta}^*$ (se)	$\overline{L}^*$ (se)	$\overline{\beta}^*$ (se)	$\overline{L}^*$ (se)
1	200	N	93.5(0.050)	4.331(0.0080)	93.1(0.054)	4.345(0.0085)
		t(5)	93.1(0.051)	4.616(0.0105)	93.2(0.052)	4.782(0.0104)
		EXP.	93.3(0.074)	5.058(0.0114)	93.0(0.073)	4.544(0.0110)
	400	N	94.2(0.039)	4.983(0.0050)	94.1(0.044)	4.971(0.0054)
		t(5)	94.1(0.040)	5.095(0.0071)	94.0(0.042)	5.123(0.0097)
		EXP.	93.9(0.058)	4.994(0.0080)	93.8(0.057)	5.061(0.0089)
3	200	N	92.4(0.055)	4.423(0.0081)	92.3(0.056)	4.575(0.0084)
		t(5)	92.3(0.057)	5.064(0.0110)	92.0(0.054)	4.896(0.0103)
		EXP.	93.6(0.077)	5.083(0.0115)	93.1(0.086)	5.225(0.0117)
	400	N	94.7(0.040)	4.963(0.0053)	94.8(0.053)	4.961(0.0057)
		t(5)	95.0(0.046)	4.934(0.0070)	94.7(0.039)	4.960(0.0099)
		EXP.	94.5(0.050)	5.015(0.0086)	94.4(0.064)	5.041(0.0102)
5	200	N	93.8(0.052)	4.968(0.0088)	93.0(0.051)	4.870(0.0083)
		t(5)	93.5(0.061)	4.850(0.0120)	92.8(0.040)	4.987(0.0106)
		EXP.	92.9(0.060)	5.082(0.0125)	93.2(0.085)	4.456(0.0119)
	400	N	94.4(0.031)	4.996(0.0051)	93.9(0.044)	5.106(0.0056)
		t(5)	94.0(0.031)	5.394(0.0071)	94.3(0.031)	4.199(0.0066)
		EXP.	94.0(0.051)	5.076(0.0087)	94.5(0.072)	5.097(0.0105)
10	200	N	92.8(0.043)	5.392(0.0084)	92.3(0.057)	5.303(0.0091)
		t(5)	92.6(0.048)	4.904(0.0122)	92.2(0.052)	5.105(0.0108)
		EXP.	93.0(0.057)	5.174(0.0132)	92.6(0.090)	4.657(0.0124)
	400	N	94.5(0.025)	4.996(0.0053)	94.0(0.036)	5.365(0.0056)
		t(5)	94.6(0.042)	5.478(0.0073)	94.4(0.043)	5.220(0.0070)
		EXP.	94.7(0.056)	5.078(0.0084)	94.5(0.081)	5.203(0.0100)

Table 4
Simulation Results for M4

Step-ahead	Sample Size	Distr.	A1		A2	
			$\bar{\beta}^*$ (se)	$\bar{L}^*$ (se)	$\bar{\beta}^*$ (se)	$\bar{L}^*$ (se)
1	200	N	93.1(0.056)	4.665(0.0090)	92.9(0.058)	4.702(0.0092)
		t(5)	93.2(0.063)	4.716(0.0120)	93.0(0.066)	4.731(0.0112)
		EXP.	92.7(0.053)	4.505(0.0123)	92.9(0.056)	4.494(0.0117)
	400	N	94.8(0.042)	4.734(0.0056)	94.4(0.044)	4.834(0.0060)
		t(5)	94.5(0.050)	4.838(0.0080)	93.9(0.051)	4.916(0.0078)
		EXP.	94.1(0.037)	4.678(0.0094)	93.9(0.032)	6.747(0.0084)
3	200	N	92.8(0.052)	4.803(0.0093)	92.8(0.060)	4.800(0.0096)
		t(5)	93.1(0.057)	4.940(0.0110)	92.1(0.063)	4.913(0.0120)
		EXP.	92.3(0.055)	4.635(0.0096)	92.1(0.058)	4.733(0.0103)
	400	N	94.8(0.040)	4.839(0.0058)	94.8(0.040)	4.892(0.0058)
		t(5)	94.2(0.053)	4.859(0.0088)	94.1(0.052)	4.982(0.0104)
		EXP.	93.9(0.037)	4.965(0.0085)	93.8(0.036)	4.972(0.0092)
5	200	N	92.7(0.056)	4.928(0.0097)	92.6(0.056)	4.880(0.0097)
		t(5)	93.1(0.067)	4.944(0.0096)	93.0(0.062)	4.919(0.0093)
		EXP.	93.0(0.054)	5.165(0.0099)	93.1(0.055)	5.210(0.0110)
	400	N	94.8(0.043)	4.866(0.0065)	93.8(0.043)	4.896(0.0065)
		t(5)	94.4(0.054)	5.075(0.009)	93.0(0.051)	5.323(0.0090)
		EXP.	94.3(0.040)	5.233(0.0079)	93.9(0.038)	5.164(0.0087)
10	200	N	92.2(0.052)	5.155(0.0098)	92.1(0.052)	4.955(0.0098)
		t(5)	92.7(0.066)	5.038(0.0100)	92.2(0.043)	4.819(0.0081)
		EXP.	92.4(0.054)	5.174(0.0085)	92.8(0.045)	4.963(0.0088)
	400	N	94.1(0.035)	5.355(0.0068)	94.1(0.035)	4.995(0.0068)
		t(5)	94.4(0.047)	5.425(0.0094)	93.0(0.031)	5.323(0.0090)
		EXP.	93.9(0.045)	5.377(0.0086)	93.8(0.040)	5.444(0.0093)

Table 5
Simulation Results for M5

Step-ahead	Distr.	n=200		n=400	
		$\bar{\beta}^*$ (se)	$\bar{L}^*$ (se)	$\bar{\beta}^*$ (se)	$\bar{L}^*$ (se)
1	N	93.5(0.051)	4.783(0.0106)	94.7(0.038)	4.968(0.0076)
	t(5)	93.7(0.054)	4.984(0.0111)	94.9(0.039)	5.097(0.0080)
	EXP.	92.9(0.063)	5.030(0.0107)	94.6(0.033)	5.130(0.0077)
3	N	93.0(0.055)	4.914(0.0109)	94.8(0.034)	5.194(0.0075)
	t(5)	93.7(0.058)	5.219(0.0115)	94.5(0.040)	5.243(0.0082)
	EXP.	94.1(0.063)	5.045(0.0110)	94.6(0.039)	5.230(0.0079)
5	N	92.9(0.056)	4.983(0.0114)	94.4(0.038)	5.238(0.0079)
	t(5)	93.2(0.064)	5.096(0.0122)	95.1(0.046)	5.275(0.0084)
	EXP.	93.3(0.062)	4.361(0.0116)	94.6(0.042)	5.120(0.0081)
10	N	93.0(0.061)	5.127(0.0118)	94.4(0.041)	5.441(0.0083)
	t(5)	92.7(0.069)	5.288(0.0130)	94.7(0.046)	5.400(0.0097)
	EXP.	92.2(0.066)	5.826(0.0114)	94.6(0.043)	5.030(0.0087)

Table 6
Simulation Results for M6

Step-ahead	Distr.	n=200		n=400	
		$\bar{\beta}^*$ (se)	$\bar{L}^*$ (se)	$\bar{\beta}^*$ (se)	$\bar{L}^*$ (se)
1	N	93.5(0.062)	4.653(0.0112)	94.8(0.043)	4.952(0.0083)
	t(5)	93.6(0.070)	4.718(0.0124)	94.7(0.052)	5.073(0.0091)
	EXP.	93.4(0.061)	4.761(0.0116)	95.3(0.051)	4.961(0.0087)
3	N	92.8(0.066)	4.614(0.0114)	94.8(0.045)	4.990(0.0086)
	t(5)	92.9(0.071)	5.024(0.0129)	95.0(0.057)	5.164(0.0101)
	EXP.	92.6(0.063)	5.030(0.0117)	94.9(0.052)	5.161(0.0096)
5	N	91.7(0.065)	5.199(0.0119)	94.4(0.050)	5.233(0.0083)
	t(5)	91.9(0.076)	5.531(0.0136)	94.8(0.058)	5.413(0.0104)
	EXP	92.2(0.070)	5.426(0.0129)	95.3(0.060)	5.361(0.0096)
10	N	90.7(0.073)	5.514(0.0127)	94.5(0.057)	5.517(0.0091)
	t(5)	91.7(0.082)	5.716(0.0145)	94.1(0.060)	5.852(0.0100)
	EXP.	91.1(0.075)	5.425(0.0134)	94.3(0.062)	5.461(0.0098)