Pak. J. Statist.
2016 Vol. 32(6), 497-515

ON THE POSTERIOR DISTRIBUTION OF LOCATION PARAMETER OF LOG-NORMAL DISTRIBUTION

Akbar Ali Khan[1], **Muhammad Aslam**[2], **Zawar Hussain**[1], **Alamgir Khalil**[3] and **Muhammad Tahir**[4]

[1] Department of Statistics, Quaid-i-Azam University, Islamabad Pakistan. Email: akbaraliwazir@yahoo.com; zhlangah@yahoo.com

[2] Department of Basic Sciences, Riphah International University Islamabad, Pakistan. Email: aslamsdqu@yahoo.com

[3] Department of Statistics, University of Peshawar, Peshawar Pakistan. Email: profalamgir@yahoo.com

[4] Department of Statistics, Government College University Faisalabad, Pakistan. Email: tahirqaustat@yahoo.com

ABSTRACT

Bayesian estimation technique is widely used for estimation purposes due to its phenomenal performance in estimating the unknown parameter(s). In this study, Bayesian estimation technique is employed to estimate the location parameter of Log-Normal distribution. Two non-informative priors (Uniform and Jeffreys), two informative priors (Normal and Cauchy) and four loss functions are utilized for this purpose. Prior predictive distributions are used to elicit the prior densities. The main purpose of this study is to search for a suitable prior and loss function for estimation of the location parameter of Log-Normal distribution. Through simulation study, comparisons are made on the basis of the posterior variances, coefficients of skewness, ex-kurtosis and Bayes risks. A real data of failure time is used to verify the simulation results. Numerical results for Cauchy prior and Quadratics Loss function have been obtained by solving the expressions numerically using the “Quadrature” method in “Mathematica 9”.

KEYWORDS

Prior distribution, Normal distribution, Elicitation, Hyper-parameter.

1. INTRODUCTION

The probability density function (pdf) of Log-Normal random variable X is:

$$f(x;\theta,\phi)=\frac{1}{x\sqrt{2\pi\phi}}\,e^{-\frac{1}{2\phi}(\ln x-\theta)^2},\ x>0,\ -\infty<\theta<\infty,\ \phi>0 \qquad (1)$$

where, θ is location and ϕ is scale parameter.

The cumulative distribution function of this distribution is given by:

$$F(x) = \frac{1}{2} + \frac{1}{2} Erf\left(\frac{\ln x - \theta}{\sqrt{2\phi}}\right) \tag{2}$$

The Log-Normal distribution has a wide range of applications in almost every field of life. It is named as "the Cinderella of distributions" by Aitchison and Brown (1957). Log-Normal is the second most used distribution, after the Normal distribution, to model real life phenomena. Extensive literature is available on Log-Normal distribution [for example, see Hill (1963), Tiku (1968), Goldstein (1973), Stedinger (1980), Becker (1991), Singh et al. (1997), Shen (1998), Furusawa et al. (2005), Mehta et al. (2007)]. After the phenomenal advancement in computer programming, Bayesian estimation tools are now widely utilized to estimate parameter(s). Khan et al. (2006), Martin and Perez (2009), Saleem and Aslam (2009b), Saleem and Aslam (2009a), Kifayat et al. (2012), Sindhu and Aslam (2013), Khan et al. (2015) and many others adopted Bayesian estimation techniques in their studies.

Loss function is a mathematical function of the difference between the estiamted and true value. Loss function plays vital role in Bayesian estimation approach. Aslam et al. (2015) used different loss functions for the estimation of parameters of the 3-component mixture of Rayleigh distribution. Sultana and Aslam (2016) considered the estimation of 3-component of inverse Rayleigh distribution in type-I censoring under the squared error loss function (SELF), precautionary loss function (PLF) and DeGroot loss function (DLF). Sindhu et al. (2016) evaluated Bayes estimators under Squared Error Loss Function (SELF), Precautionary Loss Function (PLF), Weighted Squared Error Loss Function (WSELF), Quasi-Quadratic Loss Function (QQLF), and Squared-Log Error Loss Function (SLELF).

Recently, Khan et al. (2015) used different loss functions and priors for the estimation of the scale parameter of Log-Normal distribution. Their study is concerned to look for a best loss function and suitable non-informative prior to estimate the scale parameter. However, they did not consider the estimation of location parameter of the Log-Normal distribution in their study. Therefore, this study aims to estimate the location parameter of the Log-Normal distribution under Bayesian paradigm. In this paper, the posterior distributions for the location parameter θ are derived assuming two non-informative (Uniform and Jeffreys) and two informative (Normal and Cauchy) priors. Bayes estimators (BEs) and Bayes risks (BRs) are obtained using Squared Error Loss Function (SELF), Quadratic Loss Function (QLF), Precautionary Loss Function (PLF) and DeGroot Loss Function (DLF). The hyper-parameters of the priors are elicited using the prior predictive distributions. The results for Cauchy prior and QLF are obtained by solving the mathematical expressions numerically using the Quadrature method.

Performances of priors and the loss functions are checked on the basis of posterior variances, coefficients of skewness, ex-kurtosis and Bayes risks. For the sake of numerical comparisons, simulation studies are carried out by considering different sample sizes and different choices of the scale parameter. The simulation results are verified by a real data set.

2. POSTERIOR DISTRIBUTIONS

Posterior distributions of the location parameter θ are derived assuming two non-informative (Uniform and Jeffreys) and two informative (Normal and Cauchy) priors.

The Likelihood function of the Log-normal distribution can be written as under.

$$L(\theta) = \frac{(2\pi\phi)^{-\frac{n}{2}}}{\prod_{i=1}^{n} x_i} e^{-\frac{1}{2\phi}\sum_{i=1}^{n}(\ln x_i - \theta)^2} \quad (3)$$

2.1 Posterior Distributions under Non-Informative Priors

Firstly, Uniform prior is assumed for the location parameter θ which is defined as:

$$p_U(\theta) \propto 1, \; -\infty < \theta < \infty \quad (4)$$

The posterior distribution of θ given that data $\boldsymbol{x}$ under the Uniform prior is derived as:

$$p(\theta \mid x) \propto p_U(\theta)\, L(\theta)$$

$$\propto 1 \frac{(2\pi\phi)^{-\frac{n}{2}}}{\prod_{i=1}^{n} x_i} e^{-\frac{1}{2\phi}\sum_{i=1}^{n}(\ln x_i - \theta)^2}$$

$$\propto e^{-\frac{1}{2\phi}\sum_{i=1}^{n}\left\{(\ln x_i - \ln G) - (\theta - \ln G)\right\}^2}$$

$$\propto e^{-\frac{1}{2\phi}\sum_{i=1}^{n}\left\{(\ln x_i - \ln G)^2 + (\theta - \ln G)^2 - 2(\theta - \ln G)(\ln x_i - \ln G)\right\}}$$

$$\propto e^{-\frac{1}{2\phi} n(\theta - \ln G)^2}$$

This is the density kernel of the Normal distribution.

Here, $G = e^{\frac{\sum_{i=1}^{n}\ln x_i}{n}}$ is the Geometric mean.

Therefore, the posterior distribution of θ given that data can be written as:

$$p(\theta \mid x) = \frac{1}{\sqrt{2\pi\sigma_U^2}} e^{-\frac{1}{2\sigma_U^2}(\theta - \mu_U)^2}, \; -\infty < \theta < \infty, \quad (5)$$

where $\mu_U = \frac{\sum_{i=1}^{n}\ln x_i}{n}$ and $\sigma_U^2 = \frac{\phi}{n}$.

Jeffreys prior for the location parameter θ is derived as:

$$p_J(\theta) \propto 1, -\infty < \theta < \infty \tag{6}$$

The posterior distribution using Jeffreys prior is the same as obtained under Uniform prior, so, under Jeffreys prior, the posterior distribution of θ given that data is:

$$p(\theta \mid x) = \frac{1}{\sqrt{2\pi\sigma_J^2}} e^{-\frac{1}{2\sigma_J^2}\left(\theta-\mu_J\right)^2}, \ -\infty < \theta < \infty, \tag{7}$$

where, $\mu_J = \frac{\sum_{i=1}^{n} \ln x_i}{n}$ and $\sigma_J^2 = \frac{\phi}{n}$.

2.2 Posterior Distributions under Informative Priors

We assume that the location parameter θ follows the Normal distribution having hyper-parameters a_n and b_n i.e.

$$p_N(\theta) = \frac{1}{\sqrt{2\pi b_n}} e^{-\frac{1}{2b_n}\left(\theta-a_n\right)^2}, -\infty < \theta < \infty, -\infty < a_n < \infty, b_n > 0 \tag{8}$$

This distribution acts as a natural conjugate prior (NCP) in this case. The posterior distribution under this prior can be obtained as:

$$\begin{aligned} p(\theta \mid x) &\propto p_N(\theta)\, L(\theta) \\ &\propto \frac{1}{\sqrt{2\pi b_n}} e^{-\frac{1}{2b_n}\left(\theta-a_n\right)^2} \frac{\left(2\pi\phi\right)^{-\frac{n}{2}}}{\prod_{i=1}^{n} x_i} e^{-\frac{1}{2\phi}\sum_{i=1}^{n}\left(\ln x_i-\theta\right)^2} \\ &\propto e^{-\frac{1}{2}\left(b_n^{-1}+n\phi^{-1}\right)\left\{\theta-\frac{\phi^{-1}\sum_{i=1}^{n}\ln x_i+b_n^{-1}a_n}{n\phi^{-1}+b_n^{-1}}\right\}^2}, \end{aligned}$$

which is the density kernel of the Normal Distribution.

Therefore, the posterior distribution of θ given data x is:

$$p(\theta \mid x) = \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-\frac{1}{2\sigma_n^2}\left(\theta-\mu_n\right)^2}, -\infty < \theta < \infty \tag{9}$$

where, $\mu_n = \frac{b_n \sum_{i=1}^{n} \ln x_i + a_n\phi}{nb_n + \phi}$ and $\sigma_n^2 = \frac{\phi b_n}{\phi + nb_n}$.

Next, let the parameter θ follows Cauchy distribution with hyper-parameter b_c i.e.

$$p_C(\theta) = \frac{b_c}{\pi\left(\theta^2 + b_c^2\right)}, -\infty < \theta < \infty \tag{10}$$

The posterior distribution under this prior can be obtained as:

$$p(\theta \mid x) = K \frac{e^{-\frac{1}{2\phi}\sum_{i=1}^{n}\left(\ln x_i - \theta\right)^2}}{\theta^2 + b_c^2}, -\infty < \theta < \infty,$$

where

$$K = \frac{1}{\int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2\phi}\sum_{i=1}^{n}\left(\ln x_i - \theta\right)^2}}{\theta^2 + b_c^2} d\theta}.$$

We will use the notation $p(\theta \mid x)_C$ for this posterior distribution throughout this study.

3. PRIOR PREDICTIVE DISTRIBUTIONS

The Prior Predictive Distribution (PPD) of future random variable $Y = X_{n+1}$ can be obtained using the following rule.

$$p(y) = \int_\theta p(\theta) f(y \mid \theta) d\theta \tag{11}$$

The prior predictive distributions under Normal and Cauchy priors have been derived which will be used for the elicitation of the hyper-parameters.

3.1 Prior Predictive Distribution under Normal Prior

Using (11), the PPD of the future random variable $Y = X_{n+1}$ under the Normal prior is derived as:

$$p(y)_N = \frac{y^{\frac{a_n}{b_n+\phi}-1}}{\sqrt{2\pi\left(b_n + \phi\right)}} e^{-\frac{a_n^2 + \ln^2(y)}{2\left(b_n+\phi\right)}}, \quad y > 0. \tag{12}$$

3.2 Prior Predictive Distribution under Cauchy Prior

By using the rule (11), the PPD of $Y = X_{n+1}$ under Cauchy prior is:

$$p(y)_C = \frac{b_c}{y\pi\sqrt{2\pi\phi}} \int_{-\infty}^{\infty} \frac{1}{\left(b_c^2 + \theta^2\right)} e^{-\frac{1}{2\phi}\left(\theta - \ln(y)\right)^2} d\theta, \quad y > 0 \tag{13}$$

4. ELICITATION OF THE HYPER-PARAMETERS

Using PPDs, hyper-parameters of Normal and Cauchy priors are elicited using the prior predictive distribution based on predictive probabilities, proposed by Aslam (2003). The elicitation is done in SAS using "PROC SYSLIN" command.

Tables 1, 2 and 3 consist of the elicited values of the hyper-parmeters.

Table 1
Elicited Hyper-parameters for $\phi = 0.1$

Prior	Expert Probability		Interval		Hyper-parameters	
Normal	0.2	0.06	(1,2)	(5,6)	1.122017	0.973105
Cauchy	0.15	--	(3,4)	--	0.996791	--

Table 2
Elicited Hyper-parameters for $\phi = 0.2$

Prior	Expert Probability		Interval		Hyper-parameters	
Normal	0.1	0.04	(3,4)	(6,7)	0.974139	1.035485
Cauchy	0.2	--	(2,3)	--	0.998173	--

Table 3
Elicited Hyper-parameters for $\phi = 0.3$

Prior	Expert Probability		Interval		Hyper-parameters	
Normal	0.2	0.07	(1,2)	(4,5)	1.01024	1.052667
Cauchy	0.1		(3,4)		0.99736	

5. SIMULATION STUDY

Consider the generation of random samples of size $n = 30, 50, 100, 200$ and 500 from the Log-Normal distribution assuming the scale parameter $\varphi = 0.1, 0.2, 0.3$ and the location parameter $\theta = 1$. The simulation is repeated $10,000$ times and the results have then been averaged. Posterior variance, co-efficient of skewness and ex-kurtosis are calculated. The results are displayed in Tables 4, 5 and 6

Table 4
Posterior Variance

n	Prior	ϕ = 0.1	ϕ = 0.2	ϕ = 0.3
30	Uniform/Jeffreys	0.00333000	0.00667000	0.0100000
50		0.00200000	0.00400000	0.0060000
100		0.00100000	0.00200000	0.0030000
200		0.00050000	0.00100000	0.0015000
500		0.00020000	0.00040000	0.0006000
30	Normal	0.00332000	0.00662000	0.0099100
50		0.00200000	0.00398000	0.0059700
100		0.00100000	0.00200000	0.0029900
200		0.00050000	0.00100000	0.0015000
500		0.00020000	0.00040000	0.0006000
30	Cauchy	0.00333327	0.00666597	0.0100972
50		0.00200031	0.00399984	0.0060194
100		0.00100000	0.00199998	0.0031099
200		0.00050000	0.00099999	0.0015001
500		0.00020000	0.00040000	0.0006000

Table 5
Posterior Co-efficient of Skewness

n	Prior	ϕ = 0.1	ϕ = 0.2	ϕ = 0.3
30	Uniform/Jeffreys	0.00000000	0.00000000	0.00000000
50		0.00000000	0.00000000	0.00000000
100		0.00000000	0.00000000	0.00000000
200		0.00000000	0.00000000	0.00000000
500		0.00000000	0.00000000	0.00000000
30	Normal	0.00000000	0.00000000	0.00000000
50		0.00000000	0.00000000	0.00000000
100		0.00000000	0.00000000	0.00000000
200		0.00000000	0.00000000	0.00000000
500		0.00000000	0.00000000	0.00000000
30	Cauchy	0.00019685	0.00666597	0.0100972
50		0.00074809	0.00399984	0.0060194
100		-0.0006064	0.00199998	0.0031099
200		0.00001806	0.00099999	0.0015001
500		3.24573×10^{-6}	0.00040000	0.0006000

Table 6
Co-efficient of Ex-Kurtosis

n	Prior	ϕ = 0.1	ϕ = 0.2	ϕ = 0.3
30	Uniform/Jeffreys	0.00000000	0.00000000	0.00000000
50		0.00000000	0.00000000	0.00000000
100		0.00000000	0.00000000	0.00000000
200		0.00000000	0.00000000	0.00000000
500		0.00000000	0.00000000	0.00000000
30	Normal	0.00000000	0.00000000	0.00000000
50		0.00000000	0.00000000	0.00000000
100		0.00000000	0.00000000	0.00000000
200		0.00000000	0.00000000	0.00000000
500		0.00000000	0.00000000	0.00000000
30	Cauchy	-0.00003000	-0.00008647	-0.00030190
50		-0.01969000	-0.00005841	-0.00011080
100		0.04048690	0.00002764	-0.00002691
200		-0.00057575	-0.02929750	-0.00056564
500		-0.00002293	2.37123000	0.00018693

The results of Tables 4, 5 and 6 show that Uniform, Jeffreys and Normal priors give approximately the same results with a slight difference for small sample size. The overall results suggest that Normal prior gives better results among all the priors as it provides least variance, coefficient of skewness and ex-kurtosis for small sample size.

6. BEs AND BRs UNDER DIFFERENT LOSS FUNCTIONS

Derivations of BEs and BRs are carried out under Squared Error Loss Function (SELF), Precautionary Loss Function (PLF), DeGroot Loss Function (DLF) and Quadratic Loss Function (QLF) using the assumed priors.

6.1 Squared Error Loss Function (SELF)

SELF for estimator θ^* of parameter θ is given as under.

$$L(\theta^*, \theta) = (\theta^* - \theta)^2.$$

The BE and BR under this loss function are respectively given below.

$$\theta^* = E_{\theta|x}(\theta)$$
$$\rho(\theta^*) = E_{\theta|x}(\theta^2) - (E_{\theta|x}(\theta))^2$$

The BEs and BRs under SELF for location parameter θ using different priors are given in Table. 7.

Table 7
BEs and BRs under SELF

Prior	$BE = E_{\theta\|x}(\theta)$	$BR = E_{\theta\|x}(\theta)^2 - \left(E_{\theta\|x}(\theta)\right)^2$
Uniform/ Jeffreys	$\dfrac{\sum_{i=1}^{n} \ln x_i}{n}$	$\dfrac{\phi}{n}$
Normal	$\dfrac{b_n \sum_{i=1}^{n} \ln x_i + a_n \phi}{nb_n + \phi}$	$\dfrac{\phi b_n}{\phi + nb_n}$
Cauchy	$\int_{-\infty}^{\infty} \theta p(\theta \mid x)_C \, d\theta$	$\int_{-\infty}^{\infty} \theta^2 p(\theta \mid x)_C \, d\theta - \left\{ \int_{-\infty}^{\infty} \theta p(\theta \mid x)_C \, d\theta \right\}^2$

6.2 Degrot Loss Function (DLF)

The DLF for estimator θ^* of parameter θ is expressed by the following formula.

$$L(\theta^*, \theta) = \frac{(\theta^* - \theta)^2}{\theta^{*2}}$$

The BE under DLF is given by the following formula.

$$\theta^* = \frac{E_{\theta|x}(\theta^2)}{E_{\theta|x}(\theta)}$$

The BR is:

$$\rho(\theta^*) = 1 - \frac{\{E_{\theta|x}(\theta)\}^2}{E_{\theta|x}(\theta^2)}$$

The BEs and BRs under DLF for location parameter θ using different priors are given in Table 8.

Table 8
BEs and BRs under DLF

Prior	$BE=\frac{E_{\theta\mid x}\left(\theta^2\right)}{E_{\theta\mid x}\left(\theta\right)}$	$BR=1-\frac{\left(E_{\theta\mid x}\left(\theta\right)\right)^2}{E_{\theta\mid x}\left(\theta\right)^2}$
Uniform/ Jeffreys	$\frac{n\phi+\left(\sum_{i=1}^{n}\ln x_i\right)^2}{n\sum_{i=1}^{n}\ln x_i}$	$\frac{n\phi}{n\phi+\left(\sum_{i=1}^{n}\ln x_i\right)^2}$
Normal	$\frac{b_n\phi\left(nb_n+\phi\right)\left(b_n\sum_{i=1}^{n}\ln x_i+a_n\phi\right)+\left(b_n\sum_{i=1}^{n}\ln x_i+a_n\phi\right)^3}{nb_n+\phi}$	$\frac{nb_n+\phi}{\left(nb_n+\phi\right)+\left(b_n\sum_{i=1}^{n}\ln x_i+a_n\phi\right)^2}$
Cauchy	$\frac{\int_{-\infty}^{\infty}\theta^2 p(\theta\mid x)_C\,d\theta}{\int_{-\infty}^{\infty}\theta p(\theta\mid x)_C\,d\theta}$	$1-\frac{\left\{\int_{-\infty}^{\infty}\theta p(\theta\mid x)_C\,d\theta\right\}^2}{\int_{-\infty}^{\infty}\theta^2 p(\theta\mid x)_C\,d\theta}$

6.3 Precautionary Loss Function (PLF)

The PLF is expressed by the following formula.

$$L(\theta^*,\theta)=\frac{(\theta^*-\theta)^2}{\theta^*}$$

The BE under PLF is given by the following formula.

$$\theta^*=\sqrt{E_{\theta\mid x}(\theta^2)}$$

and the BR under PLF is:

$$\rho(\theta^*)=2\left\{\sqrt{E_{\theta\mid x}(\theta^2)}-E_{\theta\mid x}(\theta)\right\}$$

The BEs and BRs for PLF are contained in Table 9.

Table 9
BEs and BRs under PLF

Prior	$BE = \sqrt{E_{\theta\|\mathbf{x}}(\theta^2)}$	$BR = 2\left\{\sqrt{E_{\theta\|\mathbf{x}}(\theta^2)} - E_{\theta\|\mathbf{x}}(\theta)\right\}$
Uniform/ Jeffreys	$\sqrt{\dfrac{n\phi+\left(\sum\limits_{i=1}^{n}\ln x_i\right)^2}{n}}$	$\dfrac{2\sqrt{n\phi+\left(\sum\limits_{i=1}^{n}\ln x_i\right)^2}-\sum\limits_{i=1}^{n}\ln x_i}{n}$
Normal	$\dfrac{\sqrt{b_n\phi(nb_n+\phi)+\left(b_n\sum\limits_{i=1}^{n}\ln x_i+a_n\phi\right)^2}}{nb_n+\phi}$	$\frac{2}{nb_n+\phi}\left\{\sqrt{b_n\phi\left(\phi+nb_n\right)+\left(b_n\sum\limits_{i=1}^{n}\ln x_i+a_n\phi\right)^2} - \left(b_n\sum\limits_{i=1}^{n}+a_n\phi\right)\right\}$
Cauchy	$\sqrt{\int\limits_{-\infty}^{\infty}\theta^2\, p(\theta\|\mathbf{x})_C d\theta}$	$2\left\{\sqrt{\int\limits_{-\infty}^{\infty}\theta^2\, p(\theta\|\mathbf{x})_C d\theta} - \int\limits_{-\infty}^{\infty}\theta\, p(\theta\|\mathbf{x})_C d\theta\right\}$

6.4 Quadratic Loss Function (QLF)

The BE under QLF can be written as:

$$L(\theta^*,\theta) = \frac{(\theta^* - \theta)^2}{\theta^2}$$

The BE under QLF is given by the following formula.

$$\theta^* = \frac{E_{\theta|x}(\theta^{-1})}{E_{\theta|x}(\theta^2)}$$

The BR under QLF is:

$$\rho(\theta^*) = 1 - \frac{\{E_{\theta|x}(\theta)^{-1}\}^2}{E_{\theta|x}(\theta^{-2})}$$

For QLF, the BEs and BRs are showcased in Table. 10.

Table 10
BEs and BRs under QLF

Prior	$BE = \frac{E_{\theta\mid\mathbf{x}}\left(\theta^{-1}\right)}{E_{\theta\mid\mathbf{x}}\left(\theta^{-2}\right)}$	$BR = 1 - \frac{\left\{E_{\theta\mid\mathbf{x}}(\theta)^{-1}\right\}^2}{E_{\theta\mid\mathbf{x}}(\theta)^{-2}}$
Uniform/ Jeffreys	$\frac{\int_{-\infty}^{\infty} \frac{1}{\theta} e^{-\frac{1}{2\sigma_u^2}(\theta-\mu_u)^2} d\theta}{\int_{-\infty}^{\infty} \frac{1}{\theta^2} e^{-\frac{1}{2\sigma_u^2}(\theta-\mu_u)^2} d\theta}$	$1 - \frac{\left\{\int_{-\infty}^{\infty} \frac{1}{\theta} e^{-\frac{1}{2\sigma_u^2}(\theta-\mu_u)^2} d\theta\right\}^2}{\int_{-\infty}^{\infty} \frac{1}{\theta^2} e^{-\frac{1}{2\sigma_u^2}(\theta-\mu_u)^2} d\theta}$
Normal	$\frac{\int_{-\infty}^{\infty} \frac{1}{\theta} e^{-\frac{1}{2\sigma_n^2}(\theta-\mu_n)^2} d\theta}{\int_{-\infty}^{\infty} \frac{1}{\theta^2} e^{-\frac{1}{2\sigma_n^2}(\theta-\mu_n)^2} d\theta}$	$1 - \frac{\left\{\int_{-\infty}^{\infty} \frac{1}{\theta} e^{-\frac{1}{2\sigma_n^2}(\theta-\mu_n)^2} d\theta\right\}^2}{\int_{-\infty}^{\infty} \frac{1}{\theta^2} e^{-\frac{1}{2\sigma_n^2}(\theta-\mu_n)^2} d\theta}$
Cauchy	$\frac{\int_{-\infty}^{\infty} \frac{1}{\theta} p(\theta\mid\mathbf{x})_C \, d\theta}{\int_{-\infty}^{\infty} \frac{1}{\theta^2} p(\theta\mid\mathbf{x})_C \, d\theta}$	$1 - \frac{\left\{\int_{-\infty}^{\infty} \frac{1}{\theta} p(\theta\mid\mathbf{x})_C \, d\theta\right\}^2}{\int_{-\infty}^{\infty} \frac{1}{\theta^2} p(\theta\mid\mathbf{x})_C \, d\theta}$

Here, $p(\theta \mid x)_C$ is the posterior distribution of θ under Cauchy prior. Numerical results of BEs and BRs under QLF and Cauchy prior will be obtained by solving the expressions numerically through “Quadrature Method” in “Mathematica 9”.

7. SIMULATION STUDY OF BEs AND BRs

Random samples of size $30, 50, 100, 200,$ and 500 have been generated from Log-Normal distribution assuming $\theta = 1$ and $\varphi = 0.1, 0.2, 0.3.$ The simulation process is repeated $10,000$ times and the results have then been averaged. Tables $11 - 14$ consist of the simulation results.

Table 11
BEs and BRs under SELF

n	Prior	ϕ = 0.1		ϕ = 0.2		ϕ = 0.3	
		BE	BR	BE	BR	BE	BR
30	Uniform/ Jeffreys	1.000250	0.00333000	0.9996200	0.00667000	0.999460	0.0100000
50		1.000130	0.00200000	0.9999700	0.00400000	0.999760	0.0060000
100		0.999900	0.00100000	0.9998300	0.00200000	1.000360	0.0030000
200		0.999860	0.00050000	1.0001500	0.00100000	1.000050	0.0015000
500		1.000020	0.00020000	1.0000300	0.00040000	0.999940	0.0006000
30	Normal	0.999580	0.00332000	0.9999600	0.00662000	1.0007100	0.0099100
50		0.999830	0.00200000	0.9996600	0.00398000	1.0011100	0.0059700
100		1.000060	0.00100000	1.0000200	0.00200000	0.9993200	0.0029900
200		1.000030	0.00050000	0.9998300	0.00100000	1.0002400	0.0015000
500		1.000050	0.00020000	1.0000700	0.00040000	1.0000900	0.0006000
30	Cauchy	0.996918	0.00333327	0.9946290	0.00666597	0.9892210	0.0100972
50		0.998168	0.00200031	0.9955400	0.00399984	0.9923280	0.0060194
100		0.999901	0.00100000	0.9979830	0.00199998	0.9960280	0.0031099
200		0.999162	0.00050000	0.9989030	0.00099999	0.9998810	0.0015001
500		0.999837	0.00020000	0.9991010	0.00040000	0.9996490	0.0006000

Table 12
BEs and BRs under PLF

n	Prior	ϕ = 0.1		ϕ = 0.2		ϕ = 0.3	
		BE	BR	BE	BR	BE	BR
30	Uniform/ Jeffreys	1.0016600	0.0033300	1.0032500	0.0066700	1.0046000	0.0100100
50		1.0010000	0.0020000	1.0022800	0.0040000	1.0028300	0.0060000
100		1.0002900	0.0010000	1.0010000	0.0020000	1.0016500	0.0030000
200		1.0002400	0.0005000	1.0003800	0.0010000	1.0006100	0.0015000
500		1.0001200	0.0002000	1.0002600	0.0004000	1.0004100	0.0006000
30	Normal	1.0016800	0.0033200	1.0034600	0.0066200	1.0050000	0.0099100
50		1.0011900	0.0019900	1.0021300	0.0039800	1.0028700	0.0059700
100		1.0006300	0.0010000	1.0014100	0.0020000	1.0015600	0.0029900
200		1.0002700	0.0005000	1.0004100	0.0010000	1.0007900	0.0015000
500		1.0000900	0.0002000	1.0001700	0.0004000	1.0002900	0.0006000
30	Cauchy	0.9983540	0.0033428	0.9952420	0.0067187	0.9980120	0.0100734
50		0.9989240	0.0020036	0.9971720	0.0040183	0.9990150	0.0060262
100		0.9998140	0.0010005	0.9992520	0.0020033	0.9994000	0.0030067
200		0.9997430	0.0005002	0.9999840	0.0010005	0.9997490	0.0015016
500		0.9998390	0.0002001	0.9999590	0.0004001	0.9996930	0.0006004

Table 13
BEs and BRs under DLF

n	Prior	ϕ = 0.1		ϕ = 0.2		ϕ = 0.3	
		BE	BR	BE	BR	BE	BR
30	Uniform/ Jeffreys	1.0033600	0.0033300	1.0070400	0.0066400	1.0085300	0.0100200
50		1.0020200	0.0020000	1.0042300	0.0039900	1.0055500	0.0060000
100		1.0011600	0.0010000	1.0018200	0.0020000	1.0032300	0.0030000
200		1.0004800	0.0005000	1.0009300	0.0010000	1.0017900	0.0015000
500		1.0002200	0.0002000	1.0004000	0.0004000	1.0008300	0.0006000
30	Normal	0.9963500	0.0033100	0.9947100	0.0066100	0.9937800	0.0098900
50		0.9980700	0.0019900	0.9966200	0.0039800	0.9955900	0.0059600
100		0.9991300	0.0010000	0.9984800	0.0019900	0.9979400	0.0029900
200		0.9995200	0.0005000	0.9994500	0.0010000	0.9990600	0.0015000
500		0.9998000	0.0002000	0.9994800	0.0004000	0.9992200	0.0006000
30	Cauchy	1.0001600	0.00334685	0.998420	0.0067612	0.9987000	0.00830702
50		1.0000300	0.00200502	0.999434	0.0040306	1.0000100	0.00232009
100		0.9998290	0.00100164	0.999272	0.0020093	0.9999800	0.00005801
200		0.9997430	0.00050058	1.000240	0.0010011	1.0006800	0.00150224
500		0.9998500	0.00020011	1.000120	0.0004005	0.9999470	0.00060074

Table 14
BEs and BRs under QLF

n	Prior	ϕ = 0.1		ϕ = 0.2		ϕ = 0.3	
		BE	BR	BE	BR	BE	BR
30	Uniform/ Jeffreys	0.9929470	0.0033961	0.9868940	0.0069224	0.9799130	0.0106344
50		0.9965520	0.0020191	0.9911910	0.0040989	0.9875110	0.0062258
100		0.9977900	0.0010058	0.9951500	0.0020262	0.9934310	0.0030581
200		0.9991240	0.0005012	0.9982670	0.0010051	0.9965030	0.0015148
500		0.9989960	0.0003007	0.9986620	0.0007055	0.9997480	0.0006008
30	Normal	0.9935620	0.0033803	0.9853940	0.0068998	0.9778750	0.0105787
50		0.9959280	0.0020176	0.9923260	0.0040732	0.9857620	0.0062160
100		0.9981130	0.0010041	0.9947610	0.0020238	0.9941300	0.0030445
200		0.9987880	0.0005013	0.9980130	0.0010046	0.9965610	0.0015125
500		0.9996110	0.0003002	0.9997050	0.0005750	0.9986120	0.0006017
30	Cauchy	0.9892600	0.0034211	0.9799420	0.0070219	0.9696560	0.0108567
50		0.9971530	0.0020314	0.9859380	0.0041472	0.9810030	0.0063106
100		0.9971530	0.0010071	0.9937150	0.0020321	0.9907290	0.0030730
200		0.9984340	0.0005019	0.9974420	0.0010068	0.9944530	0.0015212
500		0.9990170	0.0003007	0.9986110	0.0004014	0.9981760	0.0006029

The overall results of tables $11-14$ clearly show that as the sample size increases, the BEs under all the loss functions and priors approach to the true parametric value and their BRs decrease which shows that BEs under all the loss functions are consistent. Also, the increasing value of the scale parameter ϕ puts negative affect on all the BEs. The results under Uniform and Jefferys priors are same. It is interesting to note that the results under Uniform and Jefferys (non-informative) priors are better than those under Cauchy (informative) prior. The overall results suggest that Normal prior gives more efficient and consistent results. Also, the results under DeGroot loss function (DLF) are better than the rest of the considered loss functions.

From the above discussions, it can easily be concluded that Normal is the appropriate prior and DLF is the best loss function for the estimation location parameter of the Log-Normal distribution.

8. APPLICATION

To verify the simulation results of the estimation, a published real data of failure times of 15 identical electric units is used. This data is published in Volumes 1 and 2 of book titled "Reliability and Life Testing Handbook" by Kececioglu (1994).

8.1 BEs and BRs for Real Data Set

The BEs and BRs under the assumed priors and loss functions, for the real data set, are given in Tables $15-18$.

Table 15
BEs and BRs Under SELF

Prior	$\phi = 0.1$		$\phi = 0.2$		$\phi = 0.3$	
	BE	**BR**	**BE**	**BR**	**BE**	**BR**
Uniform/Jefferys	5.230280	0.006667	5.230280	0.013333	5.230280	0.020000
Normal	5.512480	0.006625	5.519210	0.013167	5.525950	0.019627
Cauchy	5.227820	0.006670	5.225350	0.013345	5.222890	0.020026

Table 16
BEs and BRs Under DLF

Prior	$\phi = 0.1$		$\phi = 0.2$		$\phi = 0.3$	
	BE	**BR**	**BE**	**BR**	**BE**	**BR**
Uniform/Jefferys	5.231550	0.000244	5.232830	0.000487	5.234100	0.000731
Normal	5.204990	0.000244	5.180040	0.000490	5.155410	0.000739
Cauchy	5.229090	0.000244	5.227910	0.000488	5.226720	0.000734

Table 17
BEs and BRs Under QLF

Prior	$\phi = 0.1$		$\phi = 0.2$		$\phi = 0.3$	
	BE	**BR**	**BE**	**BR**	**BE**	**BR**
Uniform/Jefferys	5.227730	0.000244	5.225170	0.000489	5.222610	0.000734
Normal	5.200500	0.000245	5.171080	0.000492	5.141990	0.000743
Cauchy	5.225260	0.000244	5.220240	0.000490	5.215200	0.000737

Table 18
BEs and BRs Under PLF

Prior	$\phi = 0.1$		$\phi = 0.2$		$\phi = 0.3$	
	BE	**BR**	**BE**	**BR**	**BE**	**BR**
Uniform/Jefferys	5.230920	0.001274	5.231550	0.002549	5.232190	0.003823
Normal	5.204360	0.001273	5.178770	0.002543	5.153500	0.003809
Cauchy	5.228460	0.001276	5.226630	0.002553	5.224800	0.003834

It is clear from the results contained in Tables $15-18$ that BEs are efficient under Normal prior. Also, the performance of DLF is better amongst the considered loss function. It is surprising to note that the results of Cauchy, being an informative prior, is worse than the results of the non-informative prior.

8.2 Graphical Displays

The behavior of the posterior distributions under the assumed priors are displayed for different scale values in the following figures.

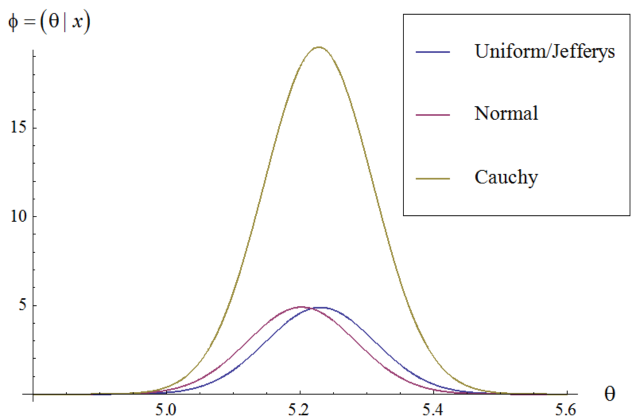

Figure 1: Posterior Distributions for $\varphi = 0.1$

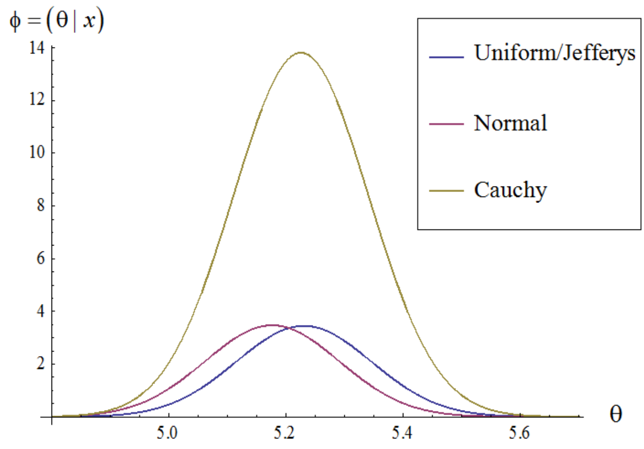

Figure 2: Posterior Distributions for $\varphi = 0.2$

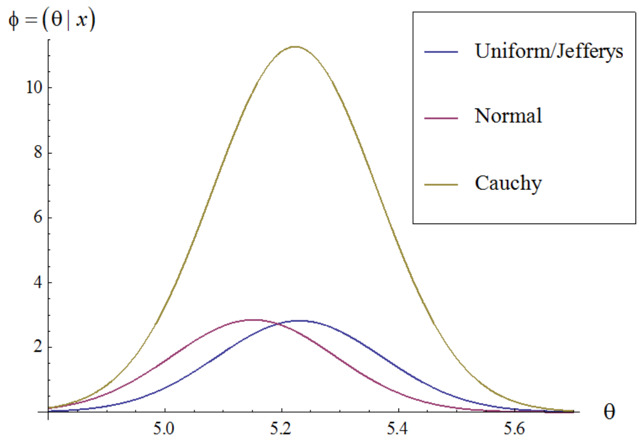

Figure 3: Graph of the Posterior Distributions for $\varphi = 0.3$

From Figures 1, 2 and 3, it is clear that the behaviors of the posterior densities are nearly similar under Uniform/Jeffreys and Normal with minor differences in the locations. Skewness and ex-kurtosis are almost zero for Uniform/Jeffreys and Normal priors. The displays of the posterior density under Cauchy prior are not attractive.

9. CONCLUSIONS

The estimation of location parameter of Log-Normal distribution is considered using Bayesian tools. Two non-informative (Uniform and Jeffreys) and two informative (Normal and Cauchy) priors are assumed for the estimation of location parameter. The posterior distributions have been derived under the assumed priors. Prior predictive distributions under Normal and Cauchy priors are derived to elicit the hyper-parameters. Four loss functions (SELF, PLF, DLF and QLF) are used for the estimation purpose. Conclusions are made on the basis of the posterior variance, skewness, ex-kurtosis and Bayes risks. Posterior variance assuming Normal prior is minimum. Skewness and ex-kurtosis under Uniform/Jeffreys and Normal priors are zero. The numerical results show that BEs are consistent because their risks decrease with increase in sample size. The small value of the scale parameter gives better results. The performance of Normal prior is convincing as it yields minimum BRs for all the BEs. The BRs of the BEs under DeGroot Loss Function (DLF) are minimum for all priors and all values of the scale parameter. It is worth to point out here that the performance of Cauchy (informative) prior is worse than the Uniform/Jeffreys (non-informative) priors. The results of the real data set verify the simulation results.

On the basis of the numerical results, Normal is suggested as a better prior and DeGroot Loss Function (DLF) is recommended as an efficient loss function for estimating the location parameter of Log-Normal distribution.

REFERENCES

1. Aitchison, J. and Brown, J. (1957). *The Lognormal Distribution*. Cambridge Univ.
2. Aslam, M. (2003). An application of prior predictive distribution to elicit the prior density. *Journal of Statistical Theory and Applications*, 2(1), 70-83.
3. Aslam, M., Tahir, M., Hussain, Z. and Al-Zahrani, B. (2015). A 3-Component mixture of Rayleigh distributions: Properties and estimation in Bayesian Framework. *PloSone*, 10(5), e0126183.
4. Becker, D.N. (1991). Statistical tests of the lognormal distribution as a basis for interest rate changes. *Transactions of the Society of Actuaries*, 43, 7-72.
5. Furusawa, C., Suzuki, T., Kashiwagi, A., Yomo, T. and Kaneko, K. (2005). Ubiquity of log-normal distributions in intracellular reaction dynamics. *Biophysics*, 1, 25-31.
6. Goldstein, G. (1973). False-alarm regulation in log-normal and Weibull clutter. *IEEE Transactions on Aerospace and Electronic Systems,* 1, 84-92.
7. Hill, B.M. (1963). The three-parameter lognormal distribution and Bayesian analysis of a point-source epidemic. *Journal of the American Statistical Association*, 58(301), 72-84.
8. Kececioglu, D. (1993 and 1994). *Reliability and Life Testing Handbook*. Vol. 1 and 2. Prentice Hall, Inc., Englewood Cliffs, New Jersey.
9. Khan, A.A., Aslam, M., Hussain, Z. and Tahir, M. (2015). Comparison of loss functions for estimating the scale parameter of log-normal distribution using non-informative priors. *Hacettepe Journal of Mathematics and Statistics*, Doi: 10.15672/HJMS.20159614094.
10. Khan, H.M., Haq, M.S. and Provost, S.B. (2006). Bayesian prediction for the log-normal model under type-II censoring. *Journal of Statistical Theory and Applications*, 5(2), 141-160.
11. Kifayat, T., Aslam, M. and Ali, S. (2012). Bayesian inference for the parameter of the power distribution. *Journal of Reliability and Statistical Studies*, 5(2), 45-58.
12. Martín, J. and Pérez, C. (2009). Bayesian analysis of a generalized lognormal distribution. *Computational Statistics & Data Analysis*, 53(4), 1377-1387.

13. Mehta, N.B., Wu, J., Molisch, A.F. and Zhang, J. (2007). Approximating a sum of random variables with a lognormal. *IEEE Transactions on Wireless Communications*, 6(7), 2690-2699.
14. Saleem, M. and Aslam, M. (2009a). Bayesian analysis of the two component mixture of the Rayleigh distribution assuming the uniform and the Jeffreys priors. *Journal of Applied Statistical Science*, 16(4), 105-113.
15. Saleem, M. and Aslam, M. (2009b). On Bayesian analysis of the Rayleigh survival time assuming the random censor time. *Pak. J. Statist*, 25(2), 71-82.
16. Shen, W.H. (1998). Estimation of parameters of a lognormal distribution. *Taiwanese Journal of Mathematics*, 2(2), 243-250.
17. Sindhu, T. and Aslam, M. (2013). Estimation of the Burrr type-VIII distribution through Bayesian framework. *Advances in Arts, Social Sciences and Education Research*, 3(2), 393-402.
18. Sindhu, T.N., Aslam, M. and Hussain, Z. (2016). Bayesian estimation on the generalized logistic distribution under left type-II censoring. *Thailand Statistician*, 14(2), 181-195.
19. Singh, A.K., Singh, A. and Engelhardt, M. (1997). *The lognormal distribution in environmental applications*. US Environmental Protection Agency, Office of Solid Waste and Emergency Response, Washington, DC.
20. Stedinger, J.R. (1980). Fitting log-normal distributions to hydrological data. *Water Resources Research*, 16(3), 481-490.
21. Sultana, T. and Aslam, M. (2016). A 3-component mixture of inverse Rayleigh distributions: properties and estimation in Bayesian framework. *International Journal of Basic and Applied Sciences*, 5(2), 120-139.
22. Tiku, M. (1968). Estimating the parameters of log-normal distribution from censored samples. *Journal of the American Statistical Association*, 63, 134-140.