Pak. J. Statist.
2016 Vol. 32(4), 247-260

NEW UNBIASED ESTIMATORS WITH THE HELP OF HARTLEY-ROSS TYPE ESTIMATORS

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ABSTRACT

For the population mean, Khoshnevisan et al. (2007) demonstrated the family of estimators via information of the known population parameters. Later, Koyuncu and Kadilar (2009) improved this family of estimators. In this study, we suggest a general family of Hartley-Ross type unbiased estimators developed from the special version of estimators in Khoshnevisan et al. (2007) and the family of estimators in Koyuncu and Kadilar (2009). The variance expressions of the suggested estimators are derived to the first order of approximation. When these findings are compared with the MSE values of the mentioned estimators, it is indicated that the proposed estimators are more effective than the estimators given by Khoshnevisan et al. (2007) and Koyuncu and Kadilar (2009) under the determined conditions. Furthermore, the obtained results are supported with a numerical demonstration.

KEYWORDS

Ratio type estimator, unbiased estimator, product type estimator, variance, Hartley-Ross type estimator, simple random sampling.

1. INTRODUCTION AND NOTATIONS

The utilization of the auxiliary information in sampling theory has recently been very popular to obtain the most efficient estimator. The auxiliary information may be helpful while choosing the type of estimators, determining the strata, selecting the sample or obtaining the estimation. The preference of the ratio and product estimators depends on the correlation between the study variable, y and the auxiliary variable, x. If the correlation is positive, the ratio estimators are utilized to estimate the population mean. Otherwise, the product estimators are used when this correlation is negative. Many authors define several estimators with the help of the known population parameters of the auxiliary variable, for example coefficient of kurtosis $(\beta_2(x))$, standard deviation (σ_x), correlation coefficient (ρ), coefficient of skewness $(\beta_1(x))$, coefficient of variation (C_x) etc.

Assume that (Y_i, X_i) are the values of y and x variables for the population of size $N(i = 1,2,\ldots,N)$, respectively. Let the sample of size, n, be drawn from this population using Simple Random Sampling Without Replacement (SRSWOR). Suppose that $\bar{y}$ and $\bar{x}$ be the unbiased estimators of population means $\bar{Y}$ and $\bar{X}$ of the study and the auxiliary variables, respectively.

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The ratio estimator for $\bar{Y}$ defined by Cochran (1940) is as follows:

$$\bar{y}_R = \bar{y}\frac{\bar{X}}{\bar{x}}.$$

Robson (1957) introduced the product estimator for $\bar{Y}$ as

$$\bar{y}_p = \bar{y}\frac{\bar{x}}{\bar{X}}.$$

Moreover, the studies in literature such as Sisodia and Dwivedi (1981), Pandey and Dubey (1988), Upadhyaya and Singh (1999), Singh and Tailor (2003), and Kadilar and Cingi (2004, 2006) have utilized the values of the known population parameters of the auxiliary variable, i.e. $C_x, \beta_2(x)$, and ρ for the improvement of the efficiencies for the ratio and product estimators.

Khoshnevisan et al. (2007) described a family of estimators for $\bar{Y}$, in the simple random sampling as follows:

$$T = \bar{y}\left[\frac{\theta\bar{X} + \varepsilon}{\alpha(\theta\bar{x} + \varepsilon) + (1-\alpha)(\theta\bar{X} + \varepsilon)}\right]^t, \tag{1.1}$$

where $\theta \neq 0$ and ε are real numbers or the functions of the population parameters of the auxiliary variable, for example $C_x, \beta_2(x), \rho, \sigma_x, \beta_1(x)$ etc., $t = -1,0,1$ and α is a suitable constant which is chosen to make the mean squared error (MSE) minimum.Some known estimators are given in Table 2, by using different values of $\theta, \varepsilon, \alpha$ and t in (1.1).

The general class of the ratio estimators when $t = 1$ and $\alpha = 1$ in (1.1), we can write as:

$$\bar{y}_{KR} = \bar{y}\frac{\theta\bar{X} + \varepsilon}{\theta\bar{x} + \varepsilon}. \tag{1.2}$$

The MSE of the this class of estimators are given by

$$MSE(\bar{y}_{KR}) = \gamma\left(S_y^2 + \delta^2 R^2 S_x^2 - 2\delta R S_{yx}\right), \tag{1.3}$$

where

$$\delta = \frac{\theta\bar{X}}{\theta\bar{X} + \varepsilon}.$$

Similarly, the general class of the product estimators when $t = -1$ and $\alpha = 1$ in (1.1), we have as follows:

$$\bar{y}_{Kp} = \bar{y}\frac{\theta\bar{x} + \varepsilon}{\theta\bar{X} + \varepsilon}. \tag{1.4}$$

The MSE of $\bar{y}_{Kp}$, is given by

$$MSE(\bar{y}_{Kp}) = \gamma\left(S_y^2 + \delta^2 R^2 S_x^2 + 2\delta R S_{yx}\right). \tag{1.5}$$

Koyuncu and Kadilar (2009) suggested a family of estimators using the known values of many population parameters for estimating $\bar{Y}$ as

$$\eta = k\bar{y}\left[\frac{\theta\bar{X} + \varepsilon}{\alpha(\theta\bar{x} + \varepsilon) + (1-\alpha)(\theta\bar{X} + \varepsilon)}\right]^t, \tag{1.6}$$

where k is a scalar that is determined suitably to make minimum value of the $MSE(\eta)$. Similarly, in Table 3 is shown some known estimators for different values of $\theta, \varepsilon, \alpha$ and t in (1.2).

When we take in consider the family of estimator in (1.6), we can write as follows:

$$\eta = k\bar{Y}(1+\tau_0)[1+\alpha\delta\tau_3]^{-t}$$

or

$$\eta \cong k\bar{Y}\left[1 - t\alpha\delta\tau_3 + \frac{t(t+1)}{2}\alpha^2\delta^2\tau_3^2 + \tau_0 - t\alpha\delta\tau_0\tau_3\right]. \tag{1.7}$$

Suppose $|\alpha\delta\tau_3| < 1$ so that $[1+\alpha\delta\tau_3]^{-t}$ is expandable.

The bias of η is given by

$$B(\eta) \cong k\bar{Y}\gamma\left[\frac{t(t+1)}{2}\alpha^2\delta^2\frac{S_x^2}{\bar{X}^2} - t\alpha\delta\frac{S_{yx}}{\bar{Y}\bar{X}}\right] + (k-1)\bar{Y} \tag{1.8}$$

and the MSE of the estimator in (1.8) is obtained as the following:

$$\begin{aligned} MSE(\eta) \cong \bar{Y}^2[&k^2\gamma C_y^2 + \left(k^2(2t^2+t) - k(t^2+t)\right)\gamma\alpha^2\delta^2C_x^2 \\ &-2(2k^2-k)\gamma t\alpha\delta C_{yx} + (k-1)^2] \end{aligned} \tag{1.9}$$

where

$$S_x^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{X})^2 \text{ and } S_{\text{yx}} = \frac{1}{N-1}\sum_{i=1}^{N}(y_i-\bar{Y})(x_i-\bar{X}).$$

We get optimal value of k, which minimizes the MSE in (1.9), as

$$k = \frac{(t^2+t)\gamma\alpha^2\delta^2C_x^2 - 2\gamma t\alpha\delta C_{yx} + 2}{2\left(\gamma C_y^2 + (2t^2+t)\gamma\alpha^2\delta^2C_x^2 - 4\gamma t\alpha\delta C_{yx} + 1\right)} = \frac{A}{2B}.$$

Thus, the minimum MSE is obtained as

$$MSE_{min}(\eta) = \bar{Y}^2\left[1 - \frac{A^2}{4B}\right]. \tag{1.10}$$

The minimum MSE equations of the ratio estimators, which are shown in Table 3, are given by

$$MSE_{min}(\eta_i) = \bar{Y}^2\left[1 - \frac{A^{+2}}{B^{+}}\right], i = 3,5,\dots,21 \tag{1.11}$$

where

$$A^{+} = \gamma\delta^2_{\frac{(i-1)}{2}}C_x^2 - \gamma\delta_{\frac{(i-1)}{2}}C_{yx} + 1, B^{+} = \gamma C_y^2 + 3\gamma\delta^2_{\frac{(i-1)}{2}}C_x^2 - 4\gamma\delta_{\frac{(i-1)}{2}}C_{yx} + 1.$$

The minimum MSE equations of the product estimators, which are shown in Table 3, become members of the same family in (1.6), given by

$$MSE_{min}(\eta_j) = \bar{Y}^2\left[1 - \frac{A^{*2}}{B^{*}}\right], j = 2,4,\dots,20 \tag{1.12}$$

where

$$A^{*}=\gamma\delta_{\frac{j}{2}}C_{yx}+1, B^{*}=\gamma C_{y}^{2}+\gamma\delta_{\frac{j}{2}}^{2}C_{x}^{2}+4\gamma\delta_{\frac{j}{2}}C_{yx}+1, \delta_{1}=\delta_{2}=\frac{\bar{X}}{\bar{X}+C_{x}},$$
$$\delta_{3}=\delta_{4}=\frac{\bar{X}}{\bar{X}+\beta_{2}(x)}, \delta_{5}=\delta_{6}=\frac{\bar{X}\beta_{2}(x)}{\bar{X}\beta_{2}(x)+C_{x}}, \delta_{7}=\delta_{8}=\frac{\bar{X}\rho}{\bar{X}\rho+C_{x}},$$
$$\delta_{9}=\delta_{10}=\frac{\bar{X}\beta_{2}(x)}{\bar{X}\beta_{2}(x)+\rho}, \delta_{11}=\delta_{12}=\frac{\bar{X}\rho}{\bar{X}\rho+\beta_{2}(x)}, \delta_{13}=\delta_{14}=\frac{\bar{X}}{\bar{X}+S_{x}},$$
$$\delta_{15}=\delta_{16}=\frac{\bar{X}\beta_{1}(x)}{\bar{X}\beta_{1}(x)+S_{x}}, \delta_{17}=\delta_{18}=\frac{\bar{X}\beta_{2}(x)}{\bar{X}\beta_{2}(x)+S_{x}} \text{ and } \delta_{19}=\delta_{20}=\frac{\bar{X}}{\bar{X}+\rho}.$$

The ratio type unbiased estimator for $\bar{Y}$ is introduced by Hartley and Ross (1954) as:

$$\bar{y}_{R}^{(u)}=\bar{r}\bar{X}+\frac{(N-1)n}{N(n-1)}(\bar{y}-\bar{r}\bar{x}),$$

where $\bar{r}=\frac{\bar{y}}{\bar{x}}$. Singh (2003) obtained the following Hartley-Ross type estimator for the classical product estimator:

$$\bar{y}_{p}^{(u)}=\bar{y}\frac{\bar{x}}{\bar{X}}-\gamma\frac{s_{yx}}{\bar{X}},$$

where

$$\gamma=\frac{N-n}{Nn} \text{ and } s_{yx}=\frac{1}{n-1}\sum_{i=1}^{n}(y_{i}-\bar{Y})(x_{i}-\bar{X}).$$

Singh et al. (2014) suggested Hartley-Ross type unbiased estimators for the population mean with the help of the estimators given by Kadilar and Cingi (2006) and Upadhyaya and Singh (1999) as follows:

$$\bar{y}_{S1}^{(u)}=\bar{r}^{\cdot}\bar{X}^{\cdot}+\frac{n(N-1)}{N(n-1)}(\bar{y}-\bar{r}^{\cdot}\bar{x}^{\cdot}) \text{ and } \bar{y}_{S2}^{(u)}=\bar{r}^{\cdot\cdot}\bar{X}^{\cdot\cdot}+\frac{n(N-1)}{N(n-1)}(\bar{y}-\bar{r}^{\cdot\cdot}\bar{x}^{\cdot\cdot}),$$

where

$$r_{i}^{\cdot}=\frac{y_{i}}{C_{x}x_{i}+\rho}=\frac{y_{i}}{x_{i}^{\cdot}}, \bar{r}^{\cdot}=\frac{\sum_{i=1}^{n}r_{i}^{\cdot}}{n}, \bar{X}^{\cdot}=C_{x}\bar{X}+\rho,$$
$$r_{i}^{\cdot\cdot}=\frac{y_{i}}{C_{x}x_{i}+\beta_{2}(x)}=\frac{y_{i}}{x_{i}^{\cdot\cdot}}, \bar{r}^{\cdot\cdot}=\frac{\sum_{i=1}^{n}r_{i}^{\cdot\cdot}}{n} \text{ and } \bar{X}^{\cdot\cdot}=C_{x}\bar{X}+\beta_{2}(x).$$

To derive the bias and variance, we have

$$\bar{y}=\bar{Y}(1+\tau_{0}), \bar{x}^{*}=\bar{X}^{*}(1+\tau_{1}), \bar{r}^{*}=\bar{R}^{*}(1+\tau_{2}), \bar{x}=\bar{X}(1+\tau_{3})$$

and

$$s_{yx}=S_{yx}(1+\tau_{4}),$$

such that

$$E(\tau_{0})=E(\tau_{1})=E(\tau_{2})=E(\tau_{3})=E(\tau_{4})=0,$$
$$E(\tau_{0}^{2})=\gamma C_{y}^{2}, E(\tau_{1}^{2})=\gamma C_{x^{*}}^{2}, E(\tau_{2}^{2})=\gamma C_{r^{*}}^{2}, E(\tau_{3}^{2})=\gamma C_{x}^{2}, E(\tau_{4}^{2})=\gamma\left(\frac{\lambda_{22}}{\rho}-1\right),$$
$$E(\tau_{0}\tau_{1})=\gamma C_{yx^{*}}, E(\tau_{0}\tau_{3})=\gamma C_{yx}, E(\tau_{1}\tau_{2})=\gamma C_{x^{*}r^{*}}, E(\tau_{1}\tau_{3})=\gamma C_{xx^{*}},$$

$$E(\tau_0\tau_4) = \gamma\frac{C_y\lambda_{21}}{\rho} \text{ and } E(\tau_3\tau_4) = \gamma\frac{C_x\lambda_{12}}{\rho},$$

where

$$C_y^2 = \frac{S_y^2}{\bar{Y}^2}, C_x^2 = \frac{S_x^2}{\bar{X}^2}, C_{x^*}^2 = \frac{S_{x^*}^2}{\bar{X}^{*2}}, C_{r^*}^2 = \frac{S_{r^*}^2}{\bar{R}^{*2}}, C_{yx} = \frac{S_{yx}}{\bar{Y}\bar{X}}, C_{yx^*} = \frac{S_{yx^*}}{\bar{Y}\bar{X}^*}, C_{xx^*} = \frac{S_{xx^*}}{\bar{X}\bar{X}^*},$$

$$C_{r^*x^*} = \frac{S_{r^*x^*}}{\bar{X}^*\bar{R}^*}, \mu_{jk} = \frac{1}{N}\sum_{i=1}^{N}(y_i - \bar{Y})^j(x_i - \bar{X})^k, \text{ and } \lambda_{jk} = \frac{\mu_{jk}}{\mu_{20}^{j/2}\mu_{02}^{k/2}}.$$

2. PROPOSED ESTIMATORS

In this section, we suggest two classes and a family of Hartley-Ross type estimators of the population mean motivated by Khoshnevisan et al. (2007) and Koyuncu and Kadilar (2009).

2.1 A Class of Proposed Hartley-Ross Type Ratio Estimators Based on Khoshnevisan et al. (2007) Estimators

Motivated by a general family of ratio estimators which is the case of $t = 1$ and $\alpha = 1$ in (1.1), we consider a class of ratio type estimators follows as:

$$\bar{y}_{GR} = \bar{r}^*\bar{X}^*, \tag{2.1}$$

where

$$\bar{r}^* = \frac{\sum_{i=1}^{n} r_i^*}{n}, r_i^* = \frac{y_i}{\theta x_i + \varepsilon} = \frac{y_i}{x_i^*} \text{ and } \bar{X}^* = \theta\bar{X} + \varepsilon.$$

We derive the following bias of the estimator in (2.1):

$$B(\bar{y}_{GR}) = \mathrm{E}(\bar{r}^*\bar{X}^* - \bar{Y}) = -\frac{N-1}{N}S_{r^*x^*}, \tag{2.2}$$

where

$$S_{r^*x^*} = \frac{1}{N-1}\sum_{i=1}^{N}(r_i^* - \bar{R}^*)(x_i^* - \bar{X}^*) \text{ and } \bar{R}^* = \frac{1}{N}\sum_{i=1}^{N} r_i^*.$$

Note that $s_{r^*x^*} = \frac{n}{(n-1)}(\bar{y} - \bar{r}^*\bar{x}^*)$is an unbiased estimator of $S_{r^*x^*}$. We replace $S_{r^*x^*}$ in (2.2) with $s_{r^*x^*}$ in order to reach an unbiased estimator of the bias of $\bar{y}_{GR}$ as

$$B(\bar{y}_{GR}) = -\frac{n(N-1)}{N(n-1)}(\bar{y} - \bar{r}^*\bar{x}^*).$$

Then, we obtain a new class of Hartley-Ross estimators, which is the proposed unbiased form of the estimator in (2.1), by using the bias, as

$$\bar{y}_{pr1} = \bar{r}^*\bar{X}^* + \frac{n(N-1)}{N(n-1)}(\bar{y} - \bar{r}^*\bar{x}^*). \tag{2.3}$$

Expressing $\bar{y}_{pr1}$ in terms of τ's, we can write

$$\bar{y}_{pr1} = \bar{R}^*(1+\tau_2)\bar{X}^* + \frac{n(N-1)}{N(n-1)}\left(\bar{Y}(1+\tau_0) - \bar{R}^*(1+\tau_2)\bar{X}^*(1+\tau_1)\right). \quad (2.4)$$

From (2.4), we get

$$\bar{y}_{pr1} - \bar{Y} = \frac{N-n}{N(n-1)}(\bar{Y} - \bar{R}^*\bar{X}^*) + \bar{R}^*\bar{X}^*\tau_2 + \frac{n(N-1)}{N(n-1)}\left(\bar{Y}\tau_0 - \bar{R}^*\bar{X}^*(\tau_1 + \tau_2 + \tau_1\tau_2)\right).$$

We assume that

$$\bar{Y} - \bar{R}^*\bar{X}^* \cong 0 \text{ and } \frac{n(N-1)}{N(n-1)} \cong 1$$

and then we have

$$\bar{y}_{pr1} - \bar{Y} \cong \bar{Y}\tau_0 - \bar{R}^*\bar{X}^*(\tau_1 + \tau_1\tau_2). \quad (2.5)$$

Squaring both sides of (2.5) and then taking expectations, the variance of $\bar{y}_{pr1}$ can be obtained, to the first order of approximation as,

$$V(\bar{y}_{pr1}) \cong \gamma\left(S_y^2 + \bar{R}^{*2}S_{x^*}^2 - 2\bar{R}^*S_{yx^*}\right), \quad (2.6)$$

where

$$S_y^2 = \frac{1}{N-1}\sum_{i=1}^{N}(y_i - \bar{Y})^2, S_{x^*}^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i^* - \bar{X}^*)^2$$

and

$$S_{yx^*} = \frac{1}{N-1}\sum_{i=1}^{N}(y_i - \bar{Y})(x_i^* - \bar{X}^*).$$

2.2 A Class of Proposed Hartley-Ross Type Product Estimators Based on Khoshnevisan et al. (2007) Estimators

Similarly, modified on a general class of estimators when $t = -1$ and $\alpha = 1$ in (1.1), we can develop a class of product type estimators as follows:

$$\bar{y}_{Gp} = \frac{\bar{r}^*}{\bar{X}^*}. \quad (2.7)$$

The bias of the estimator in (2.7) is obtained as follows:

$$B(\bar{y}_{Gp}) = E\left(\frac{\bar{r}^*}{\bar{X}^*} - \bar{Y}\right) = \frac{\bar{R}^*}{\bar{X}^*}(1 - \bar{X}^{*2}) - \frac{N-1}{N}S_{r^*x^*}. \quad (2.8)$$

We substitute $S_{r^*x^*}$and $\bar{R}^*$, respectively, in (2.8) with $s_{r^*x^*}$ and $\bar{r}^*$, which are unbiased estimators of $S_{r^*x^*}$ and $\bar{R}^*$, for deriving an unbiased estimator of the bias of $\bar{y}_{Gp}$ as

$$B(\bar{y}_{Gp}) = \frac{\bar{r}^*}{\bar{X}^*}(1 - \bar{X}^{*2}) - \frac{n(N-1)}{N(n-1)}(\bar{y} - \bar{r}^*\bar{x}^*).$$

Then, we obtain a new class of Hartley-Ross type estimators, by using the bias, as

$$\bar{y}_{pr2} = \frac{\bar{r}^*}{\bar{X}^*} - \frac{\bar{r}^*}{\bar{X}^*}(1 - \bar{X}^{*2}) + \frac{n(N-1)}{N(n-1)}(\bar{y} - \bar{r}^*\bar{x}^*)$$
$$= \bar{r}^*\bar{X}^* + \frac{n(N-1)}{N(n-1)}(\bar{y} - \bar{r}^*\bar{x}^*). \qquad (2.9)$$

It is clearly seen that the proposed product unbiased estimator in (2.9) equals to the proposed ratio estimator in (2.3). Therefore, we have

$$V(\bar{y}_{pr1}) = V(\bar{y}_{pr2}). \qquad (2.10)$$

2.3 A Family of Proposed Hartley-Ross Type Estimators Based on Koyuncu and Kadilar (2009) Estimator

Similarly, obtaining the unbiased form of the estimator in (1.2), by using the bias, we develop the unbiased version of estimators as

$$\bar{y}_{pr3} = k^{\circ}\bar{y}\left[\frac{\theta\bar{X} + \varepsilon}{\alpha(\theta\bar{x} + \varepsilon) + (1-\alpha)(\theta\bar{X} + \varepsilon)}\right]^t$$
$$- k^{\circ}\bar{y}\gamma\left[\frac{t(t+1)}{2}\alpha^2\delta^2\frac{S_x^2}{\bar{X}^2} - t\alpha\delta\frac{s_{yx}}{\bar{y}\bar{X}}\right] - (k^{\circ} - 1)\bar{y}. \qquad (2.11)$$

Note that s_{yx} and $\bar{y}$ are unbiased estimators of S_{yx} and $\bar{Y}$. We replace S_{yx} and $\bar{Y}$ in (1.8) with s_{yx} and $\bar{y}$ in order to reach an unbiased estimator of the bias of η.

Expressing the third proposed unbiased estimator in (2. 11) in terms of τ's, we get

$$\bar{y}_{pr3} \cong k^{\circ}\bar{Y}\left[1 - t\alpha\delta\tau_3 + \frac{t(t+1)}{2}\alpha^2\delta^2\tau_3^2 + \tau_0 - t\alpha\delta\tau_0\tau_3\right]$$
$$-k^{\circ}\bar{Y}(1+\tau_0)\gamma\left[\frac{t(t+1)}{2}\alpha^2\delta^2C_x^2 - t\alpha\delta\frac{S_{yx}(1+\tau_4)}{\bar{Y}\bar{X}(1+\tau_0)}\right]$$
$$-(k^{\circ} - 1)\bar{Y}(1+\tau_0). \qquad (2.12)$$

From (2.12), we get

$$\bar{y}_{pr3} - \bar{Y} \cong k^{\circ}\bar{Y}\left[-t\alpha\delta\tau_3 + \frac{t(t+1)}{2}\alpha^2\delta^2\tau_3^2 + \tau_0 - t\alpha\delta\tau_0\tau_3\right]$$
$$- k^{\circ}\bar{Y}\gamma\left[\frac{t(t+1)}{2}\alpha^2\delta^2C_x^2(1+\tau_0) - t\alpha\delta C_{yx}(1+\tau_4)\right]$$
$$-(k^{\circ} - 1)\bar{Y}\tau_0. \qquad (2.13)$$

Squaring both sides of (2.13) and then taking expectations, the variance of our third estimator is obtained to the first order of approximation, as follows:

$$V(\bar{y}_{pr3}) \cong \bar{Y}^2\gamma\left\{\left(k^{\circ 2}t^2\alpha^2\delta^2C_x^2 - 2k^{\circ}t\alpha\delta C_{yx} + C_y^2\right) - \gamma\left[k^{\circ}t\alpha\delta\left(\frac{t+1}{2}\alpha\delta C_x^2 - C_{yx}\right)\right]^2 - \gamma k^{\circ}t\alpha\delta\left[2\frac{C_{yx}}{\rho}\left(k^{\circ}t\alpha\delta C_x\lambda_{12} - C_y\lambda_{21}\right) - (t+1)\alpha\delta C_x^2\left(k^{\circ}t\alpha\delta C_{yx} - C_y^2\right)\right]\right\}. \tag{2.14}$$

Note that the term of γ^3 is ignored, because it is equal to approximately zero.

We get optimal value of k°, which minimizes the variance in (2.14), as

$$k_{opt}^{\circ} = \frac{\Gamma}{\Lambda} \tag{2.15}$$

where

$$\Gamma = t\alpha\delta\left[C_{yx}\left(1 - \frac{\gamma C_y\lambda_{21}}{\rho}\right) + \frac{(t+1)}{2}\gamma\alpha\delta C_x^2C_y^2\right]$$

and

$$\Lambda = t^2\alpha^2\delta^2\left[C_x^2 + \gamma\left(C_{yx}\left((t+1)\alpha\delta C_x^2 - 2\frac{C_x\lambda_{12}}{\rho}\right) - \left(\frac{(t+1)}{2}\alpha\delta C_x^2 - C_{yx}\right)^2\right)\right].$$

When the optimal value of k_{opt}° in (2.15) is replaced with k° in (2.14), the minimum variance is obtained by

$$V_{min}(\bar{y}_{pr3}) \cong \bar{Y}^2\gamma\left[C_y^2 - \frac{\Gamma^2}{\Lambda}\right]. \tag{2.16}$$

To derive the minimum variance of the ratio estimators, which are shown in Table 3, we consider equation in (2.14). We have

$$V(\bar{y}_{pr3i}) \cong \bar{Y}^2\gamma\left\{\left(k^{\odot 2}\delta_{\frac{(i-1)}{2}}^2C_x^2 - 2k^{\odot}\delta_{\frac{(i-1)}{2}}C_{yx} + C_y^2\right) - \gamma k^{\odot 2}\delta_{\frac{(i-1)}{2}}^2\left(\delta_{\frac{(i-1)}{2}}C_x^2 - C_{yx}\right)^2 - 2\gamma k^{\odot}\delta_{\frac{(i-1)}{2}}\left[\frac{C_{yx}}{\rho}\left(k^{\odot}\delta_{\frac{(i-1)}{2}}C_x\lambda_{12} - C_y\lambda_{21}\right) - \delta_{\frac{(i-1)}{2}}C_x^2\left(k^{\odot}\delta_{\frac{(i-1)}{2}}C_{yx} - C_y^2\right)\right]\right\}$$

$$i = 3, 5, \ldots, 21 \tag{2.17}$$

and also the minimum variance is obtained for the product estimators, given in Table 3, we can write

$$V(\bar{y}_{pr3j}) \cong \bar{Y}^2\gamma\left\{\left(k^{\ddagger 2}\delta_{\frac{j}{2}}^2C_x^2 + 2k^{\ddagger}\delta_{\frac{j}{2}}C_{yx} + C_y^2\right) - \gamma k^{\ddagger 2}\delta_{\frac{j}{2}}^2C_{yx}^2 - 2\gamma k^{\ddagger}\delta_{\frac{j}{2}}\frac{C_{yx}}{\rho}\left(k^{\ddagger}\delta_{\frac{j}{2}}C_x\lambda_{12} + C_y\lambda_{21}\right)\right\}, j = 2, 4, \ldots, 20 \tag{2.18}$$

The optimal values of k° are found as follows:

$$k_{opt}^{\odot} = \frac{\delta_{\frac{(i-1)}{2}}\left\{C_{yx} + \gamma\left(C_x^2 C_y^2 - C_{yx}\frac{C_y\lambda_{21}}{\rho}\right)\right\}}{\delta_{\frac{(i-1)}{2}}^2\left\{C_x^2 - \gamma\left(\delta_{\frac{(i-1)}{2}}C_x^2 - C_{yx}\right)^2 - 2\gamma C_{yx}C_x\left(\frac{\lambda_{12}}{\rho} - \delta_{\frac{(i-1)}{2}}C_x\right)\right\}} = \frac{\Gamma^{\odot}}{\Lambda^{\odot}}$$

and

$$k_{opt}^{\ddagger} = \frac{\delta_{\frac{j}{2}}\left\{C_{yx}\left(-1 + \gamma\frac{C_y\lambda_{21}}{\rho}\right)\right\}}{\delta_{\frac{j}{2}}^2\left\{C_x^2 - \gamma C_{yx}\left(C_{yx} + 2\frac{C_x\lambda_{12}}{\rho}\right)\right\}} = \frac{\Gamma^{\ddagger}}{\Lambda^{\ddagger}}.$$

When $k_{opt}^{\odot}$ and $k_{opt}^{\ddagger}$ are replaced with $k^{\odot}$ and $k^{\ddagger}$ in (2.17) and (2.18), respectively, we get the minimum variance for the ratio estimators as follows:

$$V_{min}(\bar{y}_{pr3i}) \cong \bar{Y}^2\gamma\left[C_y^2 - \frac{\Gamma^{\odot 2}}{\Lambda^{\odot}}\right], i = 3, 5, \ldots, 21 \quad (2.19)$$

and we obtain the minimum variance for the product estimators as:

$$V_{min}(\bar{y}_{pr3j}) \cong \bar{Y}^2\gamma\left[C_y^2 - \frac{\Gamma^{\ddagger 2}}{\Lambda^{\ddagger}}\right], j = 2, 4, \ldots, 20. \quad (2.20)$$

3. EFFICIENCY COMPARISONS

Comparing the variance of the first proposed estimators in (2.6) with the MSE of the general class of the ratio estimators in (1.3), we have the following condition:

$$V(\bar{y}_{pr1}) < MSE(\bar{y}_{KR}),$$

if

$$\bar{R}^{*2}S_{x^*}^2 - 2\bar{R}^*S_{yx^*} - \delta^2R^2S_x^2 + 2\delta RS_{yx} < 0. \quad (3.1)$$

When (3.1) is hold, the first estimators are more effective than the general family of ratio estimators.

If we compare the variance of the our second estimators in (2.10) with the MSE of the general class of the product estimators in (1.5), we obtain

$$V(\bar{y}_{pr2}) < MSE(\bar{y}_{Kp}),$$

if

$$\bar{R}^{*2}S_{x^*}^2 - 2\bar{R}^*S_{yx^*} - \delta^2R^2S_x^2 - 2\delta RS_{yx} < 0. \quad (3.2)$$

When (3.2) is hold, the second estimators are more effective than the general family of product estimators.

Finally, comparing the minimum variance of the third suggested estimators in (2.16) with the minimum MSE of the estimators suggested by Koyuncu and Kadilar (2009) in (1.10), we find

$$V_{min}(\bar{y}_{pr3}) < MSE_{min}(\eta),$$

if

$$\gamma\left[C_y^2 - \frac{\Gamma^2}{\Lambda}\right] - \left[1 - \frac{A^2}{4B}\right] < 0. \quad (3.3)$$

When the condition (3.3) is hold, the third family of estimators are more effective than the estimators given by Koyuncu and Kadilar (2009).

Moreover, for ratio estimator, if we compare the minimum variance of the third suggested estimators in (2.19) with the minimum MSE of the estimators given by Koyuncu and Kadilar (2009) in (1.11), we have

$$V_{min}\left(\bar{y}_{pr3\mathrm{i}}\right) < MSE_{min}(\eta_i)\ , i = 3, 5, \dots, 21,$$

if

$$\gamma\left[C_y^2 - \frac{\Gamma^{\odot 2}}{\Lambda^{\odot}}\right] - \left[1 - \frac{A^{+2}}{B^{+}}\right] < 0. \qquad (3.4)$$

When (3.4) is hold, the third family of ratio estimators are more effective than the estimators given by Koyuncu and Kadilar (2009).

Similarly, if we compare the minimum variance of the third suggested estimators in (2.20) for product estimators with the minimum MSE of the product estimators given by Koyuncu and Kadilar (2009) in (1.12), we obtain

$$V_{min}\left(\bar{y}_{pr3\mathrm{j}}\right) < MSE_{min}(\eta_j)\ , j = 2, 4, \dots, 20,$$

if

$$\gamma\left[C_y^2 - \frac{\Gamma^{\ddagger 2}}{\Lambda^{\ddagger}}\right] - \left[1 - \frac{A^{*2}}{B^{*}}\right] < 0. \qquad (3.5)$$

When the condition (3.5) is hold, the third family of product estimators are more effective than the product estimators given by Koyuncu and Kadilar (2009).

4. NUMERICAL ILLUSTRATION

We apply the same data set in Kadilar and Cingi (2003). It consists of apple production quantity as the study variable and count of apple trees as the auxiliary variable in 854 villages of 6 regions in Turkey. However, as we are related to the simple random sampling in this article, we only take the Mediterranean region having 94 villages. Further information about the data is given in Table 1. It is worth noting that the correlation between y and x variables is 0.90. It is a high positive relation. Therefore, we only use the ratio estimators here and we do not apply product estimators to this data.

Table 1
Descriptive Statistics of the Population

$N = 94$	$\bar{X} = 72410$	$R = 0.0779$	$\beta_2(x) = 26.136$	$C_x = 2.22$
$n = 20$	$\bar{Y} = 9384$	$\rho = 0.901$	$\beta_1(x) = 4.611$	$C_y = 3.187$

In Table 2, we give some special estimators, members of the family of estimators in (1.1), when $\alpha = 1$.

Table 2
Some Members of Ratio and Product Type Estimators of *T*

Ratio Type for $t = 1$	Product Type for $t = -1$	θ	ε
$\bar{y}_{KR1} = \bar{y}\frac{\bar{X} + C_x}{\bar{x} + C_x}$ Sisodia and Dwivedi (1981)	$\bar{y}_{Kp1} = \bar{y}\frac{\bar{x} + C_x}{\bar{X} + C_x}$ Pandey and Dubey (1988)	1	C_x
$\bar{y}_{KR2} = \bar{y}\frac{\bar{X} + \beta_2(x)}{\bar{x} + \beta_2(x)}$ Singh et al. (2004)	$\bar{y}_{Kp2} = \bar{y}\frac{\bar{x} + \beta_2(x)}{\bar{X} + \beta_2(x)}$ Singh et al. (2004)	1	$\beta_2(x)$
$\bar{y}_{KR3} = \bar{y}\frac{\bar{X}\beta_2(x) + C_x}{\bar{x}\beta_2(x) + C_x}$ Upadyhaha and Singh (1999)	$\bar{y}_{Kp3} = \bar{y}\frac{\bar{x}\beta_2(x) + C_x}{\bar{X}\beta_2(x) + C_x}$ Upadyhaha and Singh (1999)	$\beta_2(x)$	C_x
$\bar{y}_{KR4} = \bar{y}\frac{\bar{X}\rho + C_x}{\bar{x}\rho + C_x}$ Kadilar and Cingi (2006)	$\bar{y}_{Kp4} = \bar{y}\frac{\bar{x}\rho + C_x}{\bar{X}\rho + C_x}$	ρ	C_x
$\bar{y}_{KR5} = \bar{y}\frac{\bar{X}\beta_2(x) + \rho}{\bar{x}\beta_2(x) + \rho}$ Kadilar and Cingi (2006)	$\bar{y}_{Kp5} = \bar{y}\frac{\bar{x}\beta_2(x) + \rho}{\bar{X}\beta_2(x) + \rho}$	$\beta_2(x)$	ρ
$\bar{y}_{KR6} = \bar{y}\frac{\bar{X}\rho + \beta_2(x)}{\bar{x}\rho + \beta_2(x)}$ Kadilar and Cingi (2006)	$\bar{y}_{Kp6} = \bar{y}\frac{\bar{x}\rho + \beta_2(x)}{\bar{X}\rho + \beta_2(x)}$	ρ	$\beta_2(x)$
$\bar{y}_{KR7} = \bar{y}\frac{\bar{X} + S_x}{\bar{x} + S_x}$ Koyuncu and Kadilar (2009)	$\bar{y}_{Kp7} = \bar{y}\frac{\bar{x} + S_x}{\bar{X} + S_x}$ Singh (2003)	1	S_x
$\bar{y}_{KR8} = \bar{y}\frac{\bar{X}\beta_1(x) + S_x}{\bar{x}\beta_1(x) + S_x}$ Koyuncu and Kadilar (2009)	$\bar{y}_{Kp8} = \bar{y}\frac{\bar{x}\beta_1(x) + S_x}{\bar{X}\beta_1(x) + S_x}$ Singh (2003)	$\beta_1(x)$	S_x
$\bar{y}_{KR9} = \bar{y}\frac{\bar{X}\beta_2(x) + S_x}{\bar{x}\beta_2(x) + S_x}$ Koyuncu and Kadilar (2009)	$\bar{y}_{Kp9} = \bar{y}\frac{\bar{x}\beta_2(x) + S_x}{\bar{X}\beta_2(x) + S_x}$ Singh (2003)	$\beta_2(x)$	S_x
$\bar{y}_{KR10} = \bar{y}\frac{\bar{X} + \rho}{\bar{x} + \rho}$ Singh and Tailor (2003)	$\bar{y}_{Kp10} = \bar{y}\frac{\bar{x} + \rho}{\bar{X} + \rho}$ Singh and Tailor (2003)	1	ρ

In Table 3, we give some special estimators, members of the family of estimators in (1. 2), when $\alpha = 1$.

Table 3
Some Members of Ratio and Product Type Estimators of η

Ratio Type for $t = 1$	Product Type for $t = -1$	θ	ε
$\eta_1 = k\bar{y}\frac{\bar{X}+C_x}{\bar{x}+C_x}$ Koyuncu and Kadilar (2009)	$\eta_2 = k\bar{y}\frac{\bar{x}+C_x}{\bar{X}+C_x}$ Koyuncu and Kadilar (2009)	1	C_x
$\eta_3 = k\bar{y}\frac{\bar{X}+\beta_2(x)}{\bar{x}+\beta_2(x)}$ Koyuncu and Kadilar (2009)	$\eta_4 = k\bar{y}\frac{\bar{x}+\beta_2(x)}{\bar{X}+\beta_2(x)}$ Koyuncu and Kadilar (2009)	1	$\beta_2(x)$
$\eta_5 = k\bar{y}\frac{\bar{X}\beta_2(x)+C_x}{\bar{x}\beta_2(x)+C_x}$ Koyuncu and Kadilar (2009)	$\eta_6 = k\bar{y}\frac{\bar{x}\beta_2(x)+C_x}{\bar{X}\beta_2(x)+C_x}$ Koyuncu and Kadilar (2009)	$\beta_2(x)$	C_x
$\eta_7 = k\bar{y}\frac{\bar{X}\rho+C_x}{\bar{x}\rho+C_x}$ Koyuncu and Kadilar (2009)	$\eta_8 = k\bar{y}\frac{\bar{x}\rho+C_x}{\bar{X}\rho+C_x}$ Koyuncu and Kadilar (2009)	ρ	C_x
$\eta_9 = k\bar{y}\frac{\bar{X}\beta_2(x)+\rho}{\bar{x}\beta_2(x)+\rho}$ Koyuncu and Kadilar (2009)	$\eta_{10} = k\bar{y}\frac{\bar{x}\beta_2(x)+\rho}{\bar{X}\beta_2(x)+\rho}$ Koyuncu and Kadilar (2009)	$\beta_2(x)$	ρ
$\eta_{11} = k\bar{y}\frac{\bar{X}\rho+\beta_2(x)}{\bar{x}\rho+\beta_2(x)}$ Koyuncu and Kadilar (2009)	$\eta_{12} = k\bar{y}\frac{\bar{x}\rho+\beta_2(x)}{\bar{X}\rho+\beta_2(x)}$ Koyuncu and Kadilar (2009)	ρ	$\beta_2(x)$
$\eta_{13} = k\bar{y}\frac{\bar{X}+S_x}{\bar{x}+S_x}$ Koyuncu and Kadilar (2009)	$\eta_{14} = k\bar{y}\frac{\bar{x}+S_x}{\bar{X}+S_x}$ Koyuncu and Kadilar (2009)	1	S_x
$\eta_{15} = k\bar{y}\frac{\bar{X}\beta_1(x)+S_x}{\bar{x}\beta_1(x)+S_x}$ Koyuncu and Kadilar (2009)	$\eta_{16} = k\bar{y}\frac{\bar{x}\beta_1(x)+S_x}{\bar{X}\beta_1(x)+S_x}$ Koyuncu and Kadilar (2009)	$\beta_1(x)$	S_x
$\eta_{17} = k\bar{y}\frac{\bar{X}\beta_2(x)+S_x}{\bar{x}\beta_2(x)+S_x}$ Koyuncu and Kadilar (2009)	$\eta_{18} = k\bar{y}\frac{\bar{x}\beta_2(x)+S_x}{\bar{X}\beta_2(x)+S_x}$ Koyuncu and Kadilar (2009)	$\beta_2(x)$	S_x
$\eta_{19} = k\bar{y}\frac{\bar{X}+\rho}{\bar{x}+\rho}$ Koyuncu and Kadilar (2009)	$\eta_{20} = k\bar{y}\frac{\bar{x}+\rho}{\bar{X}+\rho}$ Koyuncu and Kadilar (2009)	1	ρ

We have calculated the MSE values of mentioned estimators and the variance values of proposed estimators for this data set. Table 4 and Table 5 exhibit that our estimators are more effective than concerned estimators, as our estimators always have smaller MSE values than the interested estimators in literature. From this result, we can report that using the Hartley-Ross type estimators improves the proficiencies of the estimators for the population mean.

Table 4
MSE and Variance Values of $\bar{y}_{pr1i}$ and $\bar{y}_{KRi}$ Ratio Estimators

Estimators	MSE	Estimators	Var
$\bar{y}_{KR1}$	26185456.37	$\bar{y}_{pr11}$	18259992.59
$\bar{y}_{KR2}$	26186693.72	$\bar{y}_{pr12}$	18313588.28
$\bar{y}_{KR3}$	26185348.92	$\bar{y}_{pr13}$	18255500.4
$\bar{y}_{KR4}$	26185471.7	$\bar{y}_{pr14}$	18261041.07
$\bar{y}_{KR5}$	26185346.31	$\bar{y}_{pr15}$	18255395.6
$\bar{y}_{KR6}$	26186842.01	$\bar{y}_{pr16}$	18325903.88
$\bar{y}_{KR7}$	31788340.63	$\bar{y}_{pr17}$	29929485.94
$\bar{y}_{KR8}$	28072373.81	$\bar{y}_{pr18}$	25602753.68
$\bar{y}_{KR9}$	26516955.16	$\bar{y}_{pr19}$	21617350.89
$\bar{y}_{KR10}$	26185391.04	$\bar{y}_{pr110}$	18257415.02

Table 5
MSE and Variance Values of $\bar{y}_{pr3i}$ and η_i Ratio Estimators

Estimators	MSE	Estimators	Var
η_1	7986761.78	$\bar{y}_{pr31}$	1816884
η_2	7990116.24	$\bar{y}_{pr32}$	1815876
η_3	7986462.36	$\bar{y}_{pr33}$	1816974
η_4	7986795.99	$\bar{y}_{pr34}$	1816873
η_5	7986455.29	$\bar{y}_{pr35}$	1816976
η_6	7990519.1	$\bar{y}_{pr36}$	1815755
η_7	19944245.2	$\bar{y}_{pr37}$	16984055
η_8	12741972.6	$\bar{y}_{pr38}$	3751131
η_9	8871529.46	$\bar{y}_{pr39}$	1698463
η_{10}	7986576.8	$\bar{y}_{pr310}$	1816939

CONCLUSION

We have developed the unbiased Hartley-Ross type ratio and product estimators for the population mean, and then derived their variance equations. From Table 5, we find that the our estimator, $\bar{y}_{pr39}$, has the smallest variance value within the mentioned estimators. Moreover, the estimators, $\bar{y}_{pr3i}$ have smaller MSE and variance values than η_i family of estimators for $i = 1,2,\dots,10$. Similarly, the proposed estimators, $\bar{y}_{pr1i}$, also have smaller MSE and variance values than $\bar{y}_{KRi}$ family of estimators for all $i = 1,2,\dots,10$. Therefore, all of our proposed estimators perform better than corresponding estimators in literature for this data set.

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