Pak. J. Statist.
2016 Vol. 32(3), 161-182

TRANSMUTED GOMPERTZ DISTRIBUTION: PROPERTIES AND ESTIMATION

Muhammad Shuaib Khan[1], **Robert King**[2] and **Irene Lena Hudson**[3]
School of Mathematical and Physical Sciences, The University of Newcastle
Callaghan, NSW 2308, Australia.
Email: [1]Muhammad.S.Khan@newcastle.edu.au
[2]robert.king@newcastle.edu.au
[3]Irene.Hudson@newcastle.edu.au

ABSTRACT

This article investigates the potential usefulness of the transmuted Gompertz distribution for modeling lifetime data. This distribution can be obtained by using the quadratic rank transmutation map scheme. Various structural properties of the transmuted Gompertz model are investigated including estimation of the parameters using maximum likelihood and evaluate the performance of MLE using simulation. The potential usefulness of the transmuted Gompertz model is shown by means of windshields data.

KEYWORDS

Gompertz distribution; moments; entropies; maximum likelihood estimation; order statistics; simulation.

1. INTRODUCTION

The two parameter Gompertz distribution was pioneered by Gompertz (1825) and is often used in survival analysis for modeling human mortality data. It is widely used for modeling in demography, actuarial studies and biological sciences. The Gompertz distribution belongs to the exponential family of lifetime distributions. The two parameter Gompertz distribution has the cumulative distribution function (cdf) is given by

$$F(x;\alpha,\beta) = 1 - \exp\left\{-\frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)\right\}, \tag{1}$$

for $\alpha > 0$, $\beta > 0$ and $x > 0$. The corresponding probability density function (pdf) is given as follows

$$f(x;\alpha,\beta) = \alpha e^{\beta x} \exp\left\{-\frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)\right\}, \tag{2}$$

In recent literature the transmuted family of lifetime distributions have received a great attention of the researcher for modeling lifetime data. More recently Abdul-Moniem and Seham (2015) proposed the transmuted Gompertz distribution and discussed some mathematical properties of this model which includes the moments, TL-moments and L-moments. The authors compared the transmuted Gompertz with Gompertz distributions and estimated the models parameters by using the maximum likelihood estimation. The motivation of this study is to investigate the potential usefulness of the transmuted Gompertz distribution and derive several mathematical properties of this model. Interestingly, although the transmuted method has been used extensively for the development of the new family of lifetime distribution, an attempt has been made to use this idea to explore properties of the subject model with an efficient estimates of the model parameters. By using the idea of the quadratic rank transmutation map proposed by Shaw and Buckley (2007), we can obtain the three parameter transmuted Gompertz distribution. According to this approach a random variable X is said to have a transmuted distribution if its (cdf) satisfies the following relationship

$$F(x) = (1+\lambda)G(x) - \lambda G(x)^2, \ |\lambda| \leq 1 \tag{3}$$

and

$$f(x) = g(x)\{(1+\lambda) - 2\lambda G(x)\}, \tag{4}$$

where $G(x)$ is the cdf of the baseline model, $g(x)$ and $f(x)$ are the corresponding probability density functions (pdf) associated with $G(x)$ and $F(x)$, respectively. This paper investigates the statistical properties of the transmuted Gompertz distribution. Aryal and Tsokos (2011) studied the transmuted Weibull distribution to analyse two lifetime data sets. Recently Khan and King (2013a, 2013b) proposed the transmuted modified Weibull and the transmuted generalized inverse Weibull distributions and discussed structural properties with application to reliability data. Recently Khan et al. (2014) proposed the transmuted Inverse Weibull distribution and discussed various structural properties with application to reliability data. More recently Khan et al. (2015a, 2015b, 2016) studied the transmuted generalized exponential, transmuted Weibull and transmuted generalized Gompertz distributions by using QRTM technique which extend the baseline models for modeling lifetime data. Merovci (2013) and Yuzhu et al. (2014) proposed the transmuted Rayleigh and the transmuted linear exponential distributions with a discussion on some properties of this family.

The article is organized as follows, in Section 2, we present the analytical shapes of the probability density and hazard functions of the transmuted Gompertz distribution. A range of mathematical properties are considered in Section 3, such as we formulate the moments, moment generating function, incomplete moments, Bonferroni and Lorenz curves and probability weighted moments. Entropies are derived in section 4. Order statistics and their moments are derived in Section 5. Maximum likelihood estimates (MLEs) of the unknown parameters are discussed in Section 6. In Section 7, we evaluate the performance of the MLEs using simulation. Application to the real data set is illustrated in Section 8. In Section 9, concluding remarks are addressed.

2. TRANSMUTED GOMPERTZ DISTRIBUTION

A random variable X is said to have transmuted Gompertz distribution with parameters $\alpha, \beta > 0$ and $|\lambda| \leq 1$, $x > 0$ then X has the distribution function as,

$$f(x;\alpha,\beta,\lambda) = \alpha e^{\beta x} \exp\left\{-\frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)\right\}\left[1 - \lambda + 2\lambda \exp\left\{-\frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)\right\}\right]. \tag{5}$$

The CDF corresponding to (5) is given by

$$F(x;\alpha,\beta,\lambda)=\left[1-\exp\left\{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)\right\}\right]\left[1+\lambda\exp\left\{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)\right\}\right]. \tag{6}$$

A random variable X with pdf (5) is denoted by $X\sim TG(x;\alpha,\beta,\lambda)$. When the transmuting parameter $\lambda=0$, we obtain the classical Gompertz distribution. For $\beta=1$, it reduces to the transmuted extended exponential distribution. A physical interpretation of the transmuted Gompertz distribution is possible when the parameters α and β are positive. Where α and β are the scale and shape parameters and λ is the transmuted parameter representing the different patterns of the transmuted Gompertz distribution.

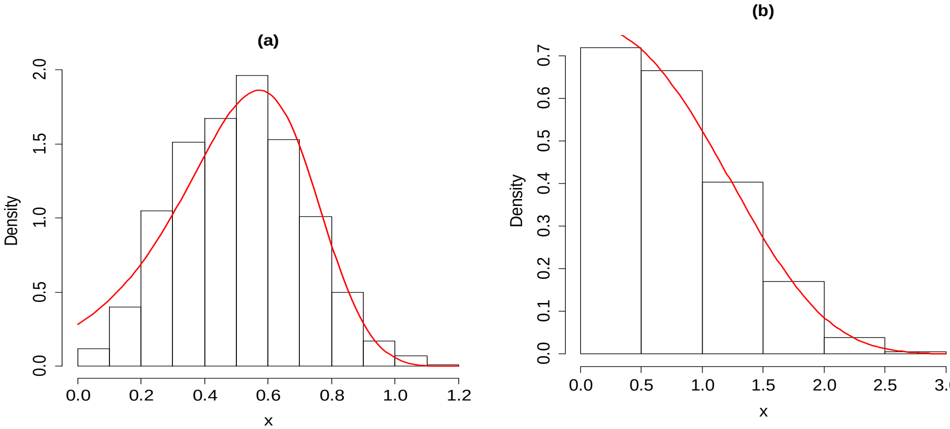


Figure 1: Plots of the TG Densities for Simulated Data Sets, (a) $\alpha=1;\beta=3;\lambda=-1$ (b) $\alpha=0.5;\beta=1;\lambda=0.5$.

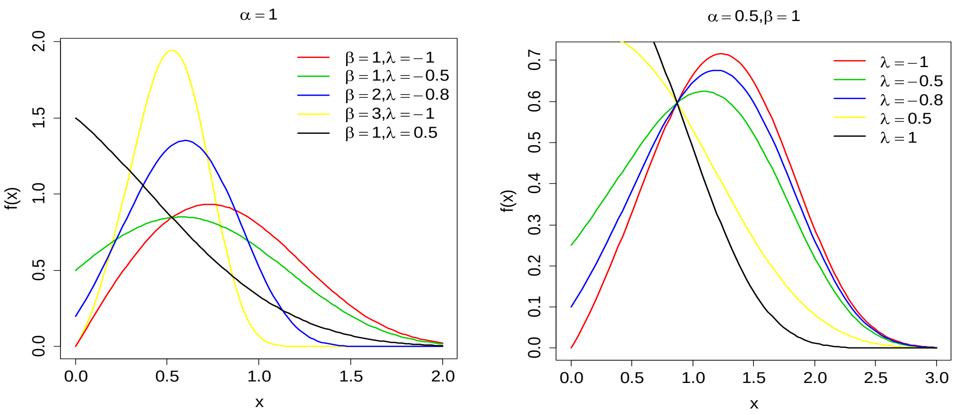


Figure 2: Plots of the TG pdf for Some Parameter Values

Figure 2 shows the shape of the transmuted Gompertz PDF with different choice of parameters. Figure 3 illustrates the instantaneous failure rate pattern of the transmuted Gompertz distribution with different choice of parameters and suggests that the distribution has increasing failure rates patterns. The reliability, quantile and hazard functions of the TG distribution are given by

$$R(x;\alpha,\beta,\lambda) = 1 - \left[1 - \exp\left\{-\frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)\right\}\right]\left[1 + \lambda \exp\left\{-\frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)\right\}\right] \quad (7)$$

$$F\left(x_q\right) = \frac{1}{\beta} ln\left[1 - \frac{\beta}{\alpha} ln\left\{1 - \frac{(1+\lambda) - \sqrt{(1+\lambda)^2 - 4\lambda q}}{2\lambda}\right\}\right], 0 < q < 1. \quad (8)$$

and

$$h(x;\alpha,\beta,\lambda) = \frac{\alpha e^{\beta x} \exp\left\{-\frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)\right\}\left[1 - \lambda + 2\lambda \exp\left\{-\frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)\right\}\right]}{1 - \left[1 - \exp\left\{-\frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)\right\}\right]\left[1 + \lambda \exp\left\{-\frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)\right\}\right]}. \quad (9)$$

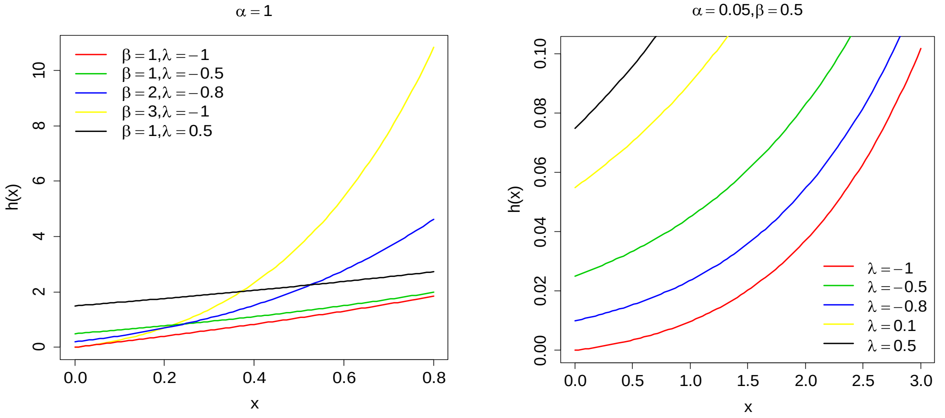


Figure 3: Plots of the TG hrf for Some Parameter Values

3. STATISTICAL PROPERTIES

In this section, we develop some structural properties of the transmuted Gompertz distribution.

3.1. *Moments*

If X has the $TG(x;\alpha,\beta,\lambda)$ with $|\lambda| \leq 1$, then the r^{th} moment of X is given as follows

$$\acute{\mu}_r = \int_0^\infty x^r\, \alpha e^{\beta x} exp\left\{-\frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)\right\}\left[1 - \lambda + 2\lambda\, exp\left\{-\frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)\right\}\right] dx.$$

The above expression reduces to

$$\acute{\mu}_r = (1-\lambda)\int_0^\infty x^r\, \alpha e^{\beta x} exp\left\{-\frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)\right\} dx$$
$$+2\lambda \int_0^\infty x^r\, \alpha e^{\beta x} exp\left\{-\frac{2\alpha}{\beta}\left(e^{\beta x} - 1\right)\right\} dx,$$

Hence, it follows that

$$\acute{\mu}_r = \sum_{i,k=0}^{\infty} \binom{i}{k} \left(\frac{\alpha}{\beta}\right)^k \frac{\alpha(-1)^{k+i+r}\Gamma(r+1)}{k!\;[\beta(i+1)]^{r+1}} (1-\lambda+\lambda 2^{k+1}). \tag{10}$$

3.2 *Moment generating function*

If X has the $TG(x;\alpha,\beta,\lambda)$ with $|\lambda| \leq 1$, then the moment generating function of X, $M_X(t)$ is given as follows

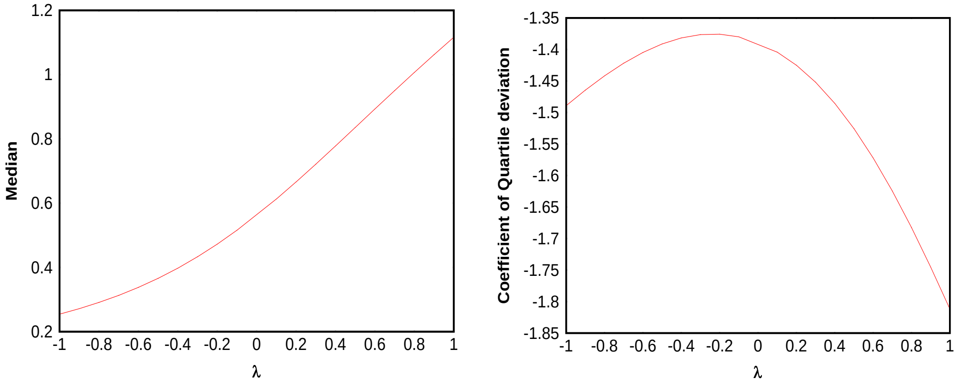
$$M_X(t) = \int_0^{\infty} \alpha e^{\beta x}\, exp\left\{tx - \frac{\alpha}{\beta}(e^{\beta x}-1)\right\} \left[1-\lambda+2\lambda\, exp\left\{-\frac{\alpha}{\beta}(e^{\beta x}-1)\right\}\right] dx,$$

using the Taylor series expansions, the above integral reduces to

$$M_X(t) = (1-\lambda)\alpha \sum_{m=0}^{\infty} \frac{t^m}{m!} \int_0^{\infty} x^m exp\left\{\beta x - \frac{\alpha}{\beta}(e^{\beta x}-1)\right\} dx$$
$$+2\lambda\alpha \sum_{m=0}^{\infty} \frac{t^m}{m!} \int_0^{\infty} x^m exp\left\{\beta x - \frac{2\alpha}{\beta}(e^{\beta x}-1)\right\} dx, \tag{11}$$

the integral in equation (11) can be finally obtained as

$$M_X(t) = \sum_{i,k,m=0}^{\infty} \frac{(t/\beta)^m}{m!} \binom{i}{k} \left(\frac{\alpha}{\beta}\right)^{k+1} \frac{(-1)^{k+i+m}\Gamma(m+1)}{k!\;(i+1)^{m+1}} (1-\lambda+\lambda 2^{k+1}). \tag{12}$$



(a) Median **(b) Coeff. of Quartile deviation**

Figure 4: Quantile Plots of the TG Distribution

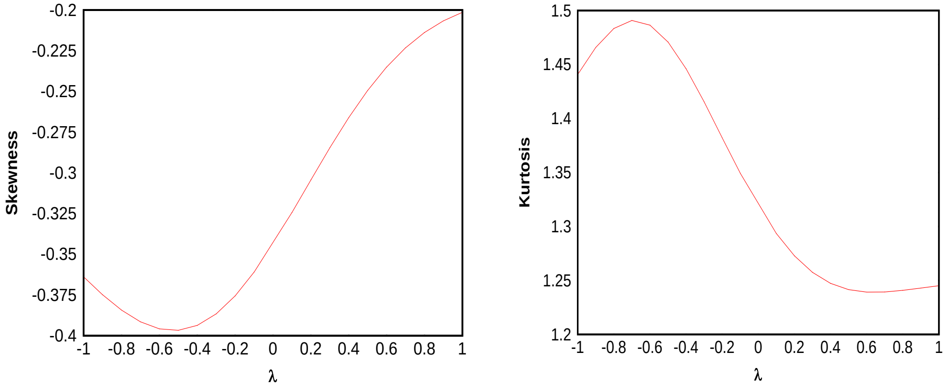


Figure 5: Skewness and Kurtosis of the TG distribution.

Table 1
Moments Calculated of the TG Distribution

α	β	λ	Estimates			
			$\acute{\mu}_1$	$\acute{\mu}_2$	$\acute{\mu}_3$	$\acute{\mu}_4$
0.5	1	-1	1.2494	1.8308	2.9811	5.2491
		-0.50	1.0864	1.5061	2.3810	4.1173
		0.50	0.7596	0.8566	1.1808	1.8539
		1	0.5963	0.5319	0.5806	0.7222
1	2	-1	0.6247	0.4577	0.3726	0.3280
		-0.50	0.5432	0.3765	0.2976	0.2573
		0.50	0.3798	0.2141	0.1476	0.1158
		1	0.2981	0.1329	0.0725	0.0451
2	2	-1	0.4156	0.2132	0.1255	0.0817
		-0.50	0.3569	0.1731	0.0990	0.0634
		0.50	0.2394	0.0928	0.0461	0.0268
		1	0.1806	0.0527	0.0196	0.0085
2	3	-1	0.3546	0.1505	0.0718	0.0374
		-0.50	0.3066	0.1230	0.0571	0.0292
		0.50	0.2106	0.0682	0.0275	0.0128
		1	0.1627	0.0408	0.0127	0.0046

Table 2
Moments based Measures of the TG Distribution

α	β	λ	Estimates				
			$Mean$	Var	CV	CS	CK
0.5	1	-1	1.2494	0.2698	0.4157	0.1393	2.5808
		-0.5	1.0864	0.3258	0.5254	0.1978	2.4201
		0.5	0.7596	0.2796	0.6961	0.7125	2.9791
		1	0.5963	0.1763	0.7042	0.7177	2.9858
1	2	-1	0.6247	0.0674	0.4157	0.1371	2.5863
		-0.5	0.5432	0.0814	0.5253	0.1986	2.4183
		0.5	0.3798	0.0698	0.6958	0.7163	2.9603
		1	0.2981	0.0440	0.7039	0.7172	3.0015
2	2	-1	0.4156	0.0404	0.4841	0.3991	2.7566
		-0.5	0.3569	0.0457	0.5991	0.4688	2.7202
		0.5	0.2394	0.0354	0.7868	1.0309	3.7415
		1	0.1806	0.0201	0.7847	0.9936	3.6265
2	3	-1	0.3546	0.0247	0.4437	0.2242	2.7253
		-0.5	0.3066	0.0289	0.5553	0.3255	2.4229
		0.5	0.2106	0.0238	0.7332	0.8397	3.3085
		1	0.1627	0.0143	0.7357	0.8158	3.4719

Table 3
Calculated Values of Rényi Entropy

α	β	λ	Estimates			
			$\rho = 2$	$\rho = 3$	$\rho = 4$	$\rho = 5$
0.5	1	-1	0.2776	0.2525	0.2367	0.2255
		-0.50	0.3195	0.3012	0.2888	0.2796
		0.50	0.2290	0.2053	0.1910	0.1812
		1	0.1249	0.1004	0.0853	0.0748
1	2	-1	-0.0234	-0.0484	-0.0643	-0.0754
		-0.50	0.0184	0.0001	-0.0123	-0.0214
		0.50	-0.0721	-0.0957	-0.1101	-0.1198
		1	-0.1761	-0.2005	-0.2157	-0.2262
2	2	-1	-0.1426	-0.1669	-0.1822	-0.1928
		-0.50	-0.1249	-0.1432	-0.1545	-0.1627
		0.50	-0.2762	-0.3135	-0.3368	-0.3532
		1	-0.3979	-0.4348	-0.4582	-0.4746
2	3	-1	-0.2430	-0.2674	-0.2828	-0.2937
		-0.50	-0.2108	-0.2285	-0.2401	-0.2486
		0.50	-0.3274	-0.3563	-0.3742	-0.3866
		1	-0.4393	-0.4689	-0.4875	-0.5006

Table 4
Calculated Values of q-Entropy

α	β	λ	Estimates			
			$q=2$	$q=3$	$q=4$	$q=5$
0.5	1	-1	0.4723	0.3437	0.2683	0.2187
		-0.50	0.5209	0.3751	0.2880	0.2309
		0.50	0.4098	0.3058	0.2442	0.2028
		1	0.2500	0.1852	0.1484	0.1245
1	2	-1	-0.0555	-0.1250	-0.0643	-0.0754
		-0.50	0.0416	0.0003	-0.0295	-0.0547
		0.50	-0.1805	-0.2769	-0.3795	-0.5040
		1	-0.5000	-0.7592	-1.1458	-1.7583
2	2	-1	-0.3888	-0.5784	-0.8398	-1.2273
		-0.50	-0.3333	-0.4667	-0.6363	-0.8689
		0.50	-0.8888	-1.6177	-3.0807	-6.2149
		1	-1.5000	-3.2037	-7.5625	-19.544
2	3	-1	-0.7500	-1.213	-2.0183	-3.4886
		-0.50	-0.6250	-0.9318	-1.4167	-2.2170
		0.50	-1.1250	-2.0797	-4.0875	-8.5497
		1	-1.7500	-3.8333	-9.3359	-24.878

3.3 *Incomplete Moment*

If $X \sim TG(x;\alpha,\beta,\lambda)$ with $|\lambda| \leq 1$, then the k^{th} incomplete moment of the TG distribution is as follows

$$\acute{\mu}_{(k)}(z) = \int_0^z x^k\, \alpha e^{\beta x}\, exp\left\{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)\right\} \left[1-\lambda+2\lambda\, exp\left\{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)\right\}\right] dx. \tag{13}$$

Using the binomial expansion equation (13) reduces to

$$\acute{\mu}_{(k)}(z) = (1-\lambda)\sum_{i,j=0}^{\infty}\binom{i}{j}\left(\frac{\alpha}{\beta}\right)^{j}\frac{\alpha(-1)^{i+j}}{j!}\int_0^z x^k exp\{(i+1)\beta x\}dx + 2\lambda\sum_{j=0}^{\infty}\binom{i}{j}\left(\frac{\alpha}{\beta}\right)^{j}\frac{\alpha(-1)^{i+j}2^{j}}{j!}\int_0^z x^k\, exp\{(i+1)\beta x\}dx\,.$$

Finally, we obtain

$$\acute{\mu}_{(k)}(z) = \sum_{i,j=0}^{\infty}\binom{i}{j}\left(\frac{\alpha}{\beta}\right)^{j+1}\frac{(-1)^{i+j+k}}{j!\,\beta^k(i+1)^{k+1}} \gamma\big(k+1,\beta z(i+1)\big)\{(1-\lambda)+\lambda 2^{j+1}\}. \tag{14}$$

The three main features of the k^{th} incomplete moment used to determine the Bonferroni and Lorenz curves, mean residual life and mean waiting time can be obtained from (14).

The degree of scatter in a population is widely measured by the totality of deviations from the mean and median. If X has the $\mathrm{TG}(x;\alpha,\beta,\lambda)$ distribution, we can then derive the mean deviation about the mean and about the median M from the following equations

$$\delta_1 = 2\{\mu\, F(\mu) - \psi(\mu)\} \text{ and } \delta_2 = \mu - 2\psi(M). \tag{15}$$

The mean is obtained from (10) with $r = 1$ and the median M is the solution of the non-linear equation from (8), where $\psi(z)$ can be obtained from (14) as

$$\psi(z) = \sum_{i,j=0}^{\infty} \binom{i}{j} \left(\frac{\alpha}{\beta}\right)^{j+1} \frac{(-1)^{i+j+1}}{j!\,\beta(i+1)^2} \gamma\big(2, \beta z(i+1)\big)\{(1-\lambda) + \lambda 2^{j+1}\}, \tag{16}$$

Hence, the measures in (15) can be obtained from (16). The quantity $\psi(z)$ can also be used to determine Bonferroni and Lorenz curves which have wide applications in econometrics and in finance. These Bonferroni and Lorenz curves equations can be calculated as

$$B(p) = \frac{1}{p\mu} \sum_{i,j=0}^{\infty} \binom{i}{j} \left(\frac{\alpha}{\beta}\right)^{j+1} \frac{(-1)^{i+j+1}}{j!\,\beta(i+1)^2} \gamma\big(2, \beta z(i+1)\big)\{(1-\lambda) + \lambda 2^{j+1}\},$$

and

$$L(p) = \frac{1}{\mu} \sum_{i,j=0}^{\infty} \binom{i}{j} \left(\frac{\alpha}{\beta}\right)^{j+1} \frac{(-1)^{i+j+1}}{j!\,\beta(i+1)^2} \gamma\big(2, \beta z(i+1)\big)\{(1-\lambda) + \lambda 2^{j+1}\},$$

where $q = F^{-1}(p)$ is calculated from (8) for a given probability p.

3.4 *Probability Weighted Moment*

If $X \sim TG(x;\alpha,\beta,\lambda)$ with $|\lambda| \le 1$, then the probability weighted moment (PWM) of the TG distribution is as follows

$$\xi_{(k,m)} = \int_0^{\infty} x^k\, F(x;\alpha,\beta,\lambda)^m\, f(x;\alpha,\beta,\lambda)dx.$$

From the above integral the expension of cdf in terms of infinite weighted sum as

$$F(x;\alpha,\beta,\lambda)^m = \sum_{p,q=0}^{\infty} \binom{m}{q}\binom{m}{p} (-1)^p \lambda^q\, exp\left\{-(p+q)\frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)\right\},$$

Using the binomial expansion PWM reduces to

$$\xi_{(k,m)} = (1-\lambda) \sum_{p,q=0}^{\infty} \binom{m}{q}\binom{m}{p} (-1)^p \lambda^q \int_0^{\infty} x^k\, exp\left\{\beta x - (p+q+1)\frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)\right\} dx$$
$$+2\lambda \sum_{p,q=0}^{\infty} \binom{m}{q}\binom{m}{p} (-1)^p \lambda^q \int_0^{\infty} x^k\, exp\left\{\beta x - (p+q+2)\frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)\right\} dx,$$

Finally, we obtain

$$\xi_{(k,m)} = (1-\lambda)\sum_{p,q=0}^{\infty}\sum_{i,j=0}^{\infty}\binom{i}{j}\left(\frac{\alpha}{\beta}\right)^{j} V_{p,q,i,j,\lambda}\,(p+q+1)^{j}\Gamma(k+1)$$
$$+2\lambda\sum_{p,q=0}^{\infty}\sum_{i,j=0}^{\infty}\binom{i}{j}\left(\frac{\alpha}{\beta}\right)^{j} V_{p,q,i,j,\lambda}\,(p+q+2)^{j}\Gamma(k+1), \quad (17)$$

where

$$V_{p,q,i,j,\lambda} = \binom{m}{q}\binom{m}{p}\frac{(-1)^{i+j+k+p}\lambda^{q}}{j!\left(\beta(i+1)\right)^{j+1}}.$$

4. ENTROPIES

The entropy of a random variable X with probability density $f(x)$ is a measure of variation of the uncertainty. A large value of entropy indicates the greater uncertainty in the data. The Rényi, A. (1961) introduced the Rényi entropy defined as

$$I_R(\rho) = \frac{1}{1-\rho}\log\left\{\int_0^{\infty} f(x)^{\rho}dx\right\}, \quad (18)$$

where $\rho > 0$ and $\rho \neq 1$. The integral in $I_R(\rho)$ of the $TG(x;\alpha,\beta,\lambda)$ can be defined as

$$\int_0^{\infty} f(x)^{\rho}dx = \int_0^{\infty}\alpha^{\rho}e^{\rho\beta x}\exp\left\{-\frac{\alpha\rho}{\beta}\left(e^{\beta x}-1\right)\right\}$$
$$\left[1-\lambda+2\lambda\exp\left\{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)\right\}\right]^{\rho}dx,$$

The above integral reduces to

$$\int_0^{\infty} f(x)^{\rho}dx = \sum_{m=0}^{\infty}\binom{\rho}{m}\left(\frac{2\lambda}{1-\lambda}\right)^{m}(1-\lambda)^{\rho}\,\alpha^{\rho}$$
$$\int_0^{\infty} e^{\rho\beta x}\exp\left(-\frac{(\rho+m)\alpha}{\beta}\left(e^{\beta x}-1\right)\right)dx,$$
$$= \sum_{m,p,q=0}^{\infty}\binom{\rho}{m}\binom{p}{q}\left(\frac{2\lambda}{1-\lambda}\right)^{m}\left(\frac{\alpha}{\beta}\right)^{p}(1-\lambda)^{\rho}\,\frac{(-1)^{p+q}\alpha^{\rho}(\rho+m)^{p}}{(p+q)\beta}, \quad (19)$$

Finally we obtain the Rényi entropy as

$$I_R(\rho) = \frac{\rho}{1-\rho}\log(\alpha) - \frac{1}{1-\rho}\log(\beta) + \frac{\rho}{1-\rho}\log(1-\lambda) + \frac{1}{1-\rho}\log$$
$$\left\{\sum_{m,p,q=0}^{\infty}\binom{\rho}{m}\binom{p}{q}\left(\frac{2\lambda}{1-\lambda}\right)^{m}\left(\frac{\alpha}{\beta}\right)^{p}\frac{(-1)^{p+q}(\rho+m)^{p}}{(p+q)}\right\}. \quad (20)$$

The q-(or α entropy) was introduced by Havrda and Charvat (1967) and is the one parameter generalization of the Shannon entropy. Ullah, 1996 stated that q-(or α entropy) measures the monotonic functions of the Rényi entropy and is defined as

$$I_H(q) = \frac{1}{q-1}\left\{1 - \int_0^\infty f(x)^q \, dx\right\}, \tag{21}$$

where $q > 0$ and $q \neq 1$. The integral in $I_H(q)$ of the TG distribution can be defined as

$$\int_0^\infty f(x)^q dx = \int_0^\infty \alpha^q e^{q\beta x} \exp\left\{-\frac{\alpha q}{\beta}\left(e^{\beta x} - 1\right)\right\} \left[1 - \lambda + 2\lambda \exp\left\{-\frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)\right\}\right]^q dx.$$

Using (19) and (21), we obtain the expression of the $q -$ entropy as

$$I_H(q) = \frac{1}{q-1}\left\{1 - \sum_{m,p,q=0}^\infty \binom{\rho}{m}\binom{p}{q}\left(\frac{2\lambda}{1-\lambda}\right)^m \left(\frac{\alpha}{\beta}\right)^p (1-\lambda)^\rho \frac{(-1)^{p+q}\alpha^\rho(\rho+m)^p}{(p+q)\beta}\right\}.$$

Shannon's introduced a probabilistic measure of uncertainty which is used in almost every branch of reliability engineering and biomedical sciences. The Shannon entropy of the TG distribution can be defined as

$$E[-log f(x)] = -log(\alpha) - \beta E(x) - \frac{\alpha}{\beta} E\left(e^{\beta x} - 1\right) + E\left[log\left\{1 - \lambda + 2\lambda \exp\left\{-\frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)\right\}\right\}\right]. \tag{22}$$

The expectation in equation (22) can be obtained by the following steps,

$$E\left[log\left\{1 - \lambda + 2\lambda \exp\left\{-\frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)\right\}\right\}\right] = \log(1-\lambda) + \sum_{k=1}^\infty \frac{(-1)^{k+1}}{k}\left(\frac{2\lambda}{1-\lambda}\right)^k I_k,$$

where

$$I_k = \int_0^\infty \exp\left\{-\frac{k\alpha}{\beta}\left(e^{\beta x} - 1\right)\right\} f(x) dx,$$

whereby the above integral can be calculated as

$$I_k = \int_0^\infty \alpha e^{\beta x} \exp\left\{-\frac{\alpha(k+1)}{\beta}\left(e^{\beta x} - 1\right)\right\} \left[1 - \lambda + 2\lambda \exp\left\{-\frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)\right\}\right] dx,$$

where

$$I_k = (1-\lambda)\int_0^\infty \alpha e^{\beta x} \exp\left\{-\frac{\alpha(k+1)}{\beta}\left(e^{\beta x} - 1\right)\right\} dx + 2\lambda \int_0^\infty \alpha e^{\beta x} \exp\left\{-\frac{\alpha(k+2)}{\beta}\left(e^{\beta x} - 1\right)\right\} dx,$$

and then

$$I_k = \sum_{i,j=0}^\infty \binom{i}{j}\frac{(-1)^{i+j}}{i!\,(j+1)}\left(\frac{\alpha}{\beta}\right)^{i+1} \{(1-\lambda)(k+1)^i + 2\lambda(k+2)^i\}.$$

Finally we obtain the Shannon entropy as

$$E[-logf(x)] = -log(\alpha) - \beta\acute{\mu}_1 - \frac{\alpha}{\beta}\left(\sum_{k=1}^{\infty}\frac{\beta^k}{k!}\acute{\mu}_k - 1\right)$$
$$+log(1-\lambda) + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\left(\frac{2\lambda}{1-\lambda}\right)^k I_k.$$

5. ORDER STATISTICS

Let $x_1, x_2, \dots, x_n$ be independently and identically distributed ordered random variables from the $TG(x;\alpha,\beta,\lambda)$ distribution having the pdf of rth order statistics is given by

$$f_{r:n}(x) = \frac{1}{B(r, n-r+1)}\sum_{m=0}^{n-r}\binom{n-r}{m}(-1)^m\big(F(x)\big)^{r+m-1}f(x), \tag{23}$$

First we evaluate the expression as

$$F(x;\alpha,\beta,\lambda)^{r+m-1} = \sum_{i,j=0}^{\infty}\mathcal{V}_{i,j,\lambda}\, exp\left\{-(i+j)\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)\right\}, \tag{24}$$

where

$$\mathcal{V}_{i,j,\lambda} = \binom{r+m-1}{i}\binom{r+m-1}{j}(-1)^i\lambda^j.$$

The pdf of rth order statistics can be written as a linear combination by inserting (5) and (24) in (23), we obtain

$$f_{r:n}(x) = n\binom{n-1}{r-1}\sum_{m=0}^{n-r}\sum_{i,j=0}^{\infty}\binom{n-r}{m}(-1)^m\,\alpha\,\mathcal{V}_{i,j,\lambda}$$
$$\exp\left\{\beta x - (i+j+1)\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)\right\}$$
$$\times\left\{1-\lambda+2\lambda\exp\left\{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)\right\}\right\}. \tag{25}$$

Using (25), we obtain the k^{th} moment of the TG distribution of $X_{i:n}$ is given by

$$E\left(X_{i:n}^k\right) = n\binom{n-1}{r-1}\sum_{m=0}^{n-r}\binom{n-r}{m}(-1)^m\xi_{(k,r+m-1)}, \tag{26}$$

where $\xi_{(k,r+m-1)}$ is given in Section 3.4 (PWM).

6. PARAMETER ESTIMATION

Consider the random samples $x_1, x_2,, x_n$ consisting of n observations from the transmuted Gompertz distribution then the log-likelihood function $\mathcal{L} = \text{lnL}$ of (5) is given by

$$\mathcal{L} = nln\alpha + \beta\sum_{i=1}^{n} x_i - \frac{\alpha}{\beta}\sum_{i=1}^{n}\left(e^{\beta x_i} - 1\right) + \sum_{i=1}^{n} ln\left\{1 - \lambda + 2\lambda \exp\left\{-\frac{\alpha}{\beta}\left(e^{\beta x_i} - 1\right)\right\}\right\}. \quad (27)$$

Let $\Theta = (\alpha, \beta, \lambda)^{\mathrm{T}}$ be the parameter vector. The associated score function is given by $U(\Theta) = (\partial\mathcal{L}/\partial\alpha, \partial\mathcal{L}/\partial\beta, \partial\mathcal{L}/\partial\lambda)^{\mathrm{T}}$, where

$$\frac{\partial\mathcal{L}}{\partial\alpha} = \frac{n}{\alpha} - \frac{1}{\beta}\sum_{i=1}^{n}\left(e^{\beta x_i} - 1\right) - \frac{2\lambda}{\beta}\sum_{i=1}^{n}\frac{exp\left\{-\frac{\alpha}{\beta}\left(e^{\beta x_i} - 1\right)\right\}\left(e^{\beta x_i} - 1\right)}{\left\{1 - \lambda + 2\lambda\, exp\left\{-\frac{\alpha}{\beta}\left(e^{\beta x_i} - 1\right)\right\}\right\}},$$

$$\frac{\partial\mathcal{L}}{\partial\beta} = \sum_{i=1}^{n} x_i - \frac{\alpha}{\beta^2}\sum_{i=1}^{n}\left\{e^{\beta x_i}(\beta x_i - 1) + 1\right\} - \frac{2\lambda\alpha}{\beta^2}\sum_{i=1}^{n}\frac{exp\left\{-\frac{\alpha}{\beta}\left(e^{\beta x_i} - 1\right)\right\}\left\{e^{\beta x_i}(\beta x_i - 1) + 1\right\}}{\left\{1 - \lambda + 2\lambda\, exp\left\{-\frac{\alpha}{\beta}\left(e^{\beta x_i} - 1\right)\right\}\right\}},$$

$$\frac{\partial\mathcal{L}}{\partial\lambda} = \sum_{i=1}^{n}\frac{2\, exp\left\{-\frac{\alpha}{\beta}\left(e^{\beta x_i} - 1\right)\right\} - 1}{\left\{1 - \lambda + 2\lambda\, exp\left\{-\frac{\alpha}{\beta}\left(e^{\beta x_i} - 1\right)\right\}\right\}}.$$

The maximum likelihood estimates (MLEs) of the parameter vector $\Theta = (\alpha, \beta, \lambda)^{\mathrm{T}}$ are obtained by solving the non-linear equations $\mathrm{U}(\Theta) = 0$. These system of non-linear equations can be solved numarically by using softwares such as R, SAS, MAPLE and OX.

For interval estimation and hypothesis tests on the model parameters of the TG distribution, we require the 3×3 unit observed information matrix as

$$I_n(\Theta) = -\begin{pmatrix} I_{\alpha,\alpha} & I_{\alpha,\beta} & I_{\alpha,\lambda} \\ I_{\alpha,\beta} & I_{\beta,\beta} & I_{\beta,\lambda} \\ I_{\alpha,\lambda} & I_{\beta,\lambda} & I_{\lambda,\lambda} \end{pmatrix},$$

The asymptotic multivariate normal $N_3(0, I_n(\Theta)^{-1})$ distribution can be used to construct the approximate confidence intervals and confidence region of individual parameters for the transmuted Gompertz distribution. We can compute the maximum values of restricted and unrestricted log-likelihood to construct the likelihood ratio (LR) statistics for testing the TG distribution and compare with other lifetime distributions. For any testing of hypothesis we formulate the null hypothesis $H_0{:}\,\Theta = \Theta_0$ versus $H_A{:}\,\Theta \neq \Theta_0$ can be performed based on the LR statistics to compare the TG distribution with other lifetime distributions. For example, the test of $H_0{:}\,\lambda = 0$ versus $H_A{:}\,H_0$ is not true to compare the TG with Gompertz distributions. In this case the likelihood ratio (LR) statistics is

$$\Lambda = 2\{l(\tilde{\alpha}, \tilde{\beta}, 0) - l(\hat{\alpha}, \hat{\beta}, \hat{\lambda})\},$$

where $\hat{\alpha}, \hat{\beta}$ and $\hat{\lambda}$ are the MLEs under H_0 and $\tilde{\alpha}$ and $\tilde{\beta}$ are the estimates under H_A.

The elements of the 3×3 information matrix $I_n(\Theta)$ are given by

$$\frac{\partial^2 \mathcal{L}}{\partial \alpha^2} = -\frac{n}{\alpha^2} + \frac{2\lambda(1-\lambda)}{\beta^2} \sum_{i=1}^{n} \frac{v_i\left(e^{\beta x_i} - 1\right)^2}{\{1 - \lambda + 2\lambda v_i\}^2},$$

$$\frac{\partial^2 \mathcal{L}}{\partial \beta^2} = -\frac{\alpha}{\beta^4} \sum_{i=1}^{n} \{\beta^3 x_i^2 e^{\beta x_i} - 2\beta u_i\}$$
$$- \frac{2\lambda\alpha}{\beta^4} \sum_{i=1}^{n} \frac{u_i\{\beta^3 x_i^2 e^{\beta x_i} - 2\beta u_i\} - \alpha v_i u_i^2}{\{1 - \lambda + 2\lambda v_i\}} - \frac{2\lambda\alpha^2}{\beta^4} \sum_{i=1}^{n} \frac{u_i^2 v_i^2}{\{1 - \lambda + 2\lambda v_i\}^2},$$

$$\frac{\partial^2 \mathcal{L}}{\partial \lambda^2} = -\sum_{i=1}^{n} \left\{\frac{2v_i - 1}{1 - \lambda + 2\lambda v_i}\right\}^2,$$

$$\frac{\partial^2 \mathcal{L}}{\partial \alpha \partial \beta} = -\frac{1}{\beta^2} \sum_{i=1}^{n} u_i + \frac{2\lambda}{\beta^2} \sum_{i=1}^{n} \frac{v_i u_i \left\{(1-\lambda)\left\{1 - \frac{\alpha}{\beta}\left(e^{\beta x_i} - 1\right)\right\} + 2\lambda v_i\right\}}{\{1 - \lambda + 2\lambda v_i\}^2},$$

$$\frac{\partial^2 \mathcal{L}}{\partial \alpha \partial \lambda} = -\frac{2(1-\lambda)}{\beta} \sum_{i=1}^{n} \frac{v_i\left(e^{\beta x_i} - 1\right)}{\{1 - \lambda + 2\lambda v_i\}^2},$$

and

$$\frac{\partial^2 \mathcal{L}}{\partial \beta \partial \lambda} = -\frac{\alpha}{\beta^2} \sum_{i=1}^{n} \frac{v_i u_i}{\{1 - \lambda + 2\lambda v_i\}^2},$$

respectively,

where $u_i = e^{\beta x_i}(\beta x_i - 1) + 1, v_i = exp\left\{-\frac{\alpha}{\beta}\left(e^{\beta x_i} - 1\right)\right\}$.

7. SIMULATION

This section evaluates the performance of MLEs by using Monte Carlo simulation. The inversion method is used to generate samples from the TG distribution for different sample sizes $n = 25, 50, 100, 200, 400$ and for different choices of parameters using equation (8). The simulation process is repeated for 1000 times using the BFGS optimization method in R and compute the MLE for the TG distribution by optimum routine were displayed in Table 5-10. Tables 5-10 show the output of the ML estimates, standard deviation, bias, mean square error (MSE), ratio, root mean square error (RMSE) vary with respect to the sample sizes. These results of the estimated values of the parameters α, β, λ, are quite promising as the sample size n increases the values of the bias and mean square error (MSE) decreases. Furthermore, the graphical comparison of these three parameters of the bias estimates for the TG distribution are displayed in Figure 6.

Table 5
Estimated Values of $\hat{\Theta}$ based on MLE

$\Theta = (\alpha, \beta, \lambda)$	n	Estimates		
		$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$
0.5, 0.5, -0.8	25	0.3085	0.8889	-0.2385
	50	0.3481	0.6441	-0.9998
	100	0.6108	0.4006	-0.9894
	200	0.3906	0.6046	-0.4974
	400	0.5629	0.4585	-0.8956
0.5, 1, -0.5	25	0.3661	1.4189	-0.1040
	50	0.4274	1.0763	-0.9327
	100	0.7020	0.7730	-0.8167
	200	0.2904	1.2820	0.0749
	400	0.5418	0.9707	-0.5426
1, 2, 0.5	25	0.9645	2.5403	0.7284
	50	0.6688	2.8042	0.1245
	100	0.9276	1.9240	0.6606
	200	0.8518	2.0221	0.7920
	400	0.9582	2.2675	0.4019
1, 3, 0.8	25	1.1423	3.8676	0.7468
	50	0.7971	4.2617	0.1238
	100	1.1080	2.9697	0.6914
	200	1.0268	3.0863	0.8147
	400	1.0262	2.8571	0.7517

Table 6
Standard Deviations of the Estimate of $\hat{\Theta}$ based on MLE

$\Theta = (\alpha, \beta, \lambda)$	n	SD		
		$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$
0.5, 0.5, -0.8	25	0.2950	0.4519	0.8800
	50	0.3152	0.4282	0.8283
	100	0.1217	0.1197	0.1642
	200	0.2092	0.2390	0.5140
	400	0.0655	0.0676	0.0999
0.5, 1, -0.5	25	0.3375	0.6105	0.8222
	50	0.2302	0.3859	0.3916
	100	0.1774	0.2031	0.2160
	200	0.1236	0.2182	0.3993
	400	0.1212	0.1548	0.2057
1, 2, 0.5	25	0.6715	1.1654	1.0169
	50	0.4158	0.7709	0.5698
	100	0.2915	0.3961	0.3939
	200	0.1482	0.3290	0.2287
	400	0.3013	0.2711	0.3636
1, 3, 0.8	25	0.7880	1.7578	1.0684
	50	0.5184	1.1024	0.5848
	100	0.3118	0.5578	0.3441
	200	0.1632	0.4443	0.1895
	400	0.1335	0.2812	0.1640

Table 7
Bias of the Estimate of $\hat{\Theta}$ based on MLE

$\Theta = (\alpha, \beta, \lambda)$	n	Bias		
		$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$
0.5, 0.5, -0.8	25	-0.1915	0.3889	0.5615
	50	-0.1519	0.1441	-0.1998
	100	0.1108	-0.0994	-0.1894
	200	-0.1094	0.1046	0.3026
	400	0.0629	-0.0415	-0.0956
0.5, 1, -0.5	25	-0.1339	0.4189	0.3960
	50	-0.0726	0.0763	-0.4327
	100	0.2020	-0.2270	-0.3167
	200	-0.2096	0.2820	0.5749
	400	0.0418	-0.0293	-0.0426
1, 2, 0.5	25	-0.0355	0.5403	0.2284
	50	-0.3312	0.8042	-0.3755
	100	-0.0724	-0.0760	0.1606
	200	-0.1482	0.0221	0.2920
	400	-0.0418	0.2675	-0.0981
1, 3, 0.8	25	0.1423	0.8676	-0.0532
	50	-0.2029	1.2617	-0.6762
	100	0.1080	-0.0303	-0.1086
	200	0.0268	0.0863	0.0147
	400	0.0262	-0.1429	-0.0483

Table 8
MSE of the Estimate of $\hat{\Theta}$ based on MLE

$\Theta = (\alpha, \beta, \lambda)$	n	MSE		
		$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$
0.5, 0.5, -0.8	25	0.1236	0.3554	1.0896
	50	0.1224	0.2041	0.7260
	100	0.0270	0.0242	0.0628
	200	0.0557	0.0680	0.3557
	400	0.0082	0.0062	0.0191
0.5, 1, -0.5	25	0.1318	0.5481	0.8328
	50	0.0582	0.1547	0.3405
	100	0.0722	0.0927	0.1469
	200	0.0592	0.1271	0.4899
	400	0.0164	0.0248	0.0441
1, 2, 0.5	25	0.4521	1.6500	1.0862
	50	0.2825	1.2410	0.4656
	100	0.0902	0.1626	0.1809
	200	0.0439	0.1087	0.1375
	400	0.0925	0.1450	0.1418
1, 3, 0.5	25	0.6411	3.8425	1.1443
	50	0.3099	2.8071	0.7992
	100	0.1088	0.3120	0.1301
	200	0.0273	0.2048	0.0361
	400	0.0185	0.0994	0.0292

Table 9
Ratio of the Estimate of $\widehat{\boldsymbol{\Theta}}$ based on MLE

$\boldsymbol{\Theta} = (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\lambda})$	$\boldsymbol{n}$	Ratio		
		$\widehat{\boldsymbol{\alpha}}/\boldsymbol{\alpha}$	$\widehat{\boldsymbol{\beta}}/\boldsymbol{\beta}$	$\widehat{\boldsymbol{\lambda}}/\boldsymbol{\lambda}$
0.5, 0.5, -0.8	25	0.6170	1.7778	0.2981
	50	0.6962	1.2882	1.2497
	100	1.2216	0.8012	1.2367
	200	0.7812	1.2092	0.6217
	400	1.1258	0.9170	1.1195
0.5, 1, -0.5	25	0.7322	1.4189	0.2080
	50	0.8548	1.0763	1.8654
	100	1.4040	0.7730	1.6334
	200	0.5808	1.2820	-0.1498
	400	1.0836	0.9707	1.0852
1, 2, 0.5	25	0.9645	1.2701	1.4568
	50	0.6688	1.4021	0.2490
	100	0.9276	0.9620	1.3212
	200	0.8518	1.0110	1.5840
	400	0.9582	1.1337	0.8038
1, 3, 0.8	25	1.1423	1.2892	0.9335
	50	0.7971	1.4205	0.1547
	100	1.1080	0.9899	0.8642
	200	1.0268	1.0287	1.0183
	400	1.0262	0.9523	0.9396

Table 10
RMSE of the estimate of $\widehat{\boldsymbol{\Theta}}$ based on MLE

$\boldsymbol{\Theta} = (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\lambda})$	$\boldsymbol{n}$	RMSE		
		$\widehat{\boldsymbol{\alpha}}$	$\widehat{\boldsymbol{\beta}}$	$\widehat{\boldsymbol{\lambda}}$
0.5, 0.5, -0.8	25	0.3517	0.5962	1.0438
	50	0.3498	0.4517	0.8520
	100	0.1645	0.1555	0.2506
	200	0.2360	0.2608	0.5964
	400	0.0908	0.0793	0.1382
0.5, 1, -0.5	25	0.3630	0.7403	0.9125
	50	0.2413	0.3933	0.5835
	100	0.2688	0.3045	0.3833
	200	0.2433	0.3565	0.6999
	400	0.1282	0.1575	0.2101
1, 2, 0.5	25	0.6724	1.2845	1.0422
	50	0.5315	1.1140	0.6824
	100	0.3003	0.4033	0.4253
	200	0.2095	0.3297	0.3709
	400	0.3041	0.3808	0.3766
1, 3, 0.8	25	0.8007	1.9602	1.0697
	50	0.5566	1.6754	0.8940
	100	0.3299	0.5586	0.3608
	200	0.1653	0.4526	0.1900
	400	0.1360	0.3154	0.1709

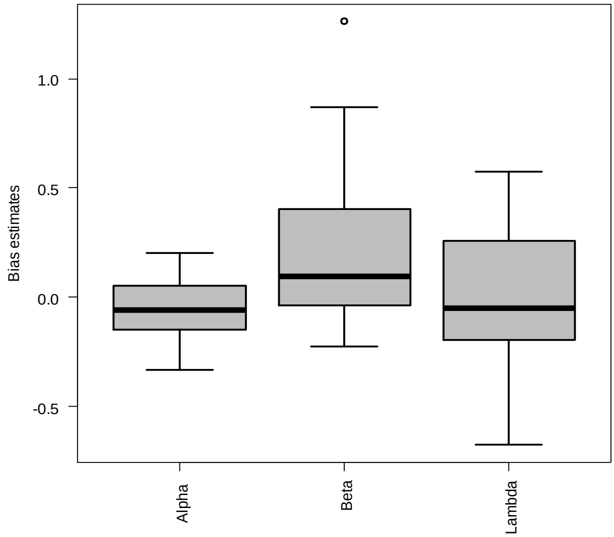


Figure 6: Boxplot of the Bias Estimates

8. APPLICATION

This section provides the data analysis in order to assess the goodness-of-fit of a model with failure times of windshields data. The data have been obtained from Murthy et al. (2004) and is given by

0.04, 0.3, 0.31, 0.557, 0.943, 1.07, 1.124, 1.248, 1.281, 1.281, 1.303, 1.432, 1.48,1.51, 1.51, 1.568, 1.615, 1.619, 1.652, 1.652, 1.757, 1.795, 1.866, 1.876, 1.899, 1.911, 1.912, 1.9141, 0.981, 2.010, 2.038, 2.085, 2.089, 2.097, 2.135, 2.154, 2.190, 2.194, 2.223, 2.224, 2.23, 2.3, 2.324, 2.349, 2.385, 2.481, 2.610, 2.625, 2.632, 2.646, 2.661, 2.688, 2.823, 2.89, 2.9, 2.934, 2.962, 2.964, 3, 3.1, 3.114, 3.117, 3.166, 3.344, 3.376, 3.385, 3.443, 3.467, 3.478, 3.578, 3.595, 3.699, 3.779, 3.924, 4.035, 4.121, 4.167, 4.240, 4.255, 4.278, 4.305, 4.376, 4.449, 4.485, 4.570, 4.602, 4.663, 4.694.

We fitted the TG, GE, WE and G distributions by the method of maximum likelihood. The required numerical evaluations were implemented using R language, R Core Team (2015). The MLEs and the values of maximized log-likelihoods for transmuted Gompertz, Generalized Exponential, Weighted Exponential and Gompertz distributions are displayed in Table 11. Table 11 gives the MLEs of the unknown parameters (with their standard errors) and the K-S test (Kolmogorov–Smirnov test), with their corresponding P-values and the estimated values of log-likelihood. The Kolmogorov–Smirnov statistic used for goodness of fit test and quantifies a distance between the empirical distribution function of the sample information and the cumulative distribution function of the TG distribution. Comparing with three distributions indicate that the TG distribution provides better fit for the windshields data. The graphical goodness of fit displayed in Figure 7 indicates that the TG distribution provides better fit for the failure times of windshields data. In Figure 8(a) & 8(b), we display the pp-plot, the empirical survival function and the estimated survival function of the TG distribution which shows the satisfactory fit.

Table 11
MLEs of the Parameters for Windshields Data, the Corresponding SEs with the K-S Test, P-value and Estimated Log-Likelihood Values

Distribution	Parameter Estimates			K-S Test	P-value	-Log-Likelihood
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$			
TG	0.0567 (0.0244)	0.8398 (0.1042)	0.3408 (0.3820)	0.0824	0.5881	134.46
GE	3.5681 (0.5999)	0.7558 (0.0751)	-	0.1071	0.2648	146.88
WE	0.0001 (0.2528)	0.7785 (0.1134)	-	0.1724	0.0107	150.26
G	0.0811 (0.0207)	0.7710 (0.0905)	-	0.0845	0.5565	134.72

To further verify which distribution provides the better estimates for windshields data, we apply the Cramér-von Mises and Anderson-Darling goodness of-fit statistics displayed in Table 12. The smaller values of these statistics indicate the better fit. We observe from Table 12 that the data points from the TG distribution has better relationship and hence the TG distribution is a good model for failure times of windshields data.

Table 12
Cramér-von Mises and Anderson-Darling Goodness of-Fit Statistics

Distribution	$\mathcal{W}$	$\mathcal{A}$
TG	0.1352	0.8428
GE	0.1961	1.5916
WE	0.1442	1.2488
G	0.1462	0.9030

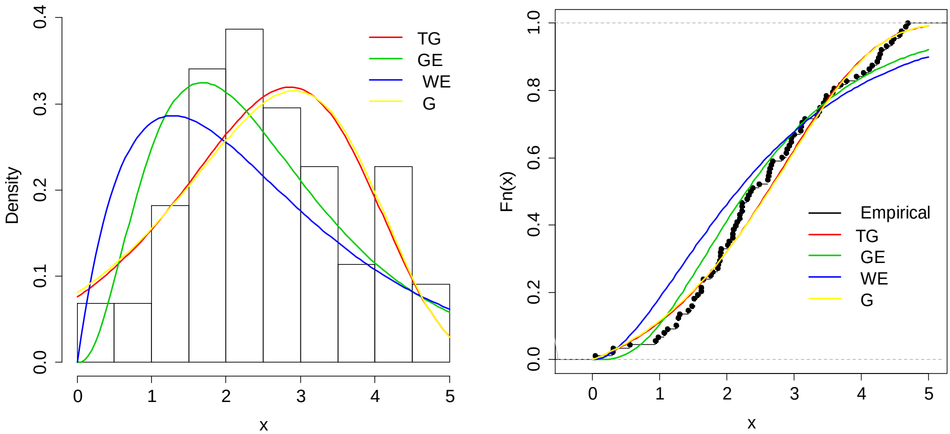
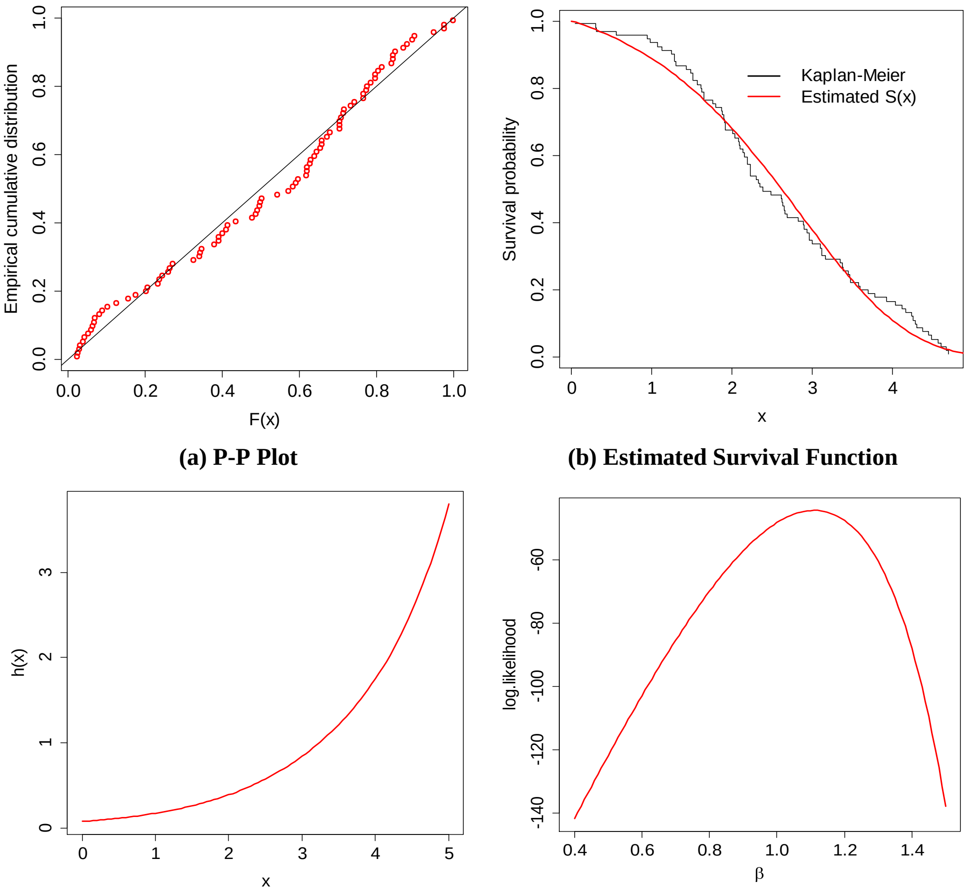


Figure 7: Plots of the Fitted pdfs and cdfs Models for Windshields Data

Figure 8(c) suggests that the failure times of the windshields data has increasing hazard function with time and follows the wear-out period of the bathtub shape failure rates. During this period the failure is generally caused by factors such as fatigue and degradation. Figure 8(d) shows the profile of log-likelihood function for the parameter β of the TG distribution for windshields data.



(a) P-P Plot **(b) Estimated Survival Function**

(c) Estimated Hazard Function **(d) Profile of Log-Likelihood for β**

Figure 8: pp-plot, Estimated Survival and Hazard Curves and the Profile of Log-Likelihood Function for the TG Distribution for Windshields data

9. CONCLUDING REMARKS

This paper studies the three parameters transmuted Gompertz distribution and derive several theoretical properties of this model with application. We obtain the analytical shapes of density and hazard functions. This model is capable of modeling increasing hazard rate function for lifetime data. We derive the explicit expressions for the moments, moment generating function, incomplete moments, probability weighted moments, quantile function, Renyi, Shannon and q-entropies, mean deviation, Bonferroni and Lorenz curves. We also derive the k^{th} moment of r^{th} order statistics. The practical relevance and applicability of the TG distribution are illustrated using windshields data. Based on the goodness of fit measures, the TG distribution provides better fit than the other three lifetime distributions. Therefore, we conclude that the model under study provides more flexibility for fitting windshields data.

ACKNOWLEDGEMENTS

The authors would like to thank the chief editor and reviewers for valuable comments and suggestions to improve the presentation of the paper.

REFERENCES

1. Abdul-Moniem, I.B. and Seham, M. (2015). Transmuted Gompertz Distribution. *Computational and Applied Mathematics*, 1(3), 88-96.
2. Aryal G.R. and Tsokos C.P. (2011). Transmuted Weibull distribution: A Generalization of the Weibull Probability Distribution. *European Journal of Pure and Applied Mathematics*, 4(2), 89-102.
3. Gompertz, B. (1825). On the Nature of the Function Expressive of the Law of Human Mortality, and on a New Mode of Determining the Value of Life Contingencies. *Philosophical Transactions of the Royal Society of London*, 115, 513-583.
4. Havrda, J. and Charvat, F. (1967). Quantification method in classification processes: concept of structural α-entropy. *Kybernetika*, 3, 30-35.
5. Khan, M.S. and King, R. (2013a). Transmuted Modified Weibull Distribution: A Generalization of the Modified Weibull Probability Distribution. *European Journal of Pure and Applied Mathematics*, 6(1), 66-88.
6. Khan, M.S. and King, R. (2013b). Transmuted generalized Inverse Weibull distribution. *Journal of Applied Statistical Sciences*, 20(3), 15-32.
7. Khan, M.S, King, R. and Hudson, I. (2014). Characterizations of the transmuted Inverse Weibull distribution. *ANZIAM J.*, 55 (EMAC2013), C197-C217.
8. Khan, M.S., King, R. and Hudson, I. (2015a). Transmuted generalized Exponential distribution: A generalization of the Exponential distribution with applications to survival data. *Communications in Statistics: Simulation and Computation*, to appear
9. Khan, M.S., King, R. and Hudson, I. (2015b). Transmuted Weibull distribution Properties and Estimation. *Communications in Statistics: theory and methods*, to appear
10. Khan, M.S., King, R. and Hudson, I. (2016). Transmuted Generalized Gompertz distribution with Application. *Journal of Statistical Theory and Applications*, to appear
11. Merovci, F. (2013). Transmuted Rayleigh distribution. *Austrian Journal of Statistics*, 42(1), 21-31.
12. Murthy, D.N.P., Bulmer, M. and Eccleston, J.A. (2004). Weibull model selection for reliability modelling, *Reliab. Eng. Syst. Saf.*, 86, 257-267.
13. Rényi, A. (1961). *On measures of entropy and information*. University of California Press, Berkeley, California, 547-561.

14. R Core Team (2015). *R: A language and environment for statistical computing*. R Foundation for Statistical Computing, Vienna, Austria. ISBN 3-900051-07-0, URL http://www.R-project.org/. R version 3.2.1.
15. Shaw, W. and Buckley, I. (2007). *The alchemy of probability distributions: beyond Gram-Charlier expansions and a skew-kurtotic-normal distribution from a rank transmutation map*. Research report, King's College, London, U.K.
16. Ullah, A. (1996). Entropy, divergence and distance measures with econometric applications. *Journal of Statistical Planning and Inference*, 49, 137-162.
17. Yuzhu, T., Maozai, T. and Qianqian, Z. (2014). Transmuted Linear Exponential Distribution: A New Generalization of the Linear Exponential Distribution. *Communications in Statistics - Simulation and Computation*, 43(10), 2661-2677.