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EFFICIENT Uk'S RE-DESCENDING M-ESTIMATOR FOR ROBUST REGRESSION

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ABSTRACT

M-estimators are used as a robust replacement of the general classical estimators used in the field of statistics. Redescending M-estimators are those estimators which rejects the extreme values completely. We developed a new redescending M-estimator "Uk's redescending M-estimator" for robust regression and detection of outlier, which will provide protection against outliers. Moreover the ψ-function of the Uk's estimator is more linear in the central segment before it redescends. Simulation studies shows that Uk's Redescending M-estimator is more efficient than the other estimators. We also have applied the estimator to the real world data taken from the literature. The newly developed Uk's Redescending M-estimator give a general idea to interconnect all the Redescending M-estimators with that of the idea used in Andrews sine function. A couple of which has been solved and the rest are under study for the mathematical solution. Another purpose of the present work is to develop a robust estimator. Like OLS estimator the formula for the estimation of standard errors is not available for robust estimators which is required for inferences purposes. So, here as an additional work, we have tried to find out standard errors of these estimators by using bootstrap method.

KEY WORDS

Robust regression, outliers, Redescending-M, OLS, Leverage Point.

1. INTRODUCTION

Linear regression is a technique which shows the relationship between the response variable y, and explanatory variable x; which can be written as

$$Y = X\beta + \varepsilon \tag{1}$$

The matrix X is showing the explanatory variable and β is a vector showing the coefficients which are to be estimated and ε is a vector of identically distributed error term which are random and having mean vector 0 and variance covariance matrix $\sigma^2 I$. In regression analysis the method of ordinary least square (OLS) is applied to estimate the parameters. The estimated regression line thus becomes

$$\hat{Y} = X\hat{\beta} \tag{2}$$

where $\hat{Y}$ is the vector of predicted or estimated values of Y. The residual r_i of the ith case is the deference between observed and estimated value of y_i:

$$r_i = y_i - \hat{y}_i \tag{3}$$

In OLS we minimize the residuals sum of square that is

$$\min_{(\hat{\beta}_0,\ldots,\hat{\beta}_P)} \sum_{i=1}^{n} (r_i)^2 \tag{4}$$

The least squares estimation technique can be extremely affected by a very few outliers meaning the assumptions are very sensitive to outliers. For this purpose robust regression method is introduced to produce an efficient estimator in the presence of outliers. Extreme observations or outliers are sample values that fall far away from the majority of the data. A regression estimator is said to be robust if it is still reliable in the presence of outliers. There are also some data points which are far away from the data in normal circumstances which have occurred due to some technical mistake. Those errors are called gross error. Gross errors often show themselves as outliers, but not all outliers are gross errors. Some outliers are genuine and may be the most important observations of the sample. "Outliers occur very frequently in the real data set and they often go unnoticed because now-a-days much data is processed by computer without careful inspection or screening" (Rousseeuw and Leroy, 2003).

The primary objective of robust statistics is to introduce a technique which executes good results in case the assumed model is appropriately reported, while being comparatively insensitive to a little departure from the assumptions of the model. Robust statistics has the ability to resist to any sudden change or deterioration to statistical procedures. It inquires the effects of deviations from the assumptions of the modelling on some known procedures in a well-organized way and if required, gives new and improved techniques (Sakata and White, 2001). Robust regression analysis has been developed for the purpose to improve the result of least square estimation in the existence of outliers. Different robust estimators are examined on the basis of their breakdown point. "Breakdown point can be explained as the largest fraction of the data which can be moved arbitrarily apart from disturbing the estimator to the boundary of the parameter space. As much the estimator is robust to outliers, the breakdown point will be higher". Other properties like influence function and efficiency are also studied to explain robustness.

There are different robust estimators that have been developed. These estimators are M-estimators, least trimmed squares (LTS), least median square (LMS) estimators, estimators, MM-estimators, S-estimators. These estimators are checked with respect to their breakdown point and efficiency. The best estimator is the one with high efficiency and high breakdown point.

M-estimator is a typical procedure introduced by Huber (1973) to handle data which contain outliers. It gives less weight to the observations that are more likely to be an outlier. M-estimators uses the maximum likelihood formulation by deriving ideal weighting for the set of data in case when the conditions are non-normal. In this type of estimation we replace the squared error term of equation (4) function by symmetric loss function $\rho(.)$ which give us:

$$\underset{\hat{\beta}}{\text{Minimize}}\sum_{i=1}^{n}\rho\left(r_i\right) \tag{5}$$

M-estimator is not robust as far as leverage points is concerned, even though it is used regularly for the analysis of data in which it can be presumed that the contamination is mostly in the y variable (i.e. in the dependent variable). Most of the M-estimators can be solved with the help of iteratively reweighted least square method. This algorithm for this method can be found in the Birch (1997) and Simpson et al. (1998).

Differentiating (5) with respect to the regression coefficients $\hat{\beta}_j$ gives us ψ -function, i.e.

$$\sum_{i=1}^{n}\psi\left(r_i\right)X_i = 0 \tag{6}$$

Divide the ψ -function by residual which gives us the weight function given below:

$$\sum_{i=1}^{n}w\left(r_i\right)X_i = 0 \tag{7}$$

“The weights, however, depend upon the residuals, the residuals depend upon the estimated coefficients, and the estimated coefficients depend upon the weights. An iterative procedure is therefore required for the solution of this kind of problem”, Insha et al. (2006).

The graph of ψ -functions four common M-estimators are given below:

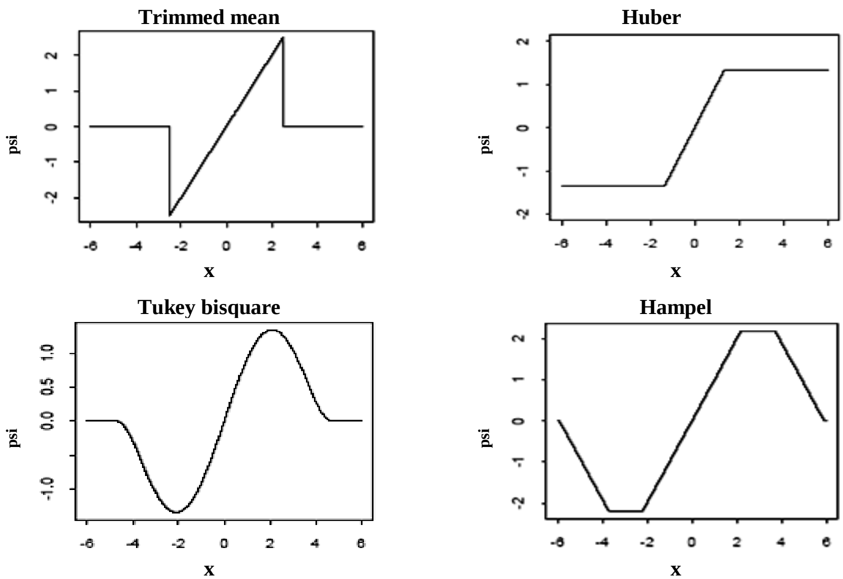

Figure 1: The ψ -Function for Four Common M-estimators

Another kind of robust estimator which was most recently developed is the Redescending M-estimator which has special robustness properties. This kind of estimators can reject completely the extreme outliers. An M-estimator is called a Redescending M-estimator, If the derivative of $\rho(.)$ redescends.

Redescending M-estimator was first introduced by Hampel (Andrews et al., 1972). Hampel used a three part-Redescending estimator. Andrews Sine function (1972), tuckey biweight (1974) and Qadir beta function (1996) are also types of Redescending M-estimators.

Redescending M-estimators having highest possible breakdown point and having favorable properties with respect to efficiency under bounded influence function (Muller, 2004). Martin, B. (2007) has found conditions for the consistency of simultaneous redescending M-estimators for location and scale.

Some of the already developed redescending robust M-estimators are Hampel's three-part function , Andrew's sine function (Andrews et al., 1972), Tukey's biweight function (Beaton and Tukey, 1974), Qadir's Beta function (Qadir, 1996), Asad's function (Ali, 2005), Insha's function (Inshaullah, 2006). Redescending M-estimator was pioneered by Hampel (Andrews et al., 1972). Hampel used later on a three part-Redescending estimator. The ψ -function of Hampel's estimator is defined by

$$\psi(r) = \begin{cases} r & for \quad |r| \le a \\ a\, sign(r) & for \quad a < |r| \le b \\ a\dfrac{c-|r|}{c-b} sign(r) & for \quad b < |r| \le c \\ 0 & for \quad |r| > c \end{cases} \tag{8}$$

where a, b, and c are constants and $0 < a \le b < c < \infty$. Another redescending M-estimator is the Andrew's sine function (Andrews et al., 1972), the ψ -function of which is given by

$$\psi(r) = \begin{cases} c\sin(r/c) & for \quad |r| \le c\pi \\ 0 & for \quad |r| > c\pi \end{cases} \tag{9}$$

And Tukey's biweight function defined by Beaten and Tukey (1974) the ψ -function given by

$$\psi(r) = \begin{cases} r\left[1-(r/c)^2\right]^2 & for \quad |r| \le c \\ 0 & for \quad |r| > c \end{cases} \tag{10}$$

Qadir (1996) introduced a redescending M-estimator based on Beta function, for which the ψ -function is given as

$$\psi(r)=\begin{cases}\dfrac{r}{16c^4}(c+r)^2(c-r)^2 & for \quad |r|\le c \\ 0 & for \quad |r|>c\end{cases}. \tag{11}$$

2. Uk's REDESCENDING M-ESTIMATOR

On the basis of the properties of the Redescending M-estimators, we propose a new Uk's Redescending M-estimator for robust regression and outlier detection. The empirical distribution of the new estimator confirmed through simulation studies is normal. The properties and shape of its ρ-function, Ψ-function and weight function will be discussed for the new Redescending M-estimator.

The objective function (ρ -function) of the new estimator is given as

$$\rho(r)=\begin{cases}\left(\dfrac{3}{2}\right)\sin\left(\dfrac{4}{9}\right)\left[\dfrac{r^{10}}{10c^8}-\dfrac{r^6}{3c^4}+\dfrac{r^2}{2}\right] & for \quad |r|\le c \\ \dfrac{3}{2}Sin\left(\dfrac{16c^2}{135}\right) & for \quad |r|>c\end{cases} \tag{12}$$

where c is tuning constant and "r" is residual of i^{th} observation. The standard properties associated to the ρ -function are

i. $\rho(r_i)\ge 0$

ii. $\rho(0)=0$

iii. $\rho(r_i)=\rho(-r_i)$

iv. $\rho(r_i)\ge\rho(r_j)$ for $|r_i|\ge|r_j|$

v. ρ is continuous (ρ is differentiable)

The graph of the objective function of the new Redescending M-estimator is as follows:

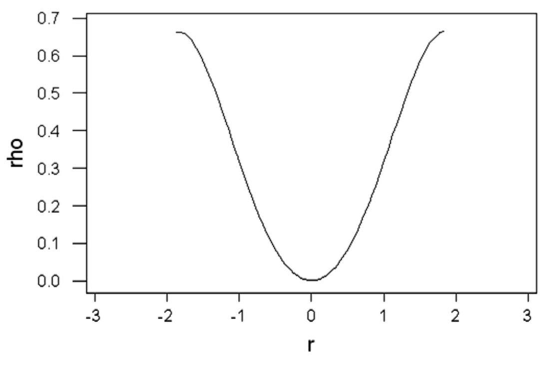

Figure 2: Graph of ρ-function of the New Redescending M-estimator

Differentiating equation (12) with respect to the residuals we get the ψ-function, which give us the following equation

$$\psi(r)=\begin{cases} r\left(\frac{3}{2}\right)\left\{1-\left(\frac{r}{c}\right)^4\right\}^2 \sin\left[\left(\frac{2}{3}\right)\left\{1-\left(\frac{r}{c}\right)^4\right\}^2\right] & for \quad |r|\le c \\ 0 & for \quad |r|>c \end{cases} \tag{13}$$

The graph of the Ψ-function of the new estimator is given as follows.

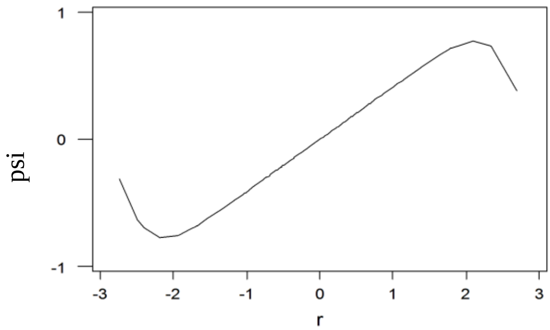

Figure 3: Graph of Ψ-function of the new Redescending M-estimator

Dividing the ψ-function on the residuals, we get the weight function which is as follows

$$w(r)=\begin{cases} \left(\frac{3}{2}\right)\left\{1-\left(\frac{r}{c}\right)^4\right\}^2 \sin\left[\left(\frac{2}{3}\right)\left\{1-\left(\frac{r}{c}\right)^4\right\}^2\right] & for \quad |r|\le c \\ 0 & for \quad |r|>c \end{cases} \tag{14}$$

The graph of weight function of the new estimator is given as

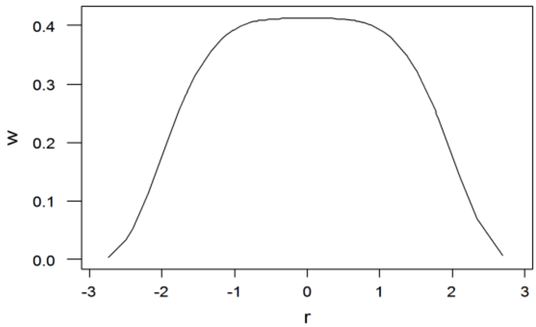

Figure 4: Graph of weight function of the new Redescending M-estimator

3. ALGORITHM FOR THE OPTIMUM CHOICE OF TUNING CONSTANT

The following algorithm has been used for the optimization of tuning constant (see Zafar Mehmood, (2010)):

1. Input / generate data
2. Initiate tuning constant (from 1.1 to 5.0 with an increment of 0.1)
3. Run leave one out cross validation (LOOCV) and remove the first observation
4. Fit the initial robust regression model by LTS technique and using Iteratively Reweighted Least Square method for estimating regression parameters
5. Run Uk's M-estimation method and compute the weighted residual sum of square
6. Return to step 4 and repeat the whole procedure by removing observation 3, instead of observation 2, and so on. Compute the absolute median prediction error.
7. Return to step 3 and using the tuning constant instead 1.2, 1.3 …..5.0. Repeat the whole procedure and select the optimum choice of "c" with the minimum value of absolute prediction error.

Simulation studies using the above algorithm shows that for Uk's redescending M-estimator, in most of the cases (Approx. 95%), the optimum choice of tuning constant lies between 3.8 and 4.2. For this reason, we have used the value tuning constant equal to 4 in our study, which gives the best results for estimating the true parameters and detecting outliers.

The results will be compared with those calculated by other redescending M-estimators which uses the proposed best choice of tuning constant by various authors. It is obvious from the results below that our Uk's estimator with optimum choice of tuning constant on the basis of the above algorithm gives better results

4. ALGORITHM FOR SIMULATION STUDIES

i) generate model based data having outliers from any distribution.
ii) fit LMS estimation procedure to obtain the initial estimates
iii) obtain the standardized residuals based on the obtained estimates.
iv) using these residuals, compute the weights based on the new redescending M-estimator
v) using these weights, compute OLS estimates iteratively
vi) repeat step iii to v.

5. SIMULATION STUDIES

Simulation studies are very important for the purpose of comparison and validation of the developed estimators. With the help of simulation studies we can check the efficiency of the newly developed Uk's Redescending M-estimator with the famous Tukey's and Andrews Redescending M-estimators. We use the below linear equation where we know the true parameters

$$y_i = \beta_0 + \beta_1 x_i + e_i \tag{15}$$

where $\beta_0 = 2$, $\beta_1 = 1$ and $e_i \sim N(0,1)$ and independent variables are generated as $x_i \sim N(20,10)$ for $j = 1,..., p$. The estimates are obtained for the generated model. As explained above the weights rely on the error term r_i, which rely on the coefficients that are estimated, and the coefficients which are estimated rely on the weights.

That's why we use an iteratively reweighted procedure for the by starting with the least square fitting. Case 1 considers the normal data case. In case 2 we replace 20% of the data by introducing outliers in the independent variable that is we generate the contaminating data as $x_i \sim N(100, 500)$. In case 3 we have replaced 20% of the data by introducing outliers in the dependent variable using an error term $e_i \sim N(50,10)$. The results shown in the following table 1 and table 2 are obtained using the average of 5000 Monte Carlo simulations, where the number of samples in table 1 and 2 are 100 and 500 respectively. Those values of tuning constants has been used which were recommended as best for each case by their corresponding authors. These values has been mentioned in the parenthesis in the table along with each case

Table 1
Estimates of Coefficients for the OLS and the Five Different Redescending M-Estimators on the Simulated Data with Only One Independent Variable and the Intercept Term, n=100

Method Used	Case 1		Case 2		Case 3	
	β0	β1	β0	β1	β0	β1
OLS	2.1548	1.0004	20.2963	0.0689	4.440	1.370
Hample(2,4,8)	2.12675	1.00249	1.91403	1.000	2.380	0.989
Qadir (4)	2.05449	1.00700	1.91208	0.9999	2.390	0.989
Andrew (3.2)	2.07908	1.00615	1.96671	1.000	2.360	0.982
Tukey (4.69)	2.08068	1.00610	2.09955	0.9999	2.380	0.980
Uk's (4)	2.06844	1.00642	1.99510	1.000	2.200	0.999

Table 2
Estimates of Coefficients for the OLS and the Five Different Redescending M-estimators on the Simulated Data with only One Independent Variable and the Intercept term, n=500

Method Used	Case 1		Case 2		Case 3	
	β0	β1	β0	β1	β0	β1
OLS	1.9117	1.0027	20.9559	0.0689	4.440	1.370
Hample(2,4,8)	2.0700	0.9999	2.04974	1.000	2.380	0.989
Qadir (4)	1.87438	1.00446	2.03842	0.9999	2.390	0.989
Andrew (3.2)	1.90692	1.00291	2.06745	1.000	2.360	0.982
Tukey (4.69)	1.85526	1.00542	2.13650	0.9999	2.380	0.980
Uk's (4)	1.90218	1.00404	2.02818	1.000	2.200	0.999

Both in Table 1 and 2 it can be seen that the values of coefficients for the Uk's estimator gives closest result to the real values in all three cases. There is no significant difference of the increase in the sample size.

6. APPLICATION TO REAL DATA, EXAMPLE 1

Consider the real data set consist of the whole number of international phone calls made from Belgium from 1950 to 1973, some quantity of outliers which exists in the data. The number of telephone calls made from Belgium is the dependent variable and the explanatory variable is the year (Rousseeuw and Leroy, 1987). The results are shown in the following table.

Table 3
Coefficients for the OLS and the Five Different Redescending M-estimators on the Belgium International Phone Calls Data

Method Used	β_0	β_1
OLS	-26.0059	0.5041
Hampel (2,4,8)	-5.22615	0.10980
Qadir (4)	-5.18905	0.10885
Andrew (3.2)	-5.16709	0.10851
Tukey (4.69)	-5.21540	0.10945
Uk's (4)	-5.16944	0.10854

On the basis of simulation results we should prefer the last row values for this data.

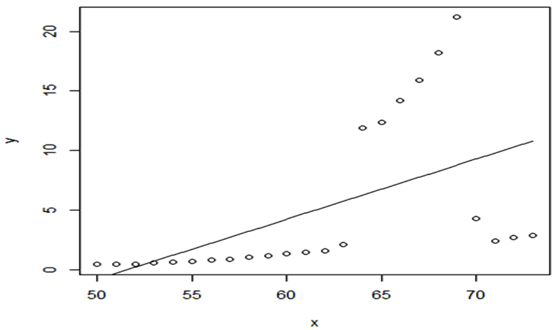

Figure 5: Least Square Fit to the Belgium International Phone Calls Data

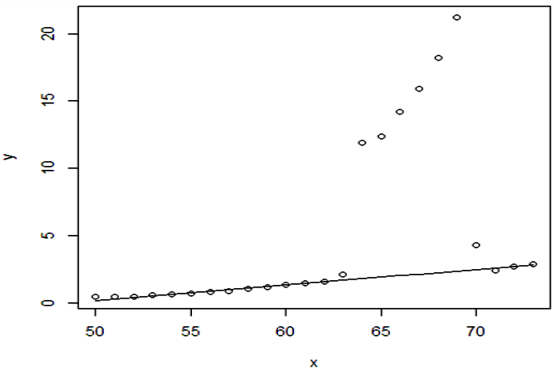

Figure 6: Robust Fit to the Belgium International Phone Calls Data

6.1 Bootstrap Distribution of Uk’s Redescending M-Estimator for Phone Call Data

Consider the data in Example 1 (International Phone Calls Data), we compute the Bootstrap Uk’s redescending estimate and approximate the standard error of Uk’s redescending M-estimator.

Norazan et al. (2009) used a new bootstrap procedure, known as weighted bootstrap with probability (WBP) for estimating the bootstrap distribution of robust estimator. WBP is based on robust estimator i.e. the weights are computed on the basis of redescending M-estimator. These weights are used to compute probabilities and the bootstrap observations in the sample are drawn according to the probabilities. We used the same WBP to compute the distribution for the new redescending Estimator. Stepwise procedure is explained as follows

6.2 Algorithm for Bootstrap Distribution

1. Fit original data with least median square (LMS). Apply Uk’s weighting function to identify outliers based on the LMS residuals. Fit a model to the remaining observations to get the parameter estimates and the fitted values.
2. Extract the residuals based on the fitted values
3. Draw a bootstrap sample from the residuals by resampling with probabilities proportional to weights. Construct fixed bootstrap values for the response variable.
4. Regress the generated bootstrap values on the fixed explanatory variable to get the bootstrap estimates of the parameter.
5. Repeat step 3 and step 4 for 2000 times
6. Computer the bootstrap based standard deviation for the estimates.

Table 4
Bootstrap Estimates and Standard Error for Phone Call Data

	β_0	β_1
Estimates	4.10897	-0.7770256
Standard Error	12.10148	0.4798208

7. APPLICATION TO REAL WORLD DATA, EXAMPLE 2

Real data set for this example has been taken from Draper and smith (1966, 1981) which has widely been used in the literature. In this data we have two explanatory variables. The predictor variable in this case is the pounds of steam that is used per month. The first explanatory variable presents the number of in service days in the month and the second presents the average atmospheric temperature in the month (in ^{0}F). The data set contain 25 observations. The results are given in the table below

Table 5
Coefficients for the OLS and the Five Different Redescending M-estimators on the Steam Data taken from Draper and Smith (1966, 1981)

Method Used	β_0	β_1	β_2
OLS	115.6599	-10.8886	1.9543
Hampel(2,4,4)	117.41465	-11.15383	2.01679
Qadir obj(4)	29.97844	-10.64140	5.92904
Andrew (3.2)	25.68118	-10.56140	6.05633
Tukey (4.69)	28.20633	-10.60295	5.98186
Uk's (4)	28.02007	-10.59521	5.99147

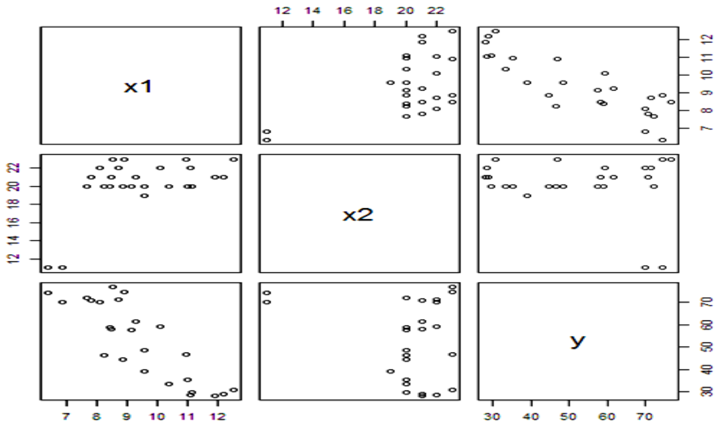

Figure 7: Scatter Plot of Belgium International Phone Call Data

7.1 Bootstrap Distribution of Uk's Redescending M-Estimator for Steam Data

Consider the data in Example 2 (Steam data), we compute the Bootstrap Uk's redescending estimate and approximate the standard error of Uk's redescending M-estimator. The same algorithm explained above for the bootstrap estimates is used here again to derive the results.

Table 6
Bootstrap Estimates and Standard Error for Steam Data

	β_0	β_1	β_2
Estimate	200.135	-21.92242	10.5181
Standard error	87.32981	5.66921	2.920358

8. APPLICATION TO REAL WORLD DATA, EXAMPLE 3

The data for the third example is taken from Brownlee (1965), the famous Stack loss data. This data has been used by a number of statisticians for different applications and especially for robust fitting. In this data we have three explanatory variables, having 21 observations, which is about the ammonia oxidation in nitric acid production.

X1 represents the air flow, X2 represents the cooling water inlet temperature, X3 represents the acid concentration. The results are shown in the following table.

Table 7
Coefficients for the OLS and the Five Different Redescending M-estimators on the Brownlee's Stack Loss Data

Method Used	β_0	β_1	β_2	B_3
OLS	-39.9197	0.7156	1.2953	-0.1521
Hampel(2,4,8)	-39.6806	0.9090	0.52867	-0.10354
Qadir (4)	-35.6953	0.71349	0.49501	-0.01548
Andrew (3.2)	-35.4073	0.74033	0.45108	-0.02642
Tukey (4.69)	-35.5259	0.68874	0.55983	-0.01679
Uk's (4)	-35.4007	0.74566	0.43503	-0.02513

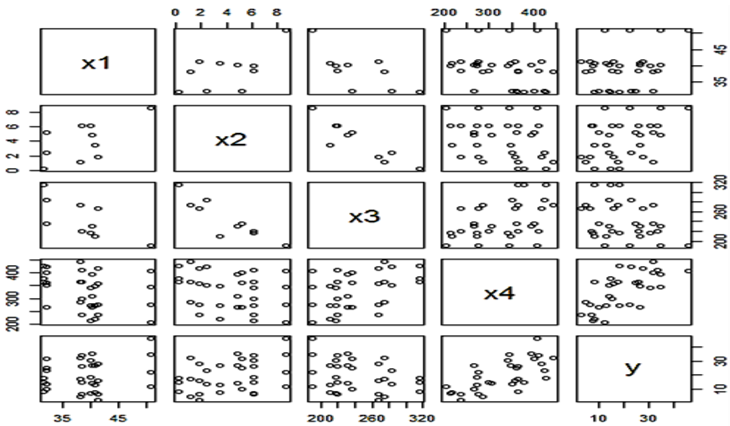

Figure 8: Scatter Plot of Stack Loss Data

8.1 Bootstrap Distribution of Uk's Redescending M-Estimator for Stack Loss Data

Consider the data in Example 3 (Stack loss data), we compute the Bootstrap Uk's redescending estimate and approximate the standard error of Uk's redescending M-estimator. Algorithm explained above is used here also to extract the results.

Table 8
Bootstrap Estimates and Standard Error for Stack Loss Data

	β_0	β_1	β_2	B_3
Estimate	-87.85191	1.04785	-0.8967403	1.144452
Standard error	28.76589	0.9195705	1.351597	0.7571046

Special care should be taken while bootstrapping contaminated data as classical bootstrap methods do not work properly in this case. For that purpose several robust bootstrap procedures have been developed an attempt to get a bootstrap sample that have the same characteristics as the original sample has. It is very clear from the results presented in this paper that the bootstrap procedure used here does not work properly for contaminated data. Due to the limited purpose of this work, we have not investigated the performance of robust type bootstrap procedures developed for contaminated data but it is a possible future work. Split sample bootstrap (Alamgir et al. (2015)) is also a possible method that can be investigated in this type of scenario.

9. CONCLUSION

A new Redescending M-estimator naming Uk's Redescending M-estimator has been proposed in this paper. We are comparing our results to the famous three estimators, i.e. Hampel three part, Andrew's and Tukey's function. The graph of its ρ-function, Ψ-function and the weight function shows that it behaves very much similar to the other famous Redescending M-estimators in addition its Ψ-function in much more continuous before it redescends. The simulation studies show that the new estimator converges much more quickly as compared to the other estimators. Although the formulae for the ρ-function is a bit more complicated due to its power but it is a matter of a second in these days for the new and advanced technology computers to solve these kinds of functions. As we require much more accuracy in our problem solving, the three examples show that the new Uk's estimator behaves efficiently on real data sets in the presence of outliers. We can see from the simulation studies where the values of the coefficients are predefined and the results from the new Uk's estimator gives much closer estimates to the real value of the coefficients as compared to the other famous robust estimation functions like Hampel, Tukey and Andrew functions. The value of the tuning constant has been used for each estimation procedure according to the best suggested value in the literature separately for each function. Six different estimation procedure has been applied in the three examples for simple and multiple regressions and compared to the rest of the Redescending M-estimators. Any statistical package can be used for the purpose of simulation. We have used R-package for the simulation and application purpose.

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