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TRANSMUTED EXPONENTIATED PARETO-I DISTRIBUTION

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ABSTRACT

In this paper, we have introduced a new generalization of the Exponentiated Pareto-I distribution named as the Transmuted Exponentiated Pareto-I distribution. The new distribution can model data with increasing, decreasing or constant hazard rate. The suitability of the distribution in modeling breaking strength of materials is highlighted with real data sets. Furthermore, we have derived some properties of the distribution including characteristic function, mean deviations, information entropies and order statistics of the new distribution. The estimation of distribution parameters has been done using Maximum Likelihood method. Finally, we have provided application of the new proposed distribution using two real data sets. It has been observed that the new distribution works better than the Pareto-I, Exponentiated Pareto-I and the Transmuted Pareto-I distributions for both data sets.

KEY WORDS

Quadratic Rank Transmutation, Exponentiated Pareto-I distribution**,** Maximum Likelihood Estimation

1. INTRODUCTION

The Pareto distribution introduced by Vilfredo Pareto, known as the power law probability distribution was first used to model the wealth distribution in the society. The Pareto distribution can also be used to describe the distribution of size of population settlements in a given area. Hard disk drive error rates also follows Pareto distribution (Schroeder et al., 2010). Size of sand particles, values of oil reserves in oil fields, file size of internet traffic which uses the TCP protocol can all be modeled by the Pareto distribution (Reed & Jorgensen, 2004).

In order to get more flexibility and accuracy, the baseline distributions can be extended to get more sophisticated models. One such technique to obtain the generalizations of existing distributions is the quadratic rank transmutation technique. Shaw & Buckley (2007) obtained the transmuted exponential distribution using the quadratic rank transmutation given as:

$$F(x) = (1+\lambda)G(x) - \lambda\left[G(x)\right]^2 \qquad , \ |\lambda| \le 1 \tag{1}$$

(Shaw & Buckley; 2007)

For $\lambda = 0$, we get the distribution of the base random variable. Where $G(x)$ is any distribution function. Abdul-Moniem and Seham (2015) obtained the Transmuted Gompertz distribution using the same technique. Oguntunde and Adejumo (2015) derived the Transmuted Inverse Exponential Distribution. Saboor, Kamal and Ahmad (2015) introduced the Transmuted Exponential–Weibull distribution. Afify, Nofal and Butt (2014) and Ahmad et al. (2014) derived the Transmuted Complementary Weibull Geometric distribution and the Transmuted Inverse Rayleigh distribution respectively. Moreover Transmuted Rayleigh distribution, Transmuted Exponentiated Modified Weibull distribution and the Transmuted Additive Weibull Distributions were introduced by Merovci (2013), Ashour and Eltehiwy (2013b) and Elbatal and Aryal (2013) respectively.

In this paper, we have introduced the Transmuted Exponentiated Pareto-I (TEP-I) distribution. The Transmuted Exponentiated Pareto-I distribution can be used to model any data with increasing, deceasing or constant failure rates hence providing greater flexibility than the Pareto and the Exponentiated Pareto distributions.

The rest of the paper is organized as: Section 2 states the distribution function, density function survival and hazard functions for the Transmuted Exponentiated Pareto-I distribution. We have also discussed the shape of the density function and hazard function in this section. In section 3 we have derived some mathematical properties of the distribution. Section 4 deals with the order statistics of the Transmuted Exponentiated Pareto-I distribution. The maximum likelihood estimates for the parameters of the distribution are given in section 5. Finally in section 6 we have provided the real data application of the newly developed distribution.

2. DISTRIBUTION, DENSITY AND HAZARD RATE FUNCTIONS

The Exponentiated Pareto-I distribution introduced by Nadarajah (2005) has the distribution function given as:

$$G(x) = 1 - k^a e^{-ax} \qquad ; \; x > \ln k, \quad a, k > 0. \tag{2}$$

Substituting the distribution function of Exponentiated Pareto-I distribution (2) in (1) and simplifying we get the distribution function of Transmuted Exponentiated Pareto-I (TEP-I) distribution given as:

$$F(x) = 1 - k^a e^{-ax}\left[1 - \lambda\left(1 - k^a e^{-ax}\right)\right] \quad ; x > \ln k, \; |\lambda| \le 1 \text{ and } a, k > 0. \tag{3}$$

Differentiating w.r.t ‘ x ’ we get the density function as:

$$f(x) = ak^a e^{-ax}\left[1 - \lambda\left(1 - 2k^a e^{-ax}\right)\right] \qquad ; \; x > \ln k, \; |\lambda| \le 1 \tag{4}$$

where a, $k > 0$ are the scale parameters and λ is the shape parameter.

Special Cases:

In the density of TEP-I distribution for $\lambda = 0$ the density reduces to the density of Exponentiated Pareto-I distribution with parameters ‘ k ’ and ‘ a ’.

Also for $\lambda = 1$ the density of TEP-I distribution reduces to the density of Exponentiated Pareto distribution with parameters ‘ k ’ and ‘ $2a$ ’ given as

$$f(x) = 2ak^{2a}e^{-2ax} \qquad ; \; x > \ln k$$

Survival and Hazard Rates:

The survival function for the TEP-I distribution is

$$S(x) = 1 - F(x)$$

$$S(x) = k^a e^{-ax}\left[1-\lambda\left(1-k^a e^{-ax}\right)\right] \quad ; For\ x > \ln k,\ a > 0,\ k > 0,\ |\lambda| \le 1. \qquad (5)$$

Now the hazard rate function for the distribution is

$$h(x) = \frac{f(x)}{S(x)}$$

$$h(x) = \frac{a\left[1-\lambda\left(1-2k^a e^{-ax}\right)\right]}{\left[1-\lambda\left(1-k^a e^{-ax}\right)\right]} \quad ; For\ x > \ln k,\ a > 0,\ k > 0,\ |\lambda| \le 1. \qquad (6)$$

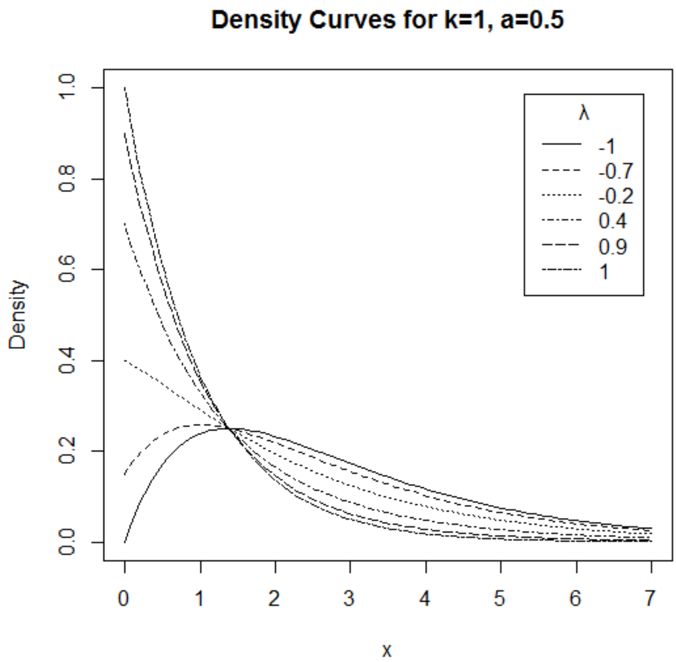

Figure 1: Density Function of Transmuted Exponentiated Pareto-I Distribution for $k = 1$, $a = 0.5$ and for different values of λ

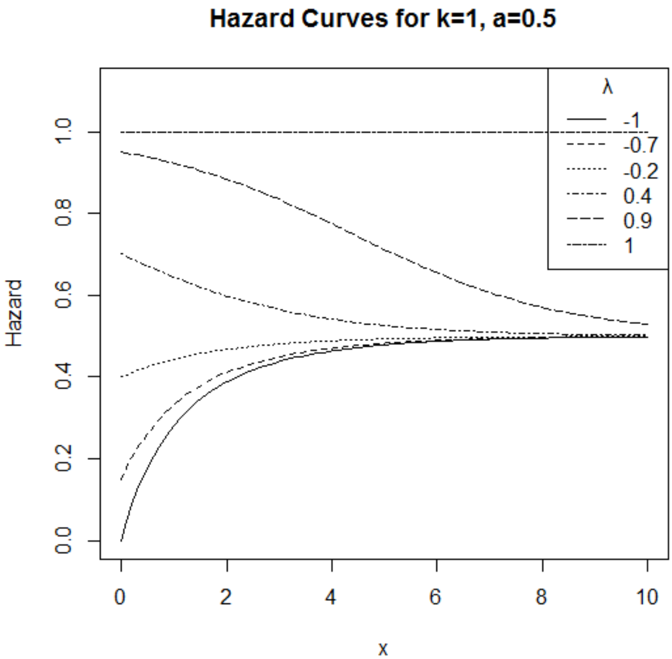

Figure 2: Hazard Function of Transmuted Exponentiated Pareto-I Distribution for $k=1$, $a=0.5$ and for different values of λ

From Figure 1 we can see that for small values of parameter λ the distribution is positively skewed. While for large values of λ the distribution is extremely positively skewed.

For negative values of 'λ' the hazard rate is increasing and for the positive values of the parameter 'λ' the hazard rate is decreasing. The hazard rate becomes constant and approaches 'a' as the variable 'X' approaches to infinity.

3. PROPERTIES

In this section, we have derived various mathematical properties of the distribution including quantile function, moment generating function, moments, mean deviations and information entropies.

3.1 Quantile Function:

The q^{th} quantile for any distribution can be obtained by solving $F\left(x_q\right)=q$ for x_q. Using the distribution function (3) of the Transmuted Exponentiated Pareto-I distribution we get:

$$q=1-k^a e^{-ax_q}\left[1-\lambda\left(1-k^a e^{-ax_q}\right)\right]$$

Letting $k^a e^{-ax_q} = t$ and simplifying we can write the above expression as:

$$\lambda t^2 - t(\lambda - 1) - 1 + q = 0$$

Solving this equation for t we get:

$$t = \frac{(\lambda - 1) + \sqrt{(1+\lambda)^2 - 4\lambda q}}{2\lambda}$$

Substituting the value of t and simplifying we get:

$$x_q = -\frac{1}{a}\ln\left[\frac{(\lambda - 1) + \sqrt{(1+\lambda)^2 - 4\lambda q}}{2\lambda k^a}\right] \tag{7}$$

Hence, Equation (7) is the quantile function for the TEP-I distribution.

The median of the distribution can be obtained by substituting $q = 0.5$ in equation (7). So the median of the distribution is:

$$x_{0.5} = -\frac{1}{a}\ln\left[\frac{(\lambda - 1) + \sqrt{(1+\lambda)^2 - 2\lambda}}{2\lambda k^a}\right] \tag{8}$$

3.2 Moment Generating Function:

The moment generating function for the TEP-I distribution can be derived as:

$$M_x(t) = \int_{\ln k}^{\infty} e^{tx} a k^a e^{-ax}\left[1 - \lambda\left(1 - 2k^a e^{-ax}\right)\right]dx$$

Using substitution

$$\left.\begin{aligned} z &= 1 - 2k^a e^{-ax} \\ dz &= 2ak^a e^{-ax}dx \end{aligned}\right\} \tag{9}$$

We get:

$$= \frac{1}{2}\int_{-1}^{1}\left[\frac{1-z}{2k^a}\right]^{-t/a}\left[1 - \lambda z\right]dz$$

$$= 2^{-t/a} k^{it}\left[\int_{-1}^{1}(1-z)^{-t/a}\,dz - \lambda\int_{-1}^{1} z(1-z)^{-t/a}\,dz\right]$$

Simplifying this expression we get the moment generating function given as

$$M_x(t)=\frac{ak^t\left[2a-t(1+\lambda)\right]}{(a-t)(2a-t)} \tag{10}$$

To get the mean and variance of the distribution we proceed as follows

Taking first derivative of the moment generating function we get

$$\frac{\partial M_x(t)}{\partial t}=\frac{ak^t}{(a-t)(2a-t)}\left[\left(2a-t(1+\lambda)\right)\left\{\ln k-\frac{2t-3a}{(a-t)(2a-t)}\right\}-(1+\lambda)\right]$$

Substituting $t = 0$ and simplifying we get the mean of the distribution given as

$$E(X)=\ln k+\frac{1}{2a}(2-\lambda) \tag{11}$$

Now to obtain the variance, taking 2nd derivative of the moment generating function given in (10) we get

$$\frac{\partial^2 M_x(t)}{\partial t^2}=\frac{ak^t}{(a-t)(2a-t)}\left[\left(2a-t(1+\lambda)\right)\left\{\ln k-\frac{2t-3a}{(a-t)(2a-t)}\right\}^2 -2(1+\lambda)\left\{\ln k-\frac{2t-3a}{(a-t)(2a-t)}\right\} +\frac{2a-t(1+\lambda)}{\left[(a-t)(2a-t)\right]^2}\left(2t^2-6at+5a^2\right)\right]$$

Substituting $t=0$ and simplifying we have

$$E\left(X^2\right)=(\ln k)^2+\frac{\ln k}{a}(2-\lambda)+\frac{1}{2a^2}(4-3\lambda) \tag{12}$$

Variance of the TEP-I distribution can be obtained using:

$$Variance=E\left(X^2\right)-\left[E(X)\right]^2 \tag{13}$$

Substituting results from (11) and (12) in (13) and simplifying we get

$$Variance=\frac{1}{4a^2}\left(4-\lambda^2-2\lambda\right) \tag{14}$$

3.3 Moments:

The r^{th} raw moment for the TEP-I distribution can be obtained as:

$$E\left(X^r\right)=\int_{\ln k}^{\infty} x^r ak^a e^{-ax}\left[1-\lambda\left(1-2k^a e^{-ax}\right)\right]dx \tag{15}$$

Substituting

$$\left.\begin{aligned} ax &= z \\ dz &= adx \end{aligned}\right\} \qquad (16)$$

Equation (15) can be written as:

$$= k^a \int_{a\ln k}^{\infty} \left(\frac{z}{a}\right)^r e^{-z}\left[1-\lambda\left(1-2k^a e^{-z}\right)\right]dz$$

Simplifying we get the r^{th} moment of our distribution as:

$$= \frac{k^a}{a^r}(1-\lambda)\int_{a\ln k}^{\infty} z^r e^{-z}dz + \frac{2\lambda k^{2a}}{a^r}\int_{a\ln k}^{\infty} z^r e^{-2z}dz$$

$$E\left(X^r\right) = \frac{k^a}{a^r}(1-\lambda)\gamma\left(r+1, a\ln k\right) + \frac{\lambda k^{2a}}{(2a)^r}\gamma\left(r+1, 2a\ln k\right); \quad a\ln k > 0. \qquad (17)$$

where $\gamma(\alpha, k) = \int_{k}^{\infty} x^{\alpha-1}e^{-x}dx$ is the incomplete gamma function.

3.4 Mean Deviations:

The mean deviations about mean and median can be obtained using:

$$MD(\mu) = 2\mu F(\mu) - 2I(\mu) \qquad (18)$$

$$MD(M) = \mu - 2I(M) \qquad (19)$$

where $I(\mu) = \int_{-\infty}^{\mu} y\, f(y)dy$ and $F(\mu) = \int_{-\infty}^{\mu} f(y)dy$.

(Ashour & Eltehiwy, 2013a)

For the TEP-I distribution we have:

$$I(w) = \int_{\ln k}^{w} xak^a e^{-ax}\left[1-\lambda\left(1-2k^a e^{-ax}\right)\right]dx$$

$$= ak^a(1-\lambda)\int_{\ln k}^{w} xe^{-ax}dx + 2ak^{2a}\lambda\int_{\ln k}^{w} xe^{-2ax}dx$$

Integrating by parts we get

$$= ak^a(1-\lambda)\left[-\frac{1}{a}\left(we^{-aw} - \frac{\ln k}{k^a}\right) + \frac{1}{a^2}\left(e^{-aw} - \frac{1}{k^a}\right)\right]$$
$$+ 2ak^{2a}\lambda\left[-\frac{1}{2a}\left(we^{-2aw} - \frac{\ln k}{k^{2a}}\right) + \frac{1}{4a^2}\left(e^{-2aw} - \frac{1}{k^{2a}}\right)\right]$$

Simplifying above expression we get the result:

$$I(w)=\ln k+\frac{1}{a}\left(1-\frac{\lambda}{2}\right)-k^{a}e^{-aw}\left[(1-\lambda)\left(w+\frac{1}{a}\right)+\lambda k^{a}e^{-aw}\left(w+\frac{1}{2a}\right)\right] \qquad (20)$$

Substituting this result in (18) and (19) along with the distribution function we get the mean deviation about mean and median respectively as:

$$MD(\mu)=2\mu-2\ln k-\frac{2}{a}\left(1-\frac{\lambda}{2}\right)+\frac{k^{a}e^{-a\mu}}{a}\left[2(1-\lambda)+\lambda k^{a}e^{-a\mu}\right] \qquad (21)$$

$$MD(M)=\mu-2\ln k-\frac{2}{a}\left(1-\frac{\lambda}{2}\right)$$
$$+2k^{a}e^{-aM}\left[(1-\lambda)\left(M+\frac{1}{a}\right)+\lambda k^{a}e^{-aM}\left(M+\frac{1}{2a}\right)\right] \qquad (22)$$

3.5 Information Entropies:

The Shannon and Renyi entropy for the transmuted Exponentiated Pareto-I distribution have been obtained herewith.

3.5.1 Shannon Entropy:

The Shannon entropy for any distribution can be defined as " $E\left[-\ln f(x)\right]$". For the TEP-I distribution the Shannon entropy is:

$$E\left[-\ln f(x)\right]=-\ln a-\ln k^{a}+aE(X)-E\left[\ln\left\{1-\lambda\left(1-2k^{a}e^{-ax}\right)\right\}\right] \qquad (23)$$

In order to evaluate (23) we need to get $E\left[\ln\left\{1-\lambda\left(1-2k^{a}e^{-ax}\right)\right\}\right]$. So we have for the TEP-I distribution:

$$E\left[\ln\left\{1-\lambda\left(1-2k^{a}e^{-ax}\right)\right\}\right]$$
$$=\int_{\ln k}^{\infty}\left[\ln\left\{1-\lambda\left(1-2k^{a}e^{-ax}\right)\right\}\right]ak^{a}e^{-ax}\left[1-\lambda\left(1-2k^{a}e^{-ax}\right)\right]dx \qquad (24)$$

Using substitution

$$\left.\begin{aligned} z&=1-\lambda\left(1-2k^{a}e^{-ax}\right)\\ \left|\frac{dz}{dx}\right|&=2a\lambda k^{a}e^{-ax}\end{aligned}\right\} \qquad (25)$$

The equation (24) reduces to:

$$E\left[\ln\left\{1-\lambda\left(1-2k^{a}e^{-ax}\right)\right\}\right]=\frac{1}{2\lambda}\int_{1-\lambda}^{1+\lambda} z\left(\ln z\right)dz$$

Integrating by parts and simplifying we get the following result

$$E\left[\ln\left\{1-\lambda\left(1-2k^{a}e^{-ax}\right)\right\}\right]=\frac{1}{4\lambda}\left[\left(1+\lambda\right)^{2}\ln\left(1+\lambda\right)-\left(1-\lambda\right)^{2}\ln\left(1-\lambda\right)-\lambda\right] \tag{26}$$

Using the result (26) in (23) and simplifying we get the Shannon entropy given as:

$$E\left[-\ln f\left(x\right)\right]=\ln\left(\frac{1}{a}\right)+\frac{1}{4}\left(5-2\lambda\right)$$
$$-\frac{1}{4\lambda}\left[\left(1+\lambda\right)^{2}\ln\left(1+\lambda\right)-\left(1-\lambda\right)^{2}\ln\left(1-\lambda\right)\right] \tag{27}$$

3.5.2 Renyi Entropy

Renyi entropy is defined as:

$$I_R\left(\gamma\right)=\frac{1}{\gamma-1}\log\int_R f^{\gamma}\left(y\right)dy\ ;\ \gamma>0\ \text{ and }\ \gamma\neq 1.$$

Now using the density function of the TEP-I distribution we get

$$\int_{\Re} f^{\gamma}\left(x\right)dx=\int_{\ln k}^{\infty} a^{\gamma}k^{\gamma a}e^{-\gamma a x}\left[1-\lambda\left(1-2k^{a}e^{-ax}\right)\right]^{\gamma}dx \tag{28}$$

Using substitution (9) we get

$$=\frac{a^{\gamma-1}}{2^{\gamma}}\int_{-1}^{1}\left(1-z\right)^{\gamma-1}\left(1-\lambda z\right)^{\gamma}dz$$

Using the result

$$\left(1-z\right)^{\gamma-1}=\sum_{j=0}^{\infty}\frac{\Gamma\left(\gamma\right)}{j!\,\Gamma\left(\gamma-j\right)}\left(-1\right)^{j}z^{j}$$

if γ is not an integer, $|z|>0$ for $|z|<1$ and $\gamma>0$

We have

$$\int_{\Re} f^{\gamma}\left(x\right)dx=\frac{a^{\gamma-1}}{2^{\gamma}}\sum_{j=0}^{\infty}\frac{\Gamma\left(\gamma+1\right)}{j!\,\Gamma\left(\gamma+1-j\right)}\left(-1\right)^{j}\lambda^{j}\int_{-1}^{1}z^{j}\left(1-z\right)^{\gamma-1}dz \tag{29}$$

Considering the integral $\int z^{j}\left(1-z\right)^{\gamma-1}dz$ and using the binomial type expansion we have:

$$\int z^{j}\left(1-z\right)^{\gamma-1}dz=\sum_{k=0}^{\infty}\frac{\Gamma\gamma}{k!\,\Gamma\gamma-k}\left(-1\right)^{k}\int_{-1}^{1}z^{j+k}dz$$

Simplifying we get

$$\int z^{j}\left(1-z\right)^{\gamma-1}dz=\frac{z^{j+1}}{\left(j+1\right)}\sum_{k=0}^{\infty}\frac{\left(-1\right)^{k}\Gamma\gamma}{\Gamma\gamma-k}\frac{z^{k}}{k!}\frac{\left(j+1\right)}{\left(j+k+1\right)}$$

Hence we can write

$$\int z^{j}\left(1-z\right)^{\gamma-1}dz=\frac{z^{j+1}\,{}_{2}F_{1}\left(1-\gamma,\,j+1;\,j+2;\,z\right)}{j+1} \tag{30}$$

where

$${}_{2}F_{1}\left(a,b;c;z\right)=\sum_{k=0}^{\infty}\frac{\left(a\right)_{k}\left(b\right)_{k}z^{k}}{k!\left(c\right)_{k}},\qquad \left|z\right|<1\vee\left|z\right|==1\wedge\operatorname{Re}\left(c-a-b\right)>0.$$

Substituting this result in equation (29) and simplifying we get:

$$\int_{\Re}f^{\gamma}\left(x\right)dx=\frac{a^{\gamma-1}}{2^{\gamma}}\sum_{j=0}^{\infty}\frac{\Gamma\gamma+1}{j!\,\Gamma\gamma+1-j}\frac{\left(-1\right)^{j}\lambda^{j}}{j+1}\left[{}_{2}F_{1}\left(1-\gamma,\,j+1;\,j+2;1\right)+\left(-1\right)^{j}\,{}_{2}F_{1}\left(1-\gamma,\,j+1;\,j+2;-1\right)\right]$$

$$\int_{\Re}f^{\gamma}\left(x\right)dx=\frac{a^{\gamma-1}}{2^{\gamma}}\sum_{j=0}^{\infty}\frac{\Gamma\gamma+1}{j!\,\Gamma\gamma+1-j}\frac{\left(-1\right)^{j}\lambda^{j}}{j+1}\left[\frac{\Gamma j+2\,\Gamma\gamma}{\Gamma\gamma+j+1}+\left(-1\right)^{j}\,{}_{2}F_{1}\left(1-\gamma,\,j+1;\,j+2;-1\right)\right] \tag{31}$$

Substituting (31) in (28) we get the Renyi entropy as:

$$I_{R}\left(\gamma\right)=\frac{1}{\gamma-1}\log\left[\frac{a^{\gamma-1}}{2^{\gamma}}\sum_{j=0}^{\infty}\frac{\left(-1\right)^{j}\lambda^{j}\Gamma\gamma+1}{\left(j+1\right)!\,\Gamma\gamma+1-j}\left[\frac{\Gamma j+2\,\Gamma\gamma}{\Gamma\gamma+j+1}+\left(-1\right)^{j}\,{}_{2}F_{1}\left(1-\gamma,\,j+1;\,j+2;-1\right)\right]\right]. \tag{32}$$

4. ORDER STATISTICS

This section provides the density of i^{th} order statistics along with its m^{th} raw moment.

In a sample of size ‘ n ’ denoted by $X_1, \ldots, X_n$ the density of i^{th} order statistics can be obtained as:

$$f\left(x_{i:n}\right)=\frac{n!}{(i-1)!(n-i)!}\left[F(x)\right]^{i-1}\left[1-F(x)\right]^{n-i} f(x) \tag{33}$$

Substituting the distribution function (3) and the density function (4) of Transmuted Exponentiated Pareto-I distribution we get:

$$f(x_{i:n})=\frac{n!}{(i-1)!(n-i)!}\left[1-k^a e^{-ax}\left[1-\lambda\left(1-k^a e^{-ax}\right)\right]\right]^{i-1}$$
$$\left[k^a e^{-ax}\left[1-\lambda\left(1-k^a e^{-ax}\right)\right]\right]^{n-i} ak^a e^{-ax}\left[1-\lambda\left(1-2k^a e^{-ax}\right)\right] \tag{34}$$

Using the binomial series expansion in the above expression and simplifying we get:

$$=\frac{n!}{(i-1)!(n-i)!}\sum_{p=0}^{i-1}\binom{i-1}{p}(-1)^p ak^{a(n-i+p+1)}e^{-ax(n-i+p+1)}$$
$$\left[1-\lambda\left(1-k^a e^{-ax}\right)\right]^{n-i+p}\left[1-\lambda\left(1-2k^a e^{-ax}\right)\right]$$
$$=\frac{n!}{(i-1)!(n-i)!}\sum_{p=0}^{i-1}\sum_{q=0}^{n-i+p}\binom{i-1}{p}\binom{n-i+p}{q}$$
$$(-1)^{p+q} ak^{a(n-i+p+1)}e^{-ax(n-i+p+1)}\lambda^q\left(1-k^a e^{-ax}\right)^q\left[1-\lambda+2\lambda k^a e^{-ax}\right]$$

Again using the binomial series expansion and simplifying, the density function of i^{th} order statistics for the Transmuted Exponentiated Pareto-I random variable is

$$f(x_{i:n})=\frac{n!}{(i-1)!(n-i)!}\sum_{p=0}^{i-1}\sum_{q=0}^{n-i+p}\sum_{r=0}^{q}\binom{i-1}{p}\binom{n-i+p}{q}\binom{q}{r}$$
$$(-1)^{p+q+r} ak^{a(n-i+p+r+1)}e^{-ax(n-i+p+r+1)}\lambda^q\left[1-\lambda+2\lambda k^a e^{-ax}\right] \quad ; x>\ln k \tag{35}$$

Now to get the m^{th} raw moment of the density of i^{th} order statistics of TEP-I distribution we proceed as:

$$E\left(X_{i:n}^m\right)=\int_{\ln k}^{\infty} x^m \frac{n!}{(i-1)!(n-i)!}\sum_{p=0}^{i-1}\sum_{q=0}^{n-i+p}\sum_{r=0}^{q}\binom{i-1}{p}\binom{n-i+p}{q}\binom{q}{r}$$
$$(-1)^{p+q+r} ak^{a(n-i+p+r+1)}e^{-ax(n-i+p+r+1)}\lambda^q\left[1-\lambda+2\lambda k^a e^{-ax}\right]dx \tag{36}$$

$$=\frac{n!}{(i-1)!(n-i)!}\sum_{p=0}^{i-1}\sum_{q=0}^{n-i+p}\sum_{r=0}^{q}\binom{i-1}{p}\binom{n-i+p}{q}\binom{q}{r}(-1)^{p+q+r}a\lambda^{q}k^{a(n-i+p+r+1)}$$

$$\left[(1-\lambda)\int_{\ln k}^{\infty}x^{m}e^{-ax(n-i+p+r+1)}dx+2\lambda k^{a}\int_{\ln k}^{\infty}x^{m}e^{-ax(n-i+p+r+2)}dx\right]$$

Taking $z=a(n-i+p+r+1)$ the above expression can be restated as:

$$=\frac{n!}{(i-1)!(n-i)!}\sum_{p=0}^{i-1}\sum_{q=0}^{n-i+p}\sum_{r=0}^{q}\binom{i-1}{p}\binom{n-i+p}{q}\binom{q}{r}(-1)^{p+q+r}a\lambda^{q}k^{az}$$

$$\left[(1-\lambda)\int_{\ln k}^{\infty}x^{m}e^{-zx}dx+2\lambda k^{a}\int_{\ln k}^{\infty}x^{m}e^{-x(z+a)}dx\right]$$

$$=\frac{n!}{(i-1)!(n-i)!}\sum_{p=0}^{i-1}\sum_{q=0}^{n-i+p}\sum_{r=0}^{q}\binom{i-1}{p}\binom{n-i+p}{q}\binom{q}{r}(-1)^{p+q+r}a\lambda^{q}k^{z}$$

$$\left[\frac{(1-\lambda)}{z^{m+1}}\gamma\left(m+1,z\ln k\right)+\frac{2\lambda k^{a}}{(z+a)^{m+1}}\gamma\left(m+1,(z+a)\ln k\right)\right]$$

Substituting back $z=a(n-i+p+r+1)$ in the above expression we get

$$E\left(X_{i:n}^{m}\right)=\frac{n!}{(i-1)!(n-i)!}\sum_{p=0}^{i-1}\sum_{q=0}^{n-i+p}\sum_{r=0}^{q}\binom{i-1}{p}\binom{n-i+p}{q}\binom{q}{r}(-1)^{p+q+r}\frac{\lambda^{q}k^{a(n-i+p+r+1)}}{a^{m}}$$

$$\left[\begin{array}{l}\frac{(1-\lambda)}{(n-i+p+r+1)^{m+1}}\gamma\left(m+1,a(n-i+p+r+1)\ln k\right)\\ \quad+\frac{2\lambda k^{a}}{(n-i+p+r+2)^{m+1}}\gamma\left(m+1,a(n-i+p+r+2)\ln k\right)\end{array}\right]$$

$$\text{for}\quad a(n-i+p+r+1)\ln k>0 \tag{37}$$

Which is the expression for m^{th} raw moment of the density of i^{th} order statistics of Transmuted Exponentiated Pareto-I distribution.

5. MAXIMUM LIKELIHOOD ESTIMATES

Here we have derived the Maximum Likelihood Estimates for the parameters of the Transmuted Exponentiated Pareto-I distribution and discussed its large sample inference.

The log-likelihood function for the single observation from the Transmuted Exponentiated Pareto-I distribution is:

$$\ln f(x) = \ln a + a \ln k - ax + \ln\left[1 - \lambda\left(1 - 2k^{a} e^{-ax}\right)\right] \tag{38}$$

Differentiating w.r.t the parameters of the distribution we get the normal equations as:

$$\frac{\partial \ln f(x)}{\partial a} = \frac{1}{a} + \ln k - x + \frac{2\lambda k^{a} e^{-ax}}{1 - \lambda\left(1 - 2k^{a} e^{-ax}\right)}\left(\ln k - x\right) \tag{39}$$

$$\frac{\partial \ln f(x)}{\partial k} = \frac{a}{k} + \frac{2a\lambda k^{a-1} e^{-ax}}{1 - \lambda\left(1 - 2k^{a} e^{-ax}\right)} \tag{40}$$

$$\frac{\partial \ln f(x)}{\partial \lambda} = \frac{2k^{a} e^{-ax} - 1}{1 - \lambda\left(1 - 2k^{a} e^{-ax}\right)} \tag{41}$$

The Maximum Likelihood Estimates (MLEs) can be obtained by solving $U_n = \sum_{i=0}^{n} U^{(i)}$ where $U^{(i)}$ is the score function described as

$$U(\underline{\alpha}) = \left[\frac{\partial l}{\partial \lambda} \quad \frac{\partial l}{\partial \theta} \quad \frac{\partial l}{\partial \beta}\right]'$$

of the i^{th} observation for $i = 1, \ldots, n$. For the Transmuted Exponentiated Pareto-I distribution the solution of $E\left[U_n\right] = 0$ does not provide closed form expressions of MLE's. So we have to use iterative procedures to get the results of MLE's.

When the sample size is large and the regularity conditions are satisfied we have

$$\sqrt{n}\left(\underline{\hat{\alpha}} - \underline{\alpha}\right) \rightarrow N_3\left(\underline{0}, I\left(\underline{\alpha}\right)^{-1}\right)$$

Also $\lim_{n\to\infty} I_n\left(\underline{\alpha}\right)^{-1} = I\left(\underline{\alpha}\right)^{-1}$. Where $I_n\left(\underline{\alpha}\right)$ is the Fisher's information matrix defined as

$$I = I_n\left(\underline{\alpha}\right) = -nE\begin{bmatrix} \frac{\partial^2 l}{\partial \lambda^2} & \frac{\partial^2 l}{\partial \lambda \partial \theta} & \frac{\partial^2 l}{\partial \lambda \partial \beta} \\ \frac{\partial^2 l}{\partial \theta \partial \lambda} & \frac{\partial^2 l}{\partial \theta^2} & \frac{\partial^2 l}{\partial \theta \partial \beta} \\ \frac{\partial^2 l}{\partial \beta \partial \lambda} & \frac{\partial^2 l}{\partial \beta \partial \theta} & \frac{\partial^2 l}{\partial \beta^2} \end{bmatrix}$$

Hence we can use the asymptotic distribution of MLE's to draw inferences about the distribution parameters. Also the $(1-\gamma)100\%$ large sample confidence intervals for the i^{th} parameter can be constructed using the given expression

$$\left(\hat{\alpha}_i - z_{1-\gamma/2}\sqrt{\hat{I}_{\alpha_i,\alpha_i}}, \hat{\alpha}_i + z_{1-\gamma/2}\sqrt{\hat{I}_{\alpha_i,\alpha_i}}\right)$$

where $\hat{I}_{\alpha_i,\alpha_i}$ is the i^{th} diagonal element of $I_n\left(\underline{\alpha}\right)^{-1}$ and $z_{1-\gamma/2}$ is the standard normal quantile for $\left(0<\gamma<1/2\right)$.

All the second order derivatives involved in Fisher's Information matrix are given below:

$$\frac{\partial^2 \ln f\left(x\right)}{\partial a^2}=-\frac{1}{a^2}+\frac{2\lambda k^a e^{-ax}\left(\ln k-x\right)^2\left(1-\lambda\right)}{\left[1-\lambda\left(1-2k^a e^{-ax}\right)\right]^2} \tag{42}$$

$$\frac{\partial^2 \ln f\left(x\right)}{\partial k^2}=-\frac{a}{k^2}+\frac{2a^2\lambda k^{a-2}e^{-ax}\left(1-\lambda\right)}{\left[1-\lambda\left(1-2k^a e^{-ax}\right)\right]^2}-\frac{2a\lambda k^{a-2}e^{-ax}}{1-\lambda\left(1-2k^a e^{-ax}\right)} \tag{43}$$

$$\frac{\partial^2 \ln f\left(x\right)}{\partial \lambda^2}=-\frac{\left(1-2k^a e^{-ax}\right)^2}{\left[1-\lambda\left(1-2k^a e^{-ax}\right)\right]^2} \tag{44}$$

$$\frac{\partial^2 \ln f\left(x\right)}{\partial a\partial k}=\frac{1}{k}+\frac{2\lambda k^{a-1}e^{-ax}}{\left[1-\lambda\left(1-2k^a e^{-ax}\right)\right]^2} \left[a\left(\ln k-x\right)\left[1-\lambda\left(1-2k^a e^{-ax}\right)\right]+1-\lambda-2\lambda k^a e^{-ax}\left(a-1\right)\right] \tag{45}$$

$$\frac{\partial^2 \ln f\left(x\right)}{\partial a\partial \lambda}=\frac{2k^a e^{-ax}\left(\ln k-x\right)}{\left[1-\lambda\left(1-2k^a e^{-ax}\right)\right]^2}\left[1+\left(1-\lambda\right)\left(1-2k^a e^{-ax}\right)\right] \tag{46}$$

$$\frac{\partial^2 \ln f\left(x\right)}{\partial k\partial \lambda}=\frac{2ak^{a-1}e^{-ax}}{\left[1-\lambda\left(1-2k^a e^{-ax}\right)\right]^2} \tag{47}$$

6. APPLICATION

In this section, we have fitted the Transmuted Exponentiated Pareto-I distribution to the two real data sets along with other distributions to show the applicability of the newly proposed model.

Data Set: 1

The first data set consists of observations on breaking stress of carbon fibers (in Gba), the data is given as:

3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2.00, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.80, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65

(Nichols & Padgett; 2006)

To check the Transmuted Exponentiated Pareto-I distribution against the Exponentiated Pareto-I distribution for this data the hypothesis to be tested is $\mathcal{H}_0: EP\ (\lambda = 0)$ against $\mathcal{H}_1: TEP\ (\lambda \neq 0)$. To test this hypothesis we have used the Likelihood Ratio Test statistic. The resulted value of the Likelihood Ratio Test statistic is **41.611** with P-value **0.0001**. Hence we can conclude that the Transmuted Exponentiated Pareto-I distribution fits better than the Exponentiated Pareto-I distribution for the data of breaking stress of carbon fibers. We have also compared the TEP-I distribution with some other distribution given in Table 1 along with the Akaike Information Criteria (AIC) and Bayesian Information Criteria (BIC) values. From the table we can observe that the TEP-I provides best fit among all the contending models since it has the lowest AIC and BIC values.

Table 1
AIC and BIC Values for the Breaking Stress of Carbon Fibers

Distribution	Parameter Estimates			Log-likelihood	AIC	BIC
	k	a	λ			
Pareto	0.3900	0.5498	-	-247.5640	499.128	504.3383
Exponentiated Pareto-I	1.4770	0.4481	-	-180.2629	364.5383	369.7487
Transmuted Pareto	0.3900	0.7899	-0.9692	-219.7713	445.5426	453.3581
Transmuted Exponentiated Pareto-I	1.4770	0.6488	-0.9601	-159.4574	324.9148	332.7303

Data Set: 2

This data set contain observations on the strengths of 1.5 cm glass fibers, measured at the National Physical Laboratory, England. The observations are:

0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2 ,0.74, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.5, 1.54, 1.6, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.5, 1.55, 1.61, 1.62, 1.66, 1.7, 1.77, 1.84, 0.84, 1.24, 1.3, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.7, 1.78, 1.89

(Smith & Naylor; 1987)

We have fitted the Pareto, Exponentiated Pareto-I, Transmuted Pareto and Transmuted Exponentiated Pareto-I distribution to this data set. The parameter estimates, log-likelihood, AIC and BIC values are reported in the Table 2. For this data the value of LR test statistic is **27.9824** with P-value **0.0001** suggesting the TEP-I distribution to be a better fitting model than the Exponentiated Pareto distribution. Also the TEP-I distribution has the minimum AIC and BIC values among all other distributions so we can conclude that the TEP-I distribution provides better fit than Pareto, Exponentiated Pareto-I and Transmuted Pareto distributions.

Table 2
AIC and BIC Values for the Strengths of 1.5 cm Glass Fibers

Distribution	Parameter Estimates			Log-likelihood	AIC	BIC
	k	*a*	λ			
Pareto	0.5500	1.0216	-	-85.6632	175.3264	179.6127
Exponentiated Pareto-I	1.7333	1.0451	-	-60.2195	124.5390	124.4390
Transmuted Pareto	0.5500	1.4589	-0.9492	-69.8640	145.7280	152.1574
Transmuted Exponentiated Pareto-I	1.7333	1.4939	-0.9423	-46.2283	98.4566	104.8860

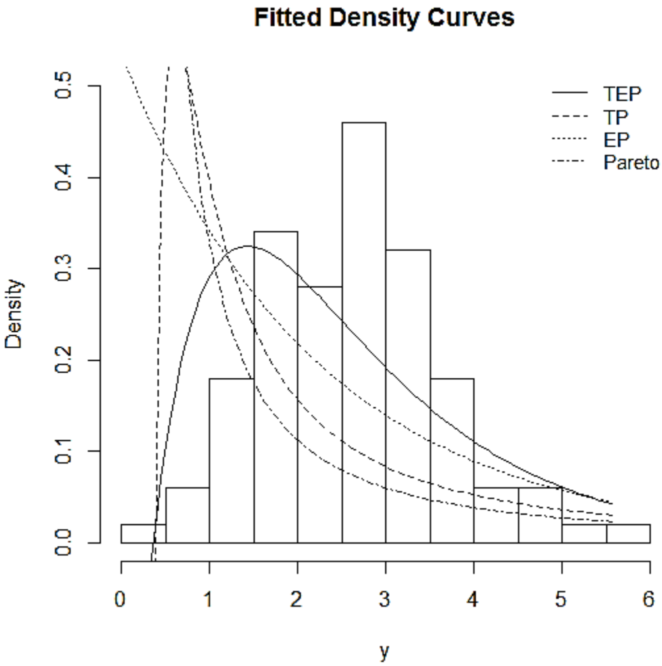

Figure 3: Breaking Stress of Carbon Fibers (in Gba) Data with Fitted Density curves.

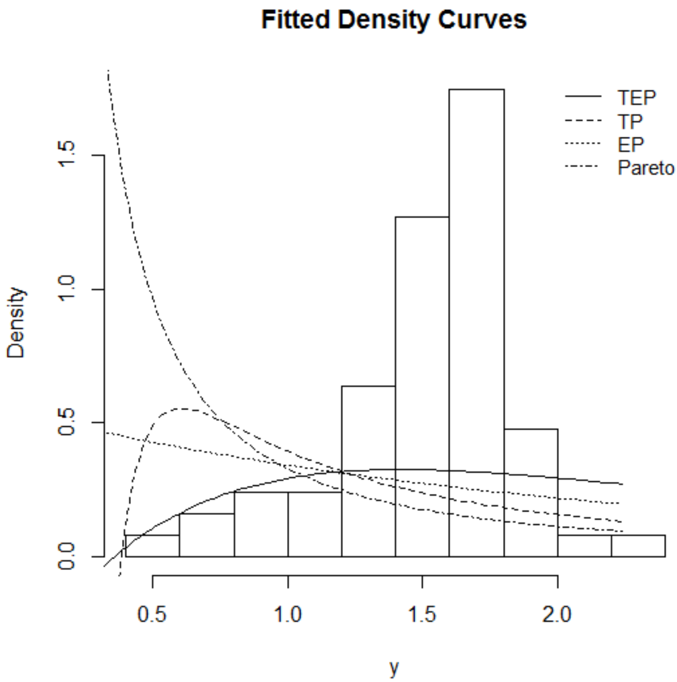

Figure 4: Data on Strengths of 1.5 cm Glass Fibers with Fitted Density Curves

CONCLUSION

In this paper the Transmuted Exponentiated Pareto-I distribution has been introduced. Its mathematical properties have been dealt with. Parameters have been estimated using the method of Maximum Likelihood. This distribution has also been applied to 2 real data sets. It has been observed that Transmuted Exponentiated Pareto-I distribution provides a better fit for data related to breaking strength of materials as compared to Pareto, Exponentiated Pareto-I and Transmuted Pareto distributions.

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