ON APPLICATION OF FRACTIONAL CALCULUS: EXACT AND APPROXIMATE SOLUTIONS FOR DRYING AND WETTING OF RIVERS

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ABSTRACT

Mathematical models for rivers differ according to the properties of the river. These models can be classified into drying–wetting, motion, masses, resources, and so on. This paper studies a system for practical water level prognostication and prediction during dry–wet seasons at the Terengganu River. The mathematical model is a generalized master equation based on fractional calculus to obtain one-dimensional shallow water equations. We introduce two exact solutions for the proposed model. The first describes the drying condition of the river, and the second characterizes the wetting status. The actual data show the effectiveness of the new model.

KEYWORDS

1. INTRODUCTION

Elaborate water level emulation is considered very important by ship operators, who need accurate mid-term prognostication to maximze the loading of ships during medium- or low-flow positions. Overflow charts can identify the fields of various overflow risks and potential damages. In addition, waterway refinements as well as their influence on economical and ecological resources require an authoritative simulation instrument for hydraulic engineering. Moreover, a robust prediction is necessary for optimizing curently available resources. Therefore, these mathematical models must be improved before they can be used to simulate natural rivers with continuous curves.

We adopt fluid dynamic calculus, which can be completed in a finite volume context (Toro & Toro, 1999). An active numerical scheme for shallow water equations on wet fields has been proposed in (Alcrudo & Garcia-Navarro, 1993). Finite volume formalism fissures the space domain in partitions called cells. Moreover, finite volume assumes averaged costs on each cell. Then, the movement of the fluid is formulated in terms of a flux between these cells. The Re-Normalized Group (RNG) $\kappa - \varepsilon$ model, which is

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extended from the standard $\kappa-\varepsilon$ model (Falconer & Lin, 1997; Versteeg & Malalasekera, 1995), is more effective than the criterion $\kappa-\varepsilon$ model, where κ is the kinetic energy and ε is the viscous of the fluid. The model achieves precision by summing an additional term in the ε-equation. It can pattern turbulent eddies with high delicacy and can imitate flows with high and low Reynolds numbers (H Jing, Guo, Li, & Zhang, 2009; Hefang Jing, Li, Guo, & Xu, 2011; Hefang Jing et al., 2013). A 2D RNG $\kappa-\varepsilon$ sediment model that involves the influence of currents is modified to pattern the residue transport and bed deformation in rivers with persistent bends (Hefang Jing et al., 2013). In physical rivers, the horizontal scale is commonly much larger than the vertical scale. Thus, employing the depth averaged 2D model is appropriate. Numerical emulation that use these systems not only demands less calculation time than 3D sediment models (specially for huge-scale simulations) but also are more accurate than 1D sediment models. RNG $\kappa-\varepsilon$ system is further modified in (Gao, 2013). This model can be represented as follows:

$$\frac{\partial \zeta}{\partial t}(t,\chi,y) + \frac{\partial \mu}{\partial \chi}(t,\chi,y) + \frac{\partial \nu}{\partial y}(t,\chi,y) = 0, \qquad (1.1)$$

where τ is time, ζ is the water level, μ and ν are velocity in χ and y directions respectively. Eq. (1.1) is modified to include water depth H in the following model:

$$\frac{\partial \zeta}{\partial t}(t,\chi,y) + \frac{\partial H\mu}{\partial \chi}(t,\chi,y) + \frac{\partial H\nu}{\partial y}(t,\chi,y) = 0. \qquad (1.2)$$

Fractional calculus (i.e., the generalization of normal calculus) has been highly useful in applications of various methods and schemes in physics, engineering, biology, chemistry, food processing, computer sciences, and medicine (Jalab, & Ibrahim, 2013; Jalab, & Ibrahim, 2014; Jalab, & Ibrahim,2015). We study a system for practical water level prognostication and prediction during dry–wet seasons at the Terengganu River. The mathematical model is a generalized Master equation based on fractional calculus, in the sense of a Riemann-Liouville differential operator, to obtain one-dimensional shallow water equations. We introduce two exact solutions for the proposed model. The first describes the dry condition of the river, and the second characterizes its wet status. The actual data shows the effectiveness of the new model.

2. MATERIALS AND METHODS

Fractional calculus (including fractional order indefinite integral and differentiation) is used to analyze phenomena with type t^{α}. Here, we apply the Riemann-Liouville operators

$$_a I_t^{\alpha} \upsilon(t) = \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \upsilon(\tau)d\tau.$$

The fractional (arbitrary) order differential of the function f of order $\alpha > 0$ is given by

$$_a D_t^{\alpha} \upsilon(t) = \frac{d}{dt} \int_a^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} \upsilon(\tau)d\tau.$$

2.1 Generalized Drying–Wetting Model

In this study, we generalize the dry–wet model (1.1-1.2) by employing the fractional derivative in the velocity as follows:

$$D_t^\alpha \zeta(t,\chi,y) + \frac{\partial\mu}{\partial\chi}(t,\chi-\chi',y) + \frac{\partial\nu}{\partial y}(t,\chi,y-y') = 0, \quad \alpha \in (0,1], a = 0, \tag{2.1}$$

subject to initial condition

$$\left(\zeta(0,\chi,y) = \lambda(\chi,y), \quad \lambda \in R^+\right).$$

We assume that

$$\Phi(\chi-\chi') := -\frac{\partial\mu}{\partial\chi}(t,\chi-\chi',y), \quad \Psi(y-y') := -\frac{\partial\nu}{\partial y}(t,\chi,y-y'),$$

where $\Phi(\chi)$ and $\Psi(y)$ are the velocity-jump rate in the two directions χ and y respectively, while $\chi-\chi'$ and $y-y'$ are the jump of length and width. For one jump Eq. (2.1) becomes

$$D_t^\alpha \zeta(t,\chi,y) = \Phi(t,\chi-\chi',y) + \Psi(t,\chi,y-y'), \quad \alpha \in (0,1]. \tag{2.2}$$

We suppose that the velocity-jump rates are associated with the the river depth

$$H(t,\theta,\vartheta) = \zeta(t,\theta,\vartheta).$$

Thus, we receive the following fractional model for multiple jumps:

$$D_t^\alpha \zeta(t,\chi,y) = \sum_{\chi'}\Phi(t,\chi-\chi',y)\zeta(t,\chi,y) + \sum_{y'}\Psi(t,\chi,y-y')\zeta(t,\chi,y). \tag{2.3}$$

We focus on Eq. (2.3) in our discussion. We find exact and approximate solutions for Eq. (2.3) in the case of drying and wetting.

2.2 Solutions of Fractional Master Equation

Operating Eq.(2.3) by I_t^α, then we obtain

$$\zeta(t,\chi,y) = \lambda(\chi,y) + I_t^\alpha \sum_{\chi'}\Phi(t,\chi-\chi',y)\zeta(t,\chi') + I_t^\alpha \sum_{y'}\Psi(t,\chi,y-y')\zeta(t,y'). \tag{2.4}$$

Fourier domain implies

$$\zeta(t,\theta,\vartheta) = 1 + I_t^\alpha \Phi(t,\theta)\zeta(t,\theta) + I_t^\alpha \Psi(t,\vartheta)\zeta(t,\vartheta), \tag{2.5}$$

where $\Phi(t,\theta)$ and $\Psi(t,\vartheta)$ are the Fourier transform of the velocity-jump rates Φ and Ψ. Laplace transform w. r. t. $(t-\tau)^{\alpha-1}$ leads to

$$\zeta(p,q,\theta,\vartheta) = \frac{p^{\alpha-1}}{p^\alpha - \Phi(\theta)} + \frac{q^{\alpha-1}}{q^\alpha - \Psi(\vartheta)}. \tag{2.6}$$

Then

$$\Phi(\theta) = -\phi\theta^2$$

and

$$\Psi(\vartheta) = -\psi\vartheta^2,$$

where ϕ and ψ are the diffusion coefficients and $\theta \geq 0, \vartheta \geq 0$. Therefore, we conclude that

$$\zeta(p,q,\theta,\vartheta) = \frac{p^{\alpha-1}}{p^\alpha + \phi\theta^2} + \frac{q^{\alpha-1}}{q^\alpha + \psi\vartheta^2}.$$

Hence the position χ' at time t achieves the relation

$$<\chi'^2> \sim t^\alpha.$$

Similarly for y at time t

$$<y'^2> \sim t^\alpha.$$

Now by utilizing the relation between the Mellin and Laplacce techniques:

$$\sigma(s) = M\left(\sigma(t), s\right) = \int_0^\infty t^{s-1}\sigma(t)dt,$$

where

$$\sigma(\omega) = L\left(\sigma(t), \omega\right) = \int_0^\infty t^{s-1}e^{-\omega t}\sigma(t)dt$$

then we have

$$M\left(\sigma(t), s\right) = \frac{1}{\Gamma(1-s)}M\left(L\left(\sigma(t),\omega\right), 1-s\right),$$

where $\Gamma(\varsigma)$ is the gamma function. A computation yields the solution

$$\zeta(s,\theta,\vartheta) = \frac{1}{\Gamma(1-s)}\int_0^\infty \frac{p^{\alpha-\theta-1}}{p^{\alpha-1} + \phi\theta^2}dp + \frac{1}{\Gamma(1-s)}\int_0^\infty \frac{q^{\alpha-\vartheta-1}}{q^{\alpha-1} + \psi\vartheta^2}dq$$

$$= \frac{1}{\alpha\Gamma(1-s)}\left(\left(\phi\theta^2\right)^{\frac{-s}{\alpha}} + \left(\psi\vartheta^2\right)^{\frac{-s}{\alpha}}\right)\beta\left[\frac{s}{\alpha},\left(1-\frac{s}{\alpha}\right)\right], \qquad (2.7)$$

where β is called the beta function. The solution is convert to time t

$$\zeta(t,\theta,\vartheta) = \frac{1}{\alpha}H_{1,2}^{1,1}\left(\left(\phi\theta^2\right)^{1/\alpha}t \mid_{(0,1/\alpha),(0,1)}^{(0,1/\alpha)}\right)$$

$$+ \frac{1}{\alpha}H_{1,2}^{1,1}\left(\left(\psi\vartheta^2\right)^{1/\alpha}t \mid_{(0,1/\alpha),(0,1)}^{(0,1/\alpha)}\right), \qquad (2.8)$$

where H is the Fox-Wright function:

$$_qH_p\left[\eta\left|\begin{matrix}(b_1,B_1),...,(b_q,B_q)\\(a_1,A_1),...,(a_p,A_p)\end{matrix}\right.\right]=_qH_p\left[\eta;(b_j,B_j)_{1,q};(a_j,A_j)_{1,p}\right]$$

$$:=\sum_{n=0}^{\infty}\frac{\Gamma(b_1+nB_1)...\Gamma(b_q+nB_q)}{\Gamma(a_1+nA_1)...\Gamma(a_q+nA_p)}\frac{\eta^n}{n!}=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{q}\Gamma(b_j+nB_j)}{\prod_{j=1}^{p}\Gamma(a_j+nA_j)}\frac{\eta^n}{n!}$$

where $a_j,b_j\in R$, $B_j>0$ for all $j=1,...,q$, $A_j>0$ for all $j=1,...,p$ and $1+\sum_{j=1}^{p}A_j-\sum_{j=1}^{q}B_j\geq0$ for $|\eta|<1$.

In series expansion, Eq. (10) can be read as

$$\zeta(t,\theta,\vartheta)=\frac{1}{\alpha}\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(2k\alpha+1)}\left(\left(\varphi\theta^2t\right)^{2\alpha k}+\left(\psi\vartheta^2t\right)^{2\alpha k}\right)\tag{2.9}$$

which satisfies

$$\left(\zeta(0,\theta,\vartheta)=\lambda(\chi,y)\right).$$

This solution decreases monotonically. Therefore, for $t\to\infty$, we have the asymptotic behavior (limit behavior)

$$\zeta(t,\theta,\vartheta)\sim t^{-\alpha}.\tag{2.10}$$

$\zeta(t,\theta,\vartheta)$ is the intermediate scattering function in the theory of fluids. It engages the density-density inconstancy (such as rocky, swing, frequency, etc.) of wave vectors in liquids. Eq. (2.10) shows that the water level is decreasing (dry case).

We proceed to describe another solution for the Eq.(2.3). If

$$\Lambda:=\sum_{\chi'}\Phi\left(t,\chi-\chi',y\right)+\sum_{y'}\Psi\left(t,\chi,y-y'\right),$$

then Eq. (2.3) can be written as the Beer-Lambert equation

$$D_t^{\alpha}\zeta(t,\chi,y)=\Lambda\zeta(t,\chi,y)\tag{2.11}$$

subjected to the initial condition

$$\left(\zeta(0,\chi,y)=\lambda(\chi,y),\quad\lambda\in R^+\right).$$

The solution of this equation can be expressed as follows:

$$\zeta(t,\chi,y)=\lambda(\chi,y)E_{\alpha}\left(\Lambda t^{\alpha}\right),\tag{2.12}$$

where $E_{\alpha}\left(\Lambda t^{\alpha}\right):=\sum_{n=0}^{\infty}\frac{\left(\Lambda t^{\alpha}\right)^n}{\Gamma\left(1+\alpha n\right)}$

is the Mittag-Leffler function. This function corresponds to the exponential function, which increases for positive data and can thus be utilized as a wet measure. Fig. (1) shows the solutions of Eq. (2.3). The left hand side is the decreasing solution (2.10) and the right hand is the increasing solution (2.12).

2.3 2D- Fractional Model

In a manner similar to the previous subsection, we generalize the 2D-system to receive the following conclusion:

$$
\left(
\begin{array}{c}
D_t^\alpha \zeta(t,\chi,y) + \dfrac{\partial \mu}{\partial \chi} \zeta\left(t, \chi - \chi', y\right) + \dfrac{\partial v}{\partial y} \zeta\left(t, \chi, y - y'\right) \\[2em]
D_t^\alpha (\zeta\mu)(t,\chi,y) + \dfrac{\partial}{\partial \chi}(\zeta\mu\mu)\left(t, \chi - \chi', y\right) + \dfrac{\partial}{\partial y}(\zeta\mu v)\left(t, \chi, y - y'\right)
\end{array}
\right)
=
\left(
\begin{array}{c}
0 \\[2em]
G(\chi, y)
\end{array}
\right),
$$

(2.13)

where $G(\chi, y)$ is a function of gravity and water viscosity, as seen in (Jing, 2013). An approximate solution for (2.13) can be obtained through the following:

Step 1. The first equation has a solution of the form (2.10) or (2.12).
Step 2. The solution of Step 1 is applied in the second equation of Eq. (2.13) to obtain approximate solutions.

In general, system (2.13) can be reduced to a non symmetric system of the form

$$
\left(
\begin{array}{c}
D_t^\alpha \zeta(t,z) + \left(\zeta(\mu+v)\right)_z \\[1.5em]
D_t^\alpha (\zeta\mu)(t,z) + \left(\mu\zeta(\mu+v)\right)_z
\end{array}
\right)
=
\left(
\begin{array}{c}
0 \\[1.5em]
G(z)
\end{array}
\right),
$$

(2.14)

where $z := (\chi, y)$. Let $\upsilon := \zeta\mu$ and $\Lambda(\zeta) := \zeta v$, then we obtain the following symmetric system:

$$
\left(
\begin{array}{c}
D_t^\alpha \zeta + (\upsilon + \Lambda)_z \\[2em]
D_t^\alpha \upsilon + \left(\dfrac{\upsilon^2}{\zeta} + \dfrac{\upsilon\Lambda}{\zeta}\right)_z
\end{array}
\right)
=
\left(
\begin{array}{c}
0 \\[2em]
G(z)
\end{array}
\right),
$$

(2.15)

subjected to the initial condition

$$
\left(\zeta(0,z) = \zeta_0, \quad \upsilon(0,z) = \upsilon_0, \zeta_0 > 0\right)
$$

where $t \in J := (0,T], T < \infty,$ $\Omega \in R^2$ is a bounded domain, the couple $(\zeta, \upsilon) \in \left(C[J,\Omega], C[J,\Omega] \right)$ is denoted the solution of system (2.15). In addition, it achieves

$$\frac{\partial \zeta}{\partial \eta} = \frac{\partial \upsilon}{\partial \eta} = 0, \quad \eta \in \partial \Omega,$$

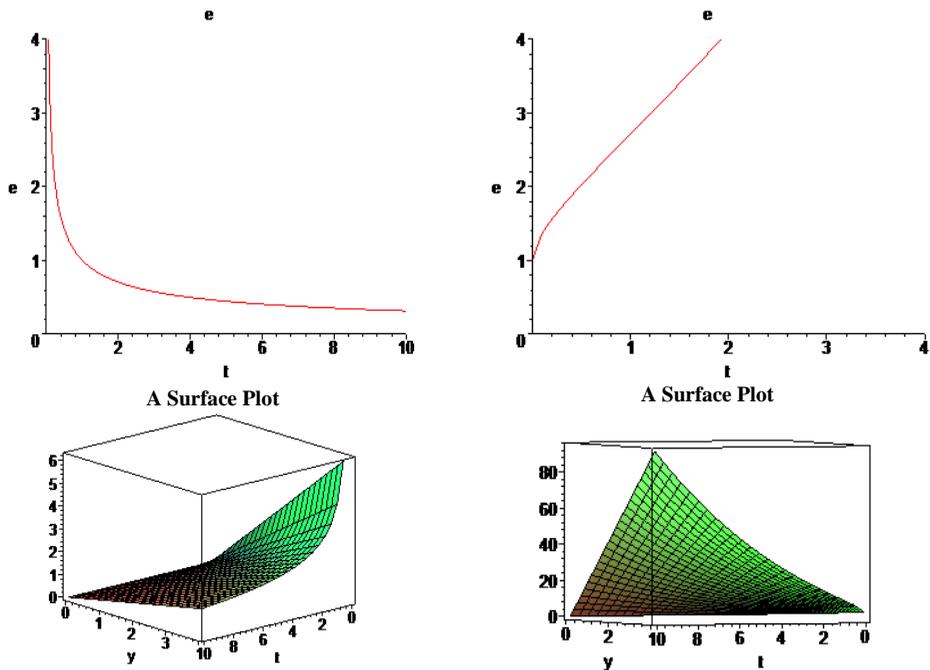with ζ, υ are smooth in J. We have the following existence result:

Theorem 3.1

Let Ω be a bounded domain in R^2 with smooth boundary $\partial \Omega$. Assume that

$$(\zeta_0, \upsilon_0) \in H^1(\Omega) \times H^1(\Omega), \quad \zeta_0 > 0, \upsilon_0 \geq 0, \text{in} \overline{\Omega}$$

where $H^1(\Omega) = \left\{ u \in L^2(\Omega) : | \nabla u | \in L^2(\Omega) \right\}$. If $\upsilon^2 \leq \zeta^2$ and $\dfrac{KT^\alpha}{\Gamma(\alpha+1)} < 1$, $K > 0$, then

there exists a unique bounded solution (ζ, υ) for the system (2.15).
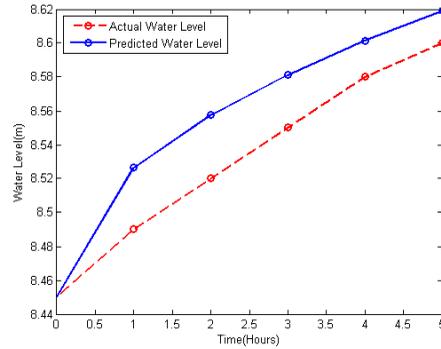
The above theorem is similar to Theorem 2.1 in (Ibrahim, and Jalab, 2014) therefore, we omit the proof.
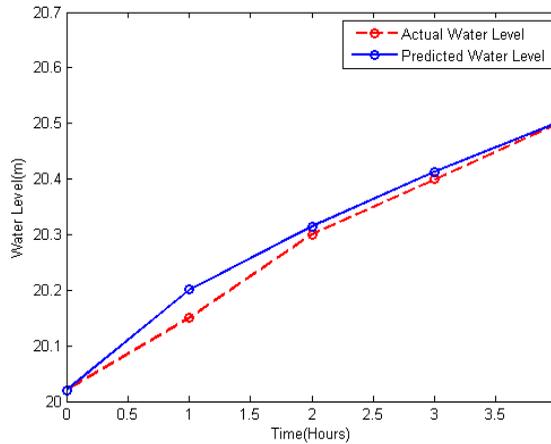

Fig. 1: Solutions of Eq. (2.3). The Left Hand Side is the Decreasing Solution (2.10) and the Right Hand is the Increasing Solution (2.12)
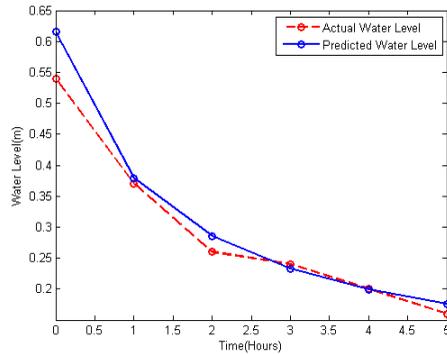
Fig. 2: Drying model Sg. Marang, Jam Peng Berangan

Fig. 3: Wetting Model Jambatan Air Putih

Fig. 4: Wetting Model Kampong Menerong

3. RESULTS AND DISCUSSION

In this section, we present numerical simulation results for the generalized river drying–wetting model. The numerical simulation uses experimental data from the Terengganu River collected from (http://infobanjir.water.gov.my/waterlevelpage. cfm?state=TRG). One set of of dry data and two sets of wet data were used to verify the analytical solution of both the Dry and Wet models of the Terengganu River. These simulations were performed witht the use of Maple and MATLAB R2010a in Windows 7. We considered the Terengganu River in this study using these two mathematical models. The Terengganu River originates from the Lake Kenyir. It flows through the state capital of Terengganu, Kuala Terengganu, and into the South China Sea. Its water level decreases over time (Milliman, Maede,1983). This research empirically studies the Kuala Berang port by using mathematical models to predict the evolution of the water level of the Terengganu River. For the time-dependent solution, we have two different

equations: the time-dependent equations for the drying and wetting areas. The results are shown for several time steps during one drying–wetting cycle applied to the Terengganu River.

Fig. 2 shows the estimation results for the testing data for the actual water level and predicted water level using the drying river model Eq. (2.10). Both results are very close, indicating that the dry model estimate probably contains slight errors. In the dry area, the results were influenced by the value of $\alpha,$ which is chosen experimentally to be equal to 0.7.

Figs. 3 and 4 show the estimation results for wetting testing data for both cases actual water level and predicted water level using the wet river model in Eq. (2.12). Moreover, the results are very close, proving that the dry model estimate and the water level have small output errors. In the wet area, the results were influenced by the value of α which is chosen experimentally as being equal to 0.49.

These findings suggest that methodologies for retrieving water levels from datasets that are only available as point of measurements must be improved. Moreover, future works should consider the application of the proposed mathematical models to consider different boundary conditions for more applications, such as flood forecasting applications.

4. CONCLUSION

New wetting and drying fractional models were proposed, analyzed, and tested. Our case study considered the Terengganu River in Malaysia. The mathematical model is a generalized master equation based on fractional calculus to obtain one-dimensional shallow water equations. We introduce two exact solutions for the proposed models. The first one describes the dry condition of the river. The second characterizes the wet condition. The actual data show the effectiveness of the proposed wetting and drying fractional models.

ACKNOWLEDGMENT

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