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THE BETA QUADRATIC HAZARD RATE DISTRIBUTION

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ABSTRACT

A five-parameter distribution so-called the beta quadratic hazard rate distribution is defined and studied. The new distribution contains, as special submodels, several important distributions, such as the Linear failure rate, Rayleigh and Exponential distributions. We derive the moments and examine the order statistics.We propose the method of maximum likelihood for estimating the model parameters and obtain the observed information matrix.

KEYWORDS

Quadratic Hazard Rate distribution, Order Statistics, Maximum Likelihood Estimation, Reliability Function.

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1. INTRODUCTION

Bain [3] presented quadratic hazard rate distribution (*QHRD*). This distribution generalizes several well known distributions. Among these distributions are the linear fialure (hazard) rate , exponential and Rayleigh distributions. Also, the *QHRD* may have an increasing (decreasing) hazard function or a bathtub shaped hazard function or an upside-down bathtub shaped hazard function. This property enables this distribution to be used in many applications in several areas, such as reliability, life testing, survival analysis and others.

A random variable X is said to have the quadratic hazard rate distribution (*QHRD*) with three parameters $\alpha, \theta,$ and β , if it has the cumulative distribution function

$$G(x,\alpha,\theta,\beta) = 1 - e^{-\left(\alpha\, x + \frac{\theta}{2} x^2 + \frac{\beta}{3} x^3\right)}, x > 0, \quad (1)$$

where $\alpha \geq 0, \beta \geq 0$ and $\theta \geq -2\sqrt{\alpha\beta}$. This restriction on the parameter space is made to be insure that the hazard function with the following form is positive, see Bain [3], $A(x,\alpha,\theta,\beta) = \alpha + \theta\, x + \beta\, x^2$, $x > 0$. The corresponding probability density function (pdf) is given by

$$g(x,\alpha,\theta,\beta) = \left(\alpha + \theta x + \beta x^2\right) e^{-\left(\alpha\, x + \frac{\theta}{2}\, x^2 + \frac{\beta}{3}\, x^3\right)}, x > 0, \tag{2}$$

Beta Generalized (Beta-G), introduced by Singh et al. [24] is a rich class of generalized distributions.This class has captured a considerable attention over the last few years. Sepanshi and Kang applied the Beta- G distribution to model the size distribution of income. This distribution has been studied in literature for various forms of G. The distributions that have been explored are the Beta Normal(BN) (Eugene et al. [14]), the Beta Gumbel (BGu) distribution (Nadarajah and Kotz [22], the Beta Frechet (BFr) distribution (Nadarajah and Gupta [21]), the Beta Exponential (BE) distribution (Nadarajah and Kotz [23]), the Beta Weibull (BW) distribution (Lee et al. [19]), and Cordeiro et al. present beta Weibull geometric [8]. Souza et al. [4] introduced the Beta Generalized Exponential (BGE) distribution and the Beta Modified Weibull (BMW) distribution was discussed by Silva et al. [24]. Beta Inverse Weibull (BIW) (Khan [11]), Beta Generalized Pareto (BGP) (Mahmoudi [20]), Beta-Lindley distribution (Merovci & Sharma [21] and Beta Dagum(BDa) (Domma and Condino [13]) are some new extensions of the Beta G class of distributions. The cumulative distribution function (cdf) of the Beta-G distribution has the form

$$\begin{aligned} F(x) &= \frac{1}{B(a,b)} \int_0^{G(x)} w^{(a-1)} (1-w)^{b-1}\, dw \\ &= \frac{B_{G(x)}(a,b)}{B(a,b)} = I_{G(x)}(a,b), \end{aligned} \tag{3}$$

where $a > 0$ and $b > 0$ are two additional parameters whose role is to introduce skewness and to vary tail weight, and $G(x)$ is an arbitrary parent baseline cdf of a random variable;

$$B_y(a,b) = \int_0^y w^{(a-1)} (1-w)^{b-1}\, dw$$

is the incomplete beta function with $B(a,b) = B_1(a,b)$ and $I_y(a,b) = \dfrac{B_y(a,b)}{B(a,b)}$ is the incomplete beta function ratio. For general a and b, we can express (1.3) in terms of the well-known hypergeometric function defined by

$${}_2F_1(\beta,\theta,\gamma;x) = \sum_{i=0}^{\infty} \frac{(\beta)_i (\theta)_i}{(\gamma)_i\, i!} x^i,$$

where $(\beta)_i = \beta(\beta+1)...(\beta+i-1)$ denotes the ascending factorial. We obtain

$$F(x) = \frac{G(x)}{aB(a,b)}\ {}_2F_1\left(a, 1-b, \theta a+1; G(x)\right).$$

The properties of $F(x)$ for any beta G distribution defined from a parent $G(x)$ in (1.3) could, in principle, follow from the properties of the hypergeometric function which are well established in the literature; see, for example, Section 9.1 of Gradshteyn and Ryzhik (2000).

The probability density function (pdf) corresponding to (1.3) is written as

$$f(x) = \frac{g(x)}{B(a,b)} G(x)^{a-1} \{1-G(x)\}^{b-1}, \tag{4}$$

we noted that $f(x)$ will be most tractable when the cdf $G(x)$ and pdf $g(x) = \frac{d}{dx}G(x)$ are cumulative distribution function and the probability density function of the parent/ baseline distribution.

The rest of the article is organized as follows. In Section 2, we define the beta quadratic hazard rate distribution , the expansion for the cumulative and density functions of the $BQHR$ distribution and some special cases. Quantile function, median, moments, moment generating function are discussed in Section 3. In Section 4 included the distribution of the order statistics. Least squares and weighted least squares estimators introduced in Section 5. Finally, maximum likelihood estimation is performed in Section 6.

2. BETA QUADRATIC HAZARD RATE DISTRIBUTION

In this section,we introduce the five$-$parameter beta quadratic hazard rate $BQHR$ (ϕ, x) distribution where $\phi = (\alpha, \theta, \beta, a, b)$. by taking $G(x)$ in Equation (1) to be the cdf of quadratic hazard rate (QHR) distribution. Using (1) and (2) in (4), the cdf of the $(BQHR)$ distribution can be written as

$$F_{BQHR}(x,\phi) = I_{\left\{1-e^{-\left(\alpha x+\frac{\theta}{2}x^2+\frac{\beta}{3}x^3\right)}\right\}}(a,b)$$

$$= \frac{1}{B(a,b)} \int_0^{1-e^{-\left(\alpha x+\frac{\theta}{2}x^2+\frac{\beta}{3}x^3\right)}} w^{(a-1)}(1-w)^{b-1}dw, x > 0, \tag{5}$$

The corresponding probability density function of the new distribution takes the form

$$f_{BQHR}(x,\phi)=\frac{\left(\alpha+\theta x+\beta x^2\right)e^{-\left(\alpha x+\frac{\theta}{2}x^2+\frac{\beta}{3}x^3\right)}}{B(a,b)}$$

$$\left[1-e^{-\left(\alpha x+\frac{\theta}{2}x^2+\frac{\beta}{3}x^3\right)}\right]^{a-1}\left\{e^{-\left(\alpha x+\frac{\theta}{2}x^2+\frac{\beta}{3}x^3\right)}\right\}^{b-1}$$

$$=\frac{\left(\alpha+\theta\, x+\beta x^2\right)}{B(a,b)}e^{-b\left(\alpha x+\frac{\theta}{2}x^2+\frac{\beta}{3}x^3\right)}\left[1-e^{-\left(\alpha x+\frac{\theta}{2}x^2+\frac{\beta}{3}x^3\right)}\right]^{a-1},x>0. \quad (6)$$

Figure 0 illustrates some of the possible shapes of the pdf of the BQHR distribution for selected values of the parameters α,θ,β,a and b, respectively.

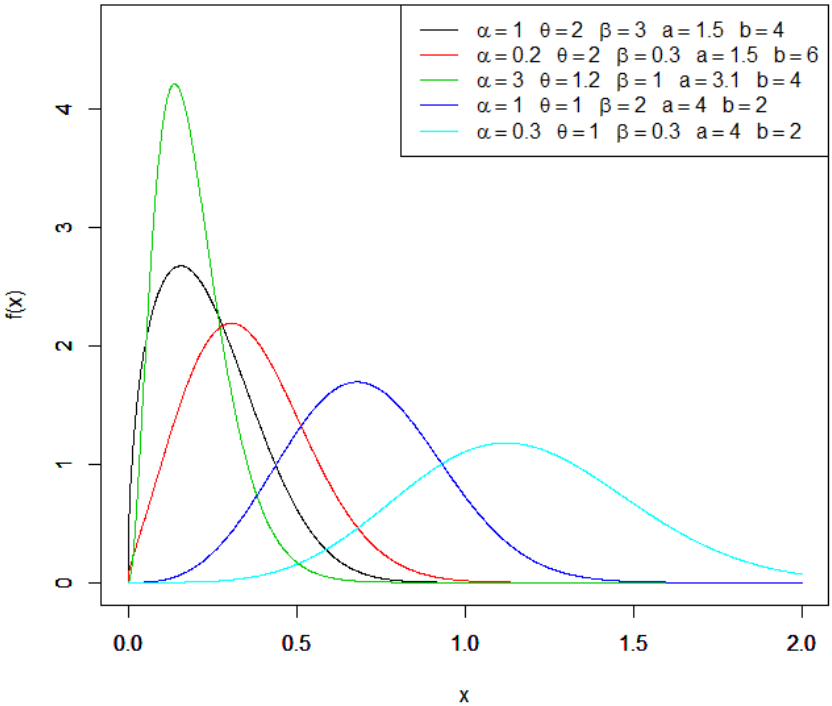

Figure 1: The pdf's of Various BQHR Distributions

The reliability (survival) function (RF) of the beta quadratic hazard rate distribution is denoted by $R_{BQHR}(x)$ also known as the survivor function and is defined as

$$R_{BQHR}(x) = 1 - F_{BQHR}(x) = I_{\left\{1-e^{-\left(\alpha x + \frac{\theta}{2}x^2 + \frac{\beta}{3}x^3\right)}\right\}}(a,b) \tag{7}$$

One of the characteristic in reliability analysis is the hazard (failure) rate function (HF) defined by

$$h_{BQHR}(x) = \frac{f_{BQHR}(x)}{1 - F_{BQHR}(x)} = \frac{\left(\alpha + \theta x + \beta x^2\right)}{B(a,b) I_{\left\{1-e^{-\left(\alpha x + \frac{\theta}{2}x^2 + \frac{\beta}{3}x^3\right)}\right\}}(a,b)} e^{-b\left(\alpha x + \frac{\theta}{2}x^2 + \frac{\beta}{3}x^3\right)} \left[1 - e^{-\left(\alpha x + \frac{\theta}{2}x^2 + \frac{\beta}{3}x^3\right)}\right]^{a-1} \tag{8}$$

2.1 Special Cases of the *BQHR* Distribution

The beta quadratic hazard rate is very flexible model that approaches to different distributions when its parameters are changed. The *BQHR* distribution contains as special- models the following well known distributions. If X is a random variable with pdf (2.2), we use the notation $X \sim BQHR(\alpha, \theta, \beta, a, b)$ then we have the following cases**.**

1. For $a = b = 1$, then (2.2) reduces to the quadratic hazard rate distribution.
2. For $\theta = \beta = 0$ we get the beta exponential distribution.
3. Beta Rayleigh distribution arises as a special case of *BQHR* by taking $\alpha = \beta = 0$.
4. Applying $\beta = 0$ we can obtain the Beta generalized linear failure rate distribution.
5. For $\beta = \theta = 0$ we can obtain the Beta linear failure rate distribution.
6. generalized linear failure rate distribution arises as special case of *BQHR* by taking $b = 1$, and $\beta = 0$.

2.2 Expansion for the Cumulative and Density Functions

In this subsection we present some representations of cdf, pdf of beta quadratic hazard rate distribution. The mathematical relation given below will be useful in this subsection. Here and henceforth, let X be a random variable having the *BQHR* density function (2.2). Equations (2.1) and (2.2) are straight forward to compute using any software with algebraic facilities. However, we can obtain expansions for $F(x)$ and $f(x)$ in terms of infinite (or finite) weighted sums of cdf's and pdf's of inverse Weibull distributions, respectively. First, for $b > 0$ real non-integer, we replace $(1-w)^{b-1}$ under the integral by the power series and integrate to obtain

$$\int_0^x w^{(a-1)}(1-w)^{b-1}\,dw = \sum_{i=0}^{\infty} \frac{(-1)^i \binom{b-1}{i}}{(a+i)} x^i,$$

where the binomial term $\binom{b-1}{i} = \frac{\Gamma(b)}{\Gamma(b-i)i!}$ is defined for any real b. From Equation (2.1) we have

$$F_{BQHR}(x,\phi) = \frac{1}{B(a,b)} \sum_{i=0}^{\infty} \frac{(-1)^i \binom{b-1}{i}}{(a+i)} \left(1-e^{-\left(\alpha x+\frac{\theta}{2}x^2+\frac{\beta}{3}x^3\right)}\right)^i, \tag{9}$$

also using the power series , the equation (2.2) becomes

$$f(x,\phi) = \frac{\sum_{j=0}^{\infty}(-1)^j \binom{b-1}{j}}{B(a,b)} \left(\alpha+\theta\, x+\beta\, x^2\right) e^{-(b+j)\left(\alpha x+\frac{\theta}{2}x^2+\frac{\beta}{3}x^3\right)}. \tag{10}$$

3. STATISTICAL PROPERTIES

In this section we discuss the statistical properties of the Beta quadratic hazard rate distribution. Specifically, moments and moment generating function. Moments are necessary and important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution (e.g., tendency, dispersion, skewness and kurtosis).

3.1 Moments

The following theorems give the r_{th} moment (μ_r) and moment generating function $M_X(t)$ of the $BQHR(x;\alpha,\theta,\beta,a,b)$.

Theorem 3.1.

If X has the $BQHR(x;\alpha,\theta,\beta,a,b)$, then the r_{th} moment of X is given by the following

$$\mu_r' = W_{j,k,m}\left[\frac{\alpha\Gamma(r+2k+3m+1)}{\left[\alpha(b+j)\right]^{r+2k+3m+1}} + \theta\frac{\Gamma(r+2k+3m+2)}{\left[\alpha(b+j)\right]^{r+2k+3m+2}} + \frac{\beta\Gamma(r+2k+3m+3)}{\left[\alpha(b+j)\right]^{r+2k+3m+3}}\right] \tag{11}$$

where

$$W_{j,k,m} = \frac{\sum_{j,k,m=0}^{\infty}(-1)^{j+k+m}\binom{b-1}{j}\theta^k\beta^m(b+j)^{k+m}}{B(a,b)k!m!2^k3^m}. \tag{12}$$

Proof:

Starting with

$$\mu_r' = \int_0^\infty x^r f_{BQHR}(x)dx$$

$$= \frac{\sum_{j=0}^{\infty}(-1)^j \binom{b-1}{j}}{B(a,b)} \int_0^\infty x^r \left(\alpha + \theta x + \beta x^2\right) e^{-(b+j)\left(\alpha x + \frac{\theta}{2}x^2 + \frac{\beta}{3}x^3\right)} dx. \qquad (13)$$

since $0 < e^{-\left(\alpha x + \frac{\theta}{2}x^2 + \frac{\beta}{3}x^3\right)} < 1$, then the series expansion of $e^{-(b+j)\frac{\theta}{2}x^2}$ and $e^{-(b+j)\frac{\beta}{3}x^3}$ are given by

$$e^{-(b+j)\frac{\theta}{2}x^2} = \sum_{k=0}^{\infty} \frac{\left(-(b+j)\frac{\theta}{2}x^2\right)^k}{k!} \text{ and } e^{-(b+j)\frac{\beta}{3}x^3} = \sum_{m=0}^{\infty} \frac{\left(-(b+j)\frac{\beta}{3}x^3\right)^m}{m!}, \quad (14)$$

we have

$$\mu_r' = \frac{\sum_{j,k,m=0}^{\infty}(-1)^{j+k+m}\binom{b-1}{j}\theta^k\beta^m(b+j)^{k+m}}{B(a,b)k!m!2^k3^m} \int_0^\infty x^{r+2k+3m}\left(\alpha+\theta x+\beta x^2\right)e^{-(b+j)(\alpha x)}dx$$

$$= W_{j,k,m}\left\{\alpha\int_0^\infty x^{r+2k+3m}e^{-\alpha(b+j)x}dx + \theta\int_0^\infty x^{r+2k+3m+1}e^{-\alpha(b+j)x}dx \right.$$

$$\left. +\beta\int_0^\infty x^{r+2k+3m+2}e^{-\alpha(b+j)x}dx\right\}$$

$$= W_{j,k,m}\left[\frac{\alpha\Gamma(r+2k+3m+1)}{\left[\alpha(b+j)\right]^{r+2k+3m+1}} + \theta\frac{\Gamma(r+2k+3m+2)}{\left[\alpha(b+j)\right]^{r+2k+3m+2}} + \frac{\beta\Gamma(r+2k+3m+3)}{\left[\alpha(b+j)\right]^{r+2k+3m+3}}\right]$$

(15)

where

$$W_{j,k,m} = \frac{\sum_{j,k,m=0}^{\infty}(-1)^{j+k+m}\binom{b-1}{j}\theta^k\beta^m(b+j)^{k+m}}{B(a,b)k!m!2^k3^m}.$$

which completes the proof.

Based on Theorem (3.1) the measures of variation, skewness and kurtosis of the $BQHR(x;\alpha,\theta,\beta,a,b)$ distribution can be obtained according to the following relation

$$CV_{BQHR} = \sqrt{\frac{\mu_2}{\mu_1} - 1},$$

$$CS_{BQHR} = \frac{\mu_3(\theta) - 3\mu_1(\theta)\mu_2(\theta) + 2\mu_1^3(\theta)}{\left[\mu_2(\theta) - \mu_1^2(\theta)\right]^{\frac{3}{2}}}$$

and

$$CK_{BQHR} = \frac{\mu_4(\theta) - 4\mu_1(\theta)\mu_3(\theta) + 6\mu_1^2(\theta)\mu_2(\theta) - 3\mu_1^4(\theta)}{\left[\mu_2(\theta) - \mu_1^2(\theta)\right]^2}.$$

3.2 Moment Generating Function

In this subsection we derived the moment generating function (mgf) of Beta quadratic hazard rate distribution.

Theorem 3.2.

If X has the $BQHR(x;\alpha,\theta,\beta,a,b)$,then the the moment generating function (mgf) of X is given as follows

$$M_X(t) = W_{j,k,m}\left[\frac{\alpha\,\Gamma(2k+3m+1)}{\left[(\alpha\,(b+j)-t)\right]^{2k+3m+1}} + \theta\frac{\Gamma(2k+3m+2)}{\left[(\alpha\,(b+j)-t)\right]^{2k+3m+2}} + \frac{\beta\,\Gamma(2k+3m+3)}{\left[(\alpha\,(b+j)-t)\right]^{2k+3m+3}}\right]. \tag{16}$$

Proof:

$$M_X(t) = \int_0^\infty e^{tx} f_{BQHR}(x)dx$$

$$= \frac{\sum_{j=0}^{\infty}(-1)^j\binom{b-1}{j}}{B(a,b)}\int_0^\infty e^{tx}\left(\alpha + \theta x + \beta x^2\right)e^{-(b+j)\left(\alpha\,x + \frac{\theta}{2}x^2 + \frac{\beta}{3}x^3\right)}dx \tag{17}$$

substituting (14) into relation (17) we get the following

$$M_X(t) = W_{j,k,m}\int_0^\infty x^{2k+3m}\left(\alpha + \theta x + \beta x^2\right)e^{-(\alpha(b+j)-t)x}dx$$

$$= W_{j,k,m}\left\{\alpha\int_0^\infty x^{2k+3m}e^{-(\alpha(b+j)-t)x}dx + \theta\int_0^\infty x^{2k+3m+1}e^{-(\alpha(b+j)-t)x}dx + \beta\int_0^\infty x^{2k+3m+2}e^{-(\alpha(b+j)-t)x}dx\right\}$$

$$= W_{j,k,m}\left[\frac{\alpha\ \Gamma(2k+3m+1)}{\left[(\alpha(b+j)-t)\right]^{2k+3m+1}} + \theta\frac{\Gamma(2k+3m+2)}{\left[(\alpha(b+j)-t)\right]^{2k+3m+2}} + \frac{\beta\ \Gamma(2k+3m+3)}{\left[(\alpha(b+j)-t)\right]^{2k+3m+3}}\right],$$

which completes the proof.

4. ORDER STATISTICS

Moments of order statistics play an important role in quality control testing and reliability, where a practitioner needs to predict the failure of future items based on the times of a few early failures. These predictors are often based on moments of order statistics. We now derive an explicit expression for the density function of the i_{th} order statistic $X_{(i:n)}$, say $f_{i::n}(x)$, in a random sample of size n from the *BQHR* distribution. Let $X_1,\ X_2,...,\ X_n$ be a simple random sample from *BQHR* $(x;\alpha,\theta,\beta,a,b)$ with cumulative distribution function and probability density function as in (2.1) and (2.2), respectively. Let $X_{(1:n)} \le X_{(2:n)} \le ... \le X_{(n:n)}$ denote the order statistics obtained from this sample. In reliability literature, $X_{(i:n)}$ denote the lifetime of an $(n-i+1)-$ out $-$ of $-\ n$ system which consists of n independent and identically components. Then the pdf of $X_{(i:n)}$, $1 \le i \le n$ is given by

$$f_{i::n}(x) = \frac{1}{\beta(i, n-i+1)}\left[F(x,\phi)\right]^{i-1}\left[1-F(x,\phi)\right]^{n-i}\ f(x,\phi) \qquad (4.1)$$

where $\phi = (\alpha,\theta,\beta,a,b)$. We defined the first order statistics $X_{(1)} = Min(X_1,\ X_2,...,\ X_n)$, the the last order statistics as $X_{(n)} = Max(X_1, X_2,..., X_n)$ and median order X_{m+1}.

The pdf of the i_{th} order statistic for Beta quadratic hazard rate distribution is given by

$$f_{i::n}(x)=\frac{1}{\beta(i,n-i+1)}\left[I_{\left\{1-e^{-\left(\alpha x_{(i)}+\frac{\theta}{2}x_{(i)}^2+\frac{\beta}{3}x_{(i)}^3\right)}\right\}}(a,b)\right]^{i-1}$$

$$\left[1-I_{\left\{1-e^{-\left(\alpha x_{(i)}+\frac{\theta}{2}x_{(i)}^2+\frac{\beta}{3}x_{(i)}^3\right)}\right\}}(a,b)\right]^{n-i}$$

$$\times\frac{\left(\alpha+\theta x_{(i)}+\beta x_{(i)}^2\right)}{B(a,b)}e^{-b\left(\alpha x_{(i)}+\frac{\theta}{2}x_{(i)}^2+\frac{\beta}{3}x_{(i)}^3\right)}\left[1-e^{-\left(\alpha x_{(i)}+\frac{\theta}{2}x_{(i)}^2+\frac{\beta}{3}x_{(i)}^3\right)}\right]^{a-1},$$

Therefore, the pdf of the largest order statistic $X_{(n)}$ is given by

$$f_{n:n}(x)=\frac{n\left(\alpha+\theta x_{(n)}+\beta x_{(n)}^2\right)}{B(a,b)}\left[I_{\left\{1-e^{-\left(\alpha x_{(n)}+\frac{\theta}{2}x_{(n)}^2+\frac{\beta}{3}x_{(n)}^3\right)}\right\}}(a,b)\right]^{n-1}$$

$$\times e^{-b\left(\alpha x_{(n)}+\frac{\theta}{2}x_{(n)}^2+\frac{\beta}{3}x_{(n)}^3\right)}\left[1-e^{-\left(\alpha x_{(n)}+\frac{\theta}{2}x_{(n)}^2+\frac{\beta}{3}x_{(n)}^3\right)}\right]^{a-1},$$

and the pdf of the smallest order statistic $X_{(1)}$ is given by

$$f_{1:n}(x)=\frac{n\left(\alpha+\theta x_{(1)}+\beta x_{(1)}^2\right)}{B(a,b)}\left[1-I_{\left\{1-e^{-\left(\alpha x_{(1)}+\frac{\theta}{2}x_{(1)}^2+\frac{\beta}{3}x_{(1)}^3\right)}\right\}}(a,b)\right]^{n-1}$$

$$\times e^{-b\left(\alpha x_{(1)}+\frac{\theta}{2}x_{(1)}^2+\frac{\beta}{3}x_{(1)}^3\right)}\left[1-e^{-\left(\alpha x_{(1)}+\frac{\theta}{2}x_{(1)}^2+\frac{\beta}{3}x_{(1)}^3\right)}\right]^{a-1}.$$

5. LEAST SQUARES AND WEIGHTED LEAST SQUARES ESTIMATORS

In this section we provide the regression based method estimators of the unknown parameters of the kumaraswamy quadratic hazard rate distribution, which was originally suggested by Swain, Venkatraman and Wilson (1988) to estimate the parameters of beta distributions. It can be used some other cases also. Suppose $X_1,...,X_n$ is a random sample of size n from a distribution function $G(.)$ and suppose $X_{(i)}$; $i=1,2,...,n$ denotes the ordered sample. The proposed method uses the distribution of $G(Y_{(i)})$. For a sample of size n, we have

$$E\left(G\left(X_{(i)}\right)\right)=\frac{i}{n+1},\quad V\left(G\left(X_{(i)}\right)\right)=\frac{i\ (n-i+1)}{(n+1)^2(n+2)}$$

and

$$Cov\left(G\left(X_{(i)}\right),G\left(X_{(j)}\right)\right)=\frac{i\ (n-j+1)}{(n+1)^2(n+2)};\quad \text{for } i<j,$$

see Johnson, Kotz and Balakrishnan (1995). Using the expectations and the variances, two variants of the least squares methods can be used.

Method 1 (Least Squares Estimators).

Obtain the estimators by minimizing

$$BQHR(\alpha,\theta,\beta,a,b)=\sum_{i=1}^{n}\left(G(X_{(i)})-\frac{i}{n+1}\right)^2, \tag{18}$$

with respect to the unknown parameters. Therefore in case of $BQHR$ distribution the least squares estimators of $\alpha,\theta,\beta,$ a and b, say $\hat{\alpha}_{LSE},\hat{\theta}_{LSE},\hat{\beta}_{LSE},\hat{a}_{LSE}$ and $\hat{b}_{LSE}$ respectively, by using (2.1) and (5.1) we have the following equation

$$BQHR(\alpha,\theta,\beta,a,b)=\sum_{i=1}^{n}\left[I_{\left\{1-e^{-\left(\alpha x_{(i)}+\frac{\theta}{2}x_{(i)}^2+\frac{\beta}{3}x_{(i)}^3\right)}\right\}}(a,b)-\frac{i}{n+1}\right]^2 \tag{19}$$

Method 2 (Weighted Least Squares Estimators).

The weighted least squares estimators can be obtained by

Minimizing

$$\sum_{i=1}^{n}w_i\left(G\left(X_{(i)}\right)-\frac{i}{n+1}\right)^2, \tag{5.3}$$

with respect to the unknown parameters, where

$$w_i = \frac{1}{V\left(G\left(X_{(i)}\right)\right)} = \frac{(n+1)^2(n+2)}{i(n-i+1)}.$$

Therefore, in case of $BQHR$ distribution the weighted least squares estimators of α, θ, β, a and b, say $\hat{\alpha}_{WLSE}, \hat{\theta}_{WLSE}, \hat{\beta}_{WLSE}, \hat{a}_{WLSE}$ and $\hat{b}_{WLSE}$ respectively, can be obtained by minimizing

$$\sum_{i=1}^{n} w_i \left[I_{\left\{1-e^{-\left(\alpha x_{(i)}+\frac{\theta}{2}x_{(i)}^2+\frac{\beta}{3}x_{(i)}^3\right)}\right\}}(a,b) - \frac{i}{n+1} \right]^2$$

with respect to the unknown parameters only.

6. MAXIMUM LIKELIHOOD ESTIMATORS

In this section we consider the maximum likelihood estimators (MLE's) of $BQHR$ distribution. Let $\phi = (\alpha, \theta, \beta, a, b)^T$, in order to estimate the parameters α, θ, β, a and b of Beta quadratic hazard rate distribution, let $x_1, ..., x_n$ be a random sample of size n from $BQHR$ $(x;\phi)$. The associated score function is given by

$$U_n(\phi) = \left[\frac{\partial L}{\partial \alpha}, \frac{\partial L}{\partial \theta}, \frac{\partial L}{\partial \beta}, \frac{\partial L}{\partial a}, \frac{\partial L}{\partial b} \right]^T$$

The log likelihood function can be written as

$$\begin{aligned} L\left(\phi, x_{(i)}\right) &= -n \ln \Gamma(a) - n \ln \Gamma(b) + n \log \Gamma(a+b) + \sum_{i=1}^{n} \log\left(\alpha + \theta x_{(i)} + \beta x_{(i)}^2\right) \\ &\quad - b\alpha \sum_{i=1}^{n} x_{(i)} - \frac{b\theta}{2} \sum_{i=1}^{n} x_{(i)}^2 - \frac{b\beta}{3} \sum_{i=1}^{n} x_{(i)}^3 \\ &\quad + (a-1) \sum_{i=1}^{n} \ln\left[1 - e^{-\left(\alpha x_{(i)} + \frac{\theta}{2} x_{(i)}^2 + \frac{\beta}{3} x_{(i)}^3\right)}\right] \end{aligned} \qquad (20)$$

Differentiating L with respect to each parameter α, θ, β, a and b and setting the result equals to zero, we obtain maximum likelihood estimates. The partial derivatives of L with respect to each parameter or the score function is given by

$$U_n(\phi) = \left(\frac{\partial L}{\partial \alpha}, \frac{\partial L}{\partial \theta}, \frac{\partial L}{\partial \beta}, \frac{\partial L}{\partial a}, \frac{\partial L}{\partial b} \right)$$

where

$$\frac{\partial L}{\partial \alpha} = \sum_{i=1}^{n} \frac{1}{\left(\alpha + \theta x_{(i)} + \beta x_{(i)}^{2}\right)} - b\sum_{i=1}^{n} x_{(i)} + (a-1)\sum_{i=1}^{n} \frac{x_{(i)} e^{-\left(\alpha x_{(i)} + \frac{\theta}{2} x_{(i)}^{2} + \frac{\beta}{3} x_{(i)}^{3}\right)}}{\left[1 - e^{-\left(\alpha x + \frac{\theta}{2} x^{2} + \frac{\beta}{3} x^{3}\right)}\right]}, \tag{21}$$

$$\frac{\partial L}{\partial \theta} = \sum_{i=1}^{n} \frac{x_{(i)}}{\left(\alpha + \theta x_{(i)} + \beta x_{(i)}^{2}\right)} - \frac{b}{2}\sum_{i=1}^{n} x_{(i)}^{2} + (a-1)\sum_{i=1}^{n} \frac{x_{(i)}^{2} e^{-\left(\alpha x_{(i)} + \frac{\theta}{2} x_{(i)}^{2} + \frac{\beta}{3} x_{(i)}^{3}\right)}}{2\left[1 - e^{-\left(\alpha x + \frac{\theta}{2} x^{2} + \frac{\beta}{3} x^{3}\right)}\right]}, \tag{22}$$

$$\frac{\partial L}{\partial \beta} = \sum_{i=1}^{n} \frac{x_{(i)}^{2}}{\left(\alpha + \theta x_{(i)} + \beta x_{(i)}^{2}\right)} - \frac{b}{3}\sum_{i=1}^{n} x_{(i)}^{3} + (a-1)\sum_{i=1}^{n} \frac{x_{(i)}^{3} e^{-\left(\alpha x_{(i)} + \frac{\theta}{2} x_{(i)}^{2} + \frac{\beta}{3} x_{(i)}^{3}\right)}}{3\left[1 - e^{-\left(\alpha x + \frac{\theta}{2} x^{2} + \frac{\beta}{3} x^{3}\right)}\right]}, \tag{23}$$

$$\frac{\partial \log L}{\partial a} = -n\psi(a) + n\psi(a+b) + \sum_{i=1}^{n} \ln\left[1 - e^{-\left(\alpha x_{(i)} + \frac{\theta}{2} x_{(i)}^{2} + \frac{\beta}{3} x_{(i)}^{3}\right)}\right], \tag{24}$$

and

$$\frac{\partial L}{\partial b} = -n\psi(b) + n\psi(a+b) - \sum_{i=1}^{n} \left(\alpha x_{(i)} + \frac{\theta}{2} x_{(i)}^{2} + \frac{\beta}{3} x_{(i)}^{3}\right). \tag{25}$$

where $\psi(t) = \frac{d \log \Gamma(t)}{dt}$ is the digamma function.

The maximum likelihood estimation (MLE) of ϕ, say $\hat{\phi}$, is obtained by solving the nonlinear system $U_n(\phi) = 0$. To solve the equations (21) through (25), it is usually more convenient to use nonlinear optimization algorithms such as quasi-Newton algorithm to numerically maximize the log-likelihood function. In order to compute the standard errors and asymptotic confidence intervals we use the usual large sample approximation, in which the MLEs can be treated as being approximately trivariate normal.

Hence as $n \to \infty$, the asymptotic distribution of the MLE is given by, see Zaindin et al. (2009):

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\theta} \\ \hat{\beta} \\ \hat{a} \\ \hat{b} \end{pmatrix} \sim \left[\begin{pmatrix} \hat{\alpha} \\ \hat{\theta} \\ \hat{\beta} \\ \hat{a} \\ \hat{b} \end{pmatrix}, \begin{pmatrix} \hat{V}_{11} & \hat{V}_{12} & \hat{V}_{13} & \hat{V}_{14} & \hat{V}_{15} \\ \hat{V}_{21} & \hat{V}_{22} & \hat{V}_{23} & \hat{V}_{24} & \hat{V}_{25} \\ \hat{V}_{31} & \hat{V}_{32} & \hat{V}_{33} & \hat{V}_{34} & \hat{V}_{35} \\ \hat{V}_{41} & \hat{V}_{42} & \hat{V}_{43} & \hat{V}_{44} & \hat{V}_{45} \\ \hat{V}_{51} & \hat{V}_{52} & \hat{V}_{53} & \hat{V}_{54} & \hat{V}_{55} \end{pmatrix} \right],$$

where $(\hat{V}_{ij} = V_{ij} | \varphi = \hat{\varphi})$, and

$$\begin{pmatrix} V_{11} & V_{12} & V_{13} & V_{14} & V_{15} \\ V_{21} & V_{22} & V_{23} & V_{24} & V_{25} \\ V_{31} & V_{32} & V_{33} & V_{34} & V_{35} \\ V_{41} & V_{42} & V_{43} & V_{44} & V_{45} \\ V_{51} & V_{52} & V_{53} & V_{54} & V_{55} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} \end{pmatrix},$$

is the approximate variance-covariance matrix with its elements obtained from

$$A_{11} = \frac{\partial^2 \mathcal{L}}{\partial \alpha^2} \quad A_{12} = \frac{\partial^2 \mathcal{L}}{\partial \alpha \partial \alpha} \quad A_{13} = \frac{\partial^2 \mathcal{L}}{\partial \alpha \partial \beta} \quad A_{14} = \frac{\partial^2 \mathcal{L}}{\partial \alpha \partial k} \quad A_{15} = \frac{\partial^2 \mathcal{L}}{\partial \alpha \partial \eta}$$

$$A_{22} = \frac{\partial^2 \mathcal{L}}{\partial \theta^2} \quad A_{23} = \frac{\partial^2 \mathcal{L}}{\partial \theta \partial \beta} \quad A_{24} = \frac{\partial^2 \mathcal{L}}{\partial \theta \partial a} \quad A_{25} = \frac{\partial^2 \mathcal{L}}{\partial \theta \partial b}$$

$$A_{33} = \frac{\partial^2 \mathcal{L}}{\partial \beta^2} \quad A_{34} = \frac{\partial^2 \mathcal{L}}{\partial \beta \partial a} \quad A_{35} = \frac{\partial^2 \mathcal{L}}{\partial \beta \partial b}$$

$$A_{44} = \frac{\partial^2 \mathcal{L}}{\partial a^2} \quad A_{45} = \frac{\partial^2 \mathcal{L}}{\partial a \partial b}$$

$$A_{55} = \frac{\partial^2 \mathcal{L}}{\partial b^2}$$

By solving this inverse dispersion matrix, these solutions will yield the asymptotic variances and covariances of these MLEs for θ, α, β, a and b.

Approximate $100(1-\alpha)\%$ confidence intervals for $\theta, \alpha, \beta, k,$ and η can be determined as

$$\hat{\alpha} \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{V}_{11}}, \quad \hat{\theta} \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{V}_{22}}, \quad \hat{\beta} \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{V}_{33}}, \quad \hat{a} \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{V}_{44}}, \quad \hat{b} \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{V}_{55}},$$

where $Z_{\frac{\alpha}{2}}$ is the upper α^{th} percentile of the standard normal distribution.

7. APPLICATIONS

In this section, we use real data sets to show that the BQHR distribution can be a better model than one based on the QHR distribution. Data set: This data set gives the time to failure ($10^3 h$) of turbocharger of one type of engine given in Xu et al.[26].

The data set is: 1.6, 2.0, 2.6, 3.0, 3.5, 3.9, 4.5, 4.6, 4.8, 5.0, 5.1, 5.3, 5.4, 5.6, 5.8, 6.0, 6.0, 6.1, 6.3, 6.5, 6.5, 6.7, 7.0, 7.1, 7.3, 7.3, 7.3, 7.7, 7.7, 7.8, 7.9, 8.0, 8.1, 8.3, 8.4, 8.4, 8.5, 8.7, 8.8, 9.0.

The variance covariance matrix $I(\hat{\phi})^{-1}$ of the MLEs under the KEMW distribution is computed as

$$\begin{pmatrix} 3.429e{-}03 & -8.939e{-}04 & 1.008e{-}05 & 0.409 & 0.390 \\ -8.939e{-}04 & 2.772e{-}04 & -4.148e{-}06 & -0.124 & -0.1433 \\ 1.008e{-}05 & -4.148e{-}06 & 1.302e{-}06 & 0.008 & 0.013 \\ 4.090e{-}01 & -1.242e{-}01 & 8.0521e{-}03 & 32.924 & 19.232 \\ 3.906e{-}01 & -1.433e{-}01 & 1.368e{-}02 & 19.232 & 2.385 \end{pmatrix}$$

The variances of the MLE of α, θ, β, a and b is $var(\hat{\alpha}) = 3.429 \cdot 10^{-3}$, $var(\hat{\theta}) = 2.772 \cdot 10^{-4}$, $var(\hat{\beta}) = 1.302 \cdot 10^{-6}$, $var(\hat{a}) = 32.924$ and $var(\hat{b}) = 2.385$. Therefore, 95% confidence intervals for α, θ, β, a and b are $[0.163, 0.274]$, $[-0.098, -0.066]$, $[0.008, 0.011]$, $[4.976, 15.879]$ and $[11.237, 14.1715]$.

Table 1
The ML Estimates, Log-likelihood, AIC and AICC for Data Set

Model	a	b	θ	α	β	**AIC**	**BIC**	**CAIC**
BQHRD	10.428	12.704	-0.082	0.218	0.009	168.413	176.857	170.177
QHRD	1	1	-0.024	0.006	0.009	173.143	178.21	173.810
BGE	0.171	91.54	0.224	32.27	1	170.307	177.062	171.449
BW	0.907	0.380	.177	1	4.282	172.258	179.014	173.401
GW	1	1	0.481	11.93	1.010	186.859	191.926	187.526
BE	7.701	17.47	.059	1	1	180.929	185.995	181.595
GE	1	1	0.449	9.514	1	184.285	187.663	184.609

In order to compare the two distribution models, we consider criteria like $-\ell$, AIC (Akaike information criterion), AICC (corrected Akaike information criterion) and BIC (Bayesian information criterion) for the data set. The better distribution corresponds to smaller -2ℓ, AIC,AICC and BIC values:‘

$$\text{AIC} = 2k - 2\mathcal{L}, \text{AICC} = \text{AIC} + \frac{2k(k+1)}{n-k-1},$$

$$\text{BIC} = 2\mathcal{L} + k \times \log n,$$

where k is the number of parameters in the statistical model, n the sample size and ℓ is the maximized value of the log-likelihood function under the considered model.

Table 1 shows parameter MLEs to each one of the three fitted distributions for data set and the values of $-2\log(L)$, AIC and AICC values. The values in Table 1, indicate that the BQHRD is a strong competitor to other distribution used here for fitting data set.

A density plot compares the fitted densities of the models with the empirical histogram of the observed data (Fig. 2). The fitted density for the BQHRD model is closer to the empirical histogram than the fits of the other models used here for fitting data set.

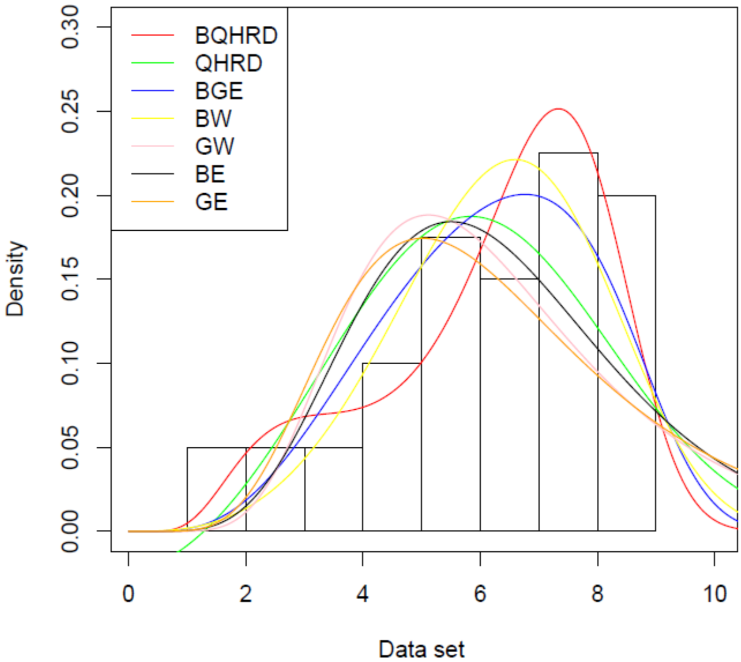

Figure 2: Estimated Densities of the Data Set

8. CONCLUSION

Here, we propose a new model, the so-called the BQHR distribution which extends the QHR distribution in the analysis of data with real support. An obvious reason for generalizing a standard distribution is because the generalized form provides larger flexibility in modelling real data. We derive expansions for the moments and for the moment generating function. The estimation of parameters is approached by the method of maximum likelihood, also the information matrix is derived. We consider the likelihood ratio statistic to compare the model with its baseline model. An application of the BQHR distribution to real data show that the new distribution can be used quite effectively to provide better fits than the QHR distribution.

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APPENDIX

$$V_{\alpha\alpha} = \frac{\partial^2 L}{\partial \alpha^2} = -\sum_{i=1}^{n}\left(\alpha+\theta x_i+\beta x_i^2\right)^{-2}$$

$$-(a-1)\sum_{i=1}^{n}\frac{x_i^2 e^{-\alpha x_i-1/2\theta x_i^2-1/3\beta x_i^3}}{\left(-1+e^{-\alpha x_i-1/2\theta x_i^2-1/3\beta x_i^3}\right)^2},$$

$$V_{\theta\theta} = \frac{\partial^2 L}{\partial \theta^2} = -\sum_{i=1}^{n}\frac{x_i^2}{\left(\alpha+\theta x_i+\beta x_i^2\right)^2}$$

$$+(a-1)\sum_{i=1}^{n}1/4\frac{x_i^4 e^{-\alpha x_i-1/2\theta x_i^2-1/3\beta x_i^3}\left(-1+2e^{-\alpha x_i-1/2\theta x_i^2-1/3\beta x_i^3}\right)}{\left(-1+e^{-\alpha x_i-1/2\theta x_i^2-1/3\beta x_i^3}\right)^2}$$

$$V_{\beta\beta} = \frac{\partial^2 L}{\partial \beta^2} = -\sum_{i=1}^{n}\frac{x_i^4}{\left(\alpha+\theta x_i+\beta x_i^2\right)^2}$$

$$-\frac{(a-1)}{9}\sum_{i=1}^{n}\frac{x_i^6 e^{-\alpha x_i-1/2\theta x_i^2-1/3\beta x_i^3}}{\left(-1+e^{-\alpha x_i-1/2\theta x_i^2-1/3\beta x_i^3}\right)^2}$$

$$V_{aa} = \frac{\partial^2 L}{\partial a^2} = -n\psi'\left(a\right)+n\psi'\left(a+b\right)$$

$$V_{bb} = \frac{\partial^2 L}{\partial b^2} = -n\psi'\left(b\right)+n\psi'\left(a+b\right)$$

$$V_{\alpha\theta} = \frac{\partial^2 L}{\partial \alpha \partial \theta} = -\sum_{i=1}^{n}\frac{x_i}{\left(\alpha+\theta x_i+\beta x_i^2\right)^2}-\frac{a-1}{2}\sum_{i=1}^{n}\frac{x_i^3 e^{-\alpha x_i-1/2\theta x_i^2-1/3\beta x_i^3}}{\left(-1+e^{-\alpha x_i-1/2\theta x_i^2-1/3\beta x_i^3}\right)^2}$$

$$V_{a\alpha} = \frac{\partial^2 L}{\partial \alpha \partial a} = \sum_{i=1}^{n}\frac{x_i e^{-\alpha x_i-1/2\theta x_i^2-1/3\beta x_i^3}}{1-e^{-\alpha x_i-1/2\theta x_i^2-1/3\beta x_i^3}}$$

$$V_{\beta b} = \frac{\partial^2 L}{\partial\beta\partial b} = -1/3\sum_{i=1}^{n} x_i^{\,3}$$

$$V_{ab} = \frac{\partial^2 L}{\partial a\partial b} = n\psi'\left(a+b\right)$$

$$V_{\beta\alpha} = \frac{\partial^2 L}{\partial\beta\partial\alpha} = -\sum_{i=1}^{n}\frac{x_i^{\,2}}{\left(\alpha+\theta x_i+\beta x_i^{\,2}\right)^2} - \frac{a-1}{3}\sum_{i=1}^{n}\frac{x_i^{\,4}\mathrm{e}^{-\alpha x_i-1/2\theta x_i^{\,2}-1/3\beta x_i^{\,3}}}{\left(-1+\mathrm{e}^{-\alpha x_i-1/2\theta x_i^{\,2}-1/3\beta x_i^{\,3}}\right)^2}$$

$$V_{a,\theta} = -1/2\sum_{i=1}^{n}\frac{x_i^{\,2}\mathrm{e}^{-1/2x_i\left(\beta x_i^{\,2}+\theta x_i+2\alpha\right)}}{-1+\mathrm{e}^{-1/2x_i\left(\beta x_i^{\,2}+\theta x_i+2\alpha\right)}}$$

$$V_{\beta a)} = \sum_{i=1}^{n}1/2\frac{x_i^{\,3}\mathrm{e}^{-\alpha x_i-1/2\theta x_i^{\,2}-1/2\beta x_i^{\,3}}}{1-\mathrm{e}^{-\alpha x_i-1/2\theta x_i^{\,2}-1/2\beta x_i^{\,3}}}$$

$$V_{\theta b} = -1/2\sum_{i=1}^{n} x_i^{\,2}$$

$$V_{\theta\beta} = -\sum_{i=1}^{n}\frac{x_i^{\,3}}{\left(\beta x_i^{\,2}+\theta x_i+\alpha\right)^2} - 1/4(a+1)\sum_{i=1}^{n}\frac{x_i^{\,5}\mathrm{e}^{-1/2x_i\left(\beta x_i^{\,2}+\theta x_i+2\alpha\right)}}{\left(-1+\mathrm{e}^{-1/2x_i\left(\beta x_i^{\,2}+\theta x_i+2\alpha\right)}\right)^2}$$

$$V_{\alpha\beta} = \sum_{i=1}^{n} -\frac{x_i^{\,2}}{\left(\beta x_i^{\,2}+\theta x_i+\alpha\right)^2} - \frac{a+1}{2}\sum_{i=1}^{n}\frac{x_i^{\,4}\mathrm{e}^{-1/2x_i\left(\beta x_i^{\,2}+\theta x_i+2\alpha\right)}}{\left(-1+\mathrm{e}^{-1/2x_i\left(\beta x_i^{\,2}+\theta x_i+2\alpha\right)}\right)^2}.$$