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SIMPLE CIRCULAR REGRESSION MODEL ASSUMING WRAPPED CAUCHY ERROR

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ABSTRACT

Studying the association between random variables is an important subject in statistical analysis. In the case of circular variables, where their ranges are bounded, special type of regression models is needed.

This paper proposes a new simple linear model for circular variables with a wrapped Cauchy error due to its desirable characteristics of having a heavy tailed. The maximum likelihood estimates for model parameters are obtained via an iterative procedure and the standard error of the estimates as well as their confidence intervals are obtained by bootstrapping methods. The properties of estimates are investigated via a simulation study, where the unbiasness, consistency and robustness are verified.

For illustration purposes, a real data set contains the average monthly wind directions in two main cities in the Gaza Strip-Palestine have been analyzed and modeled by the proposed method.

KEYWORDS

Bootstrapping, robust, von Mises, wind directions.

2010 Mathematics Subject Classification: 62H11, 62G08.

1. INTRODUCTION

The study on circular regression started four decades ago, where several circular regression models were proposed. None of the circular regression models performed superiorly comparing to others, where every model has its strict assumptions and relevant applications. See (Gould, 1969; Downs and Mardia, 2002 and Kato et al. 2008).

In cases when two circular variables X and Y are linearly correlated, Hussin, et al. (2004) proposed the simple linear regression model for circular variables in the following form

$$y_i = \alpha + \beta x_i + \varepsilon_i \pmod{2\pi}, \quad i = 1, \ldots, n. \tag{1}$$

where ε_i is the circular random error having a von Mises distribution with circular mean 0 and concentration parameter κ. One of the significant applications of model (1) is the calibration of an alternative instrument with the standard one.

Circular regression models as any other types of statistical models are affected with the existence of outliers. Several procedures where developed to detect possible outliers in model (1), see (Abuzaid, et al., 2008 and Abuzaid, et al., 2013).

The analysis of residuals term is an essential procedure for the evaluation of the goodness of any regression model via checking its underlying assumptions. Based on the importance of this role and to enhance the modeling of circular data, this paper proposes an alternative choice by assuming the circular error to follow the wrapped Cauchy distribution, $WC(\mu,\rho)$ with probability density function:

$$f\left(\theta,\mu,\rho\right)=\frac{1}{2\pi}\frac{1-\rho^2}{1+\rho^2-2\rho\cos\left(\theta-\mu\right)},\quad 0\le\theta,\mu<2\pi,\quad 0\le\rho<1, \tag{2}$$

where μ is circular mean and ρ is the concentration parameter. The *WC* distribution is a unimodal and symmetric about μ. Furthermore, (Kolassa and McCullagh, 1990) illustrated that the *WC* distribution enjoys the additive property and the central limit theorem, on other words, the convolution of the wrapped Cauchy distributions $WC\left(\mu_1,\rho_1\right)$ and $WC\left(\mu_2,\rho_2\right)$ is the wrapped Cauchy distribution $WC\left(\mu_1+\mu_2,\rho_1\rho_2\right)$.

One of the main characteristics of the *WC* distribution that it has a heavy tailed even for large concentrations, such characteristic can be employed to adapt some extreme values properly, and it is expected to make the circular regression model more robust.

The rest of the paper is organized as follows, Section 2 presents the formulation and the maximum likelihood estimates of parameters. Section 3 investigates the properties of estimators via simulation study. An illustrative example is discussed in Section 4.

2. REGRESSION MODEL WITH WRAPPED CAUCHY ERROR

Consider n pairs of circular observations $\left(x_1,y_1\right),...,\left(x_n,y_n\right)$ of two circular variables X and Y with a linear relationship. Then the simple circular regression model is given by:

$$y_i=\alpha+\beta x_i+\varepsilon_i\left(\bmod 2\pi\right),\quad i=1,...,n, \tag{3}$$

where α is the intercept parameter, β is the slope parameter and ε is a circular random error having a wrapped Cauchy distribution with mean 0 and concentration parameter, ρ. The proposed model is a rotation-invariant model. The following subsections obtain the maximum likelihood estimation, asymptotic variances and the confidence intervals.

2.1 Parameter Estimation

Based on the wrapped Cauchy probability density function in (2), the density of the error in model (3) is given by:

$$f\left(x,y;\alpha,\beta,\rho\right)=\frac{1}{2\pi}\frac{1-\rho^{2}}{1+\rho^{2}-2\rho\cos\left(y_{i}-\alpha-\beta x_{i}\right)} \tag{4}$$

The probability density function in (4) can be written in the following form

$$f\left(x,y;\alpha_1,\alpha_2,\beta\right)=\frac{1}{2\pi}\ \frac{c}{\left(1-\alpha_1\cos\left(y_i-\beta x_i\right)-\alpha_2\sin\left(y_i-\beta x_i\right)\right)}$$

where

$$c=\frac{1-\rho^2}{1+\rho^2}=\sqrt{1-\alpha_1^2-\alpha_2^2},\ \ \alpha_1=\frac{2\rho}{1+\rho^2}\cos\alpha \ \text{ and } \ \alpha_2=\frac{2\rho}{1+\rho^2}\sin\alpha \tag{5}$$

Then the log-likelihood function for model (3) is given by:

$$\log L=-n\ln 2\pi+\frac{n}{2}\ln\left(1-\alpha_1^2-\alpha_2^2\right) -\sum_i\ln\left(1-\alpha_1\cos\left(y_i-\beta x_i\right)-\alpha_2\sin\left(y_i-\beta x_i\right)\right) \tag{6}$$

The maximum likelihood estimates of α_1 and α_2 are obtained by differentiating the log-likelihood function in (6) with respect to α_1 and α_2, respectively and equating to zero as follows

$$\frac{\partial\log L}{\partial\alpha_1}=-\frac{n\ \alpha_1}{\left(1-\alpha_1^2-\alpha_2^2\right)}+\sum_i\frac{\cos\left(y_i-\beta x_i\right)}{\left(1-\alpha_1\cos\left(y_i-\beta x_i\right)-\alpha_2\sin\left(y_i-\beta x_i\right)\right)}=0 \tag{7}$$

and

$$\frac{\partial\log L}{\partial\alpha_2}=-\frac{n\ \alpha_2}{\left(1-\alpha_1^2-\alpha_2^2\right)}+\sum_i\frac{\sin\left(y_i-\beta x_i\right)}{\left(1-\alpha_1\cos\left(y_i-\beta x_i\right)-\alpha_2\sin\left(y_i-\beta x_i\right)\right)}=0 \tag{8}$$

From (7) and (8) we have for β known,

$$\hat{\alpha}_1=\frac{c^2}{n}\sum_i w_i\cos\left(y_i-\beta x_i\right), \tag{9}$$

and

$$\hat{\alpha}_2=\frac{c^2}{n}\sum_i w_i\sin\left(y_i-\beta x_i\right). \tag{10}$$

where $w_i=\left(1-\alpha_1\cos\left(y_i-\beta x_i\right)-\alpha_2\sin\left(y_i-\beta x_i\right)\right)^{-1}$

The maximum likelihood estimate of the slope parameter β is also obtained after equating the partial derivative of the $Log\ L$ with respect to β by zero,

$$\frac{\partial L\left(\alpha_1,\alpha_2,\beta;x,y\right)}{\partial\beta}=\sum_i w_i\left[\alpha_1 x_i\sin\left(y_i-\beta x_i\right)-\alpha_2 x_i\cos\left(y_i-\beta x_i\right)\right]=0 \tag{11}$$

Equation (11) can be solved iteratively by giving some initial starting values. For an initial value $\beta^{[0]}$ we have

$$(y_i - \beta x_i) = \left(y_i - \beta^{[0]} x_i\right) + x_i\left(\beta^{[0]} - \beta\right) = \left(y_i - \beta^{[0]} x_i\right) + \Delta x_i,$$

where $\Delta = \left(\beta^{[0]} - \beta\right)$. Since

$$\sin(y_i - \beta x_i) = \sin\left(y_i - \beta^{[0]} x_i\right)\cos \Delta x_i + \cos\left(y_i - \beta^{[0]} x_i\right)\sin \Delta x_i, \tag{12}$$

and

$$\cos(y_i - \beta x_i) = \cos\left(y_i - \beta^{[0]} x_i\right)\cos \Delta x_i - \sin\left(y_i - \beta^{[0]} x_i\right)\sin \Delta x_i, \tag{13}$$

For small Δ: $\cos \Delta x_i \approx 1$ and $\sin \Delta x_i \approx \Delta x_i$.

So, (12) and (13) becomes

$$\sin(y_i - \beta x_i) = \sin\left(y_i - \beta^{[0]} x_i\right) + \cos\left(y_i - \beta^{[0]} x_i\right)\Delta x_i, \tag{14}$$

and

$$\cos(y_i - \beta x_i) = \cos\left(y_i - \beta^{[0]} x_i\right) - \sin\left(y_i - \beta^{[0]} x_i\right)\Delta x_i, \tag{15}$$

respectively, by substituting (14) and (15) in (11) , then, an improved estimate of $\hat{\beta}^{[1]}$ becomes

$$\hat{\beta}^{[1]} \cong \hat{\beta}^{[0]} + \frac{\sum_i x_i \left(\alpha_1 \sin\left(y_i - \beta^{[0]} x_i\right) - \alpha_2 \cos\left(y_i - \beta^{[0]} x_i\right)\right)}{\sum_i x_i^2 \left(\alpha_1 \cos\left(y_i - \beta^{[0]} x_i\right) + \alpha_2 \sin\left(y_i - \beta^{[0]} x_i\right)\right)}. \tag{16}$$

The iterative re-weighting algorithm for the maximum likelihood estimation obtained step by step as follows:

***Step 1*:**

Initialize $\alpha_1^{[0]}$, $\alpha_2^{[0]}$ and $\beta^{[0]}$ with $\alpha_1^{[0]} + \alpha_2^{[0]} < 1$

***Step 2*:**

Given $\alpha_1^{[k-1]}$, $\alpha_2^{[k-1]}$, and $\beta^{[k-1]}$ at iteration k, calculate $\alpha_1^{[k]}$, $\alpha_2^{[k]}$ and $\beta^{[k]}$ using equations (9), (10) and (16), respectively.

***Step 3*:**

Repeat *Step 2* until the algorithm converges.

***Step 4*:**

The value of $\hat{\alpha}$ is obtained by solving two equations in (5), and it is given by:

$$\hat{\alpha} = \begin{cases} \tan^{-1}\left(\dfrac{\hat{\alpha}_2}{\hat{\alpha}_1}\right), & \text{if} \quad \hat{\alpha}_1 > 0, \hat{\alpha}_2 > 0, \\ \tan^{-1}\left(\dfrac{\hat{\alpha}_2}{\hat{\alpha}_1}\right) + \pi, & \text{if} \quad \hat{\alpha}_1 < 0, \\ \tan^{-1}\left(\dfrac{\hat{\alpha}_2}{\hat{\alpha}_1}\right) + 2\pi, & \text{if} \quad \hat{\alpha}_1 > 0, \hat{\alpha}_2 < 0, \\ \text{undefined}, & \text{if} \quad \hat{\alpha}_1 = 0, \hat{\alpha}_2 = 0. \end{cases}$$

Furthermore, the estimate of the concentration parameter ρ is given by

$$\hat{\rho} = \frac{1 - \sqrt{1 - \hat{\alpha}_1^2 - \hat{\alpha}_2^2}}{\sqrt{\hat{\alpha}_1^2 + \hat{\alpha}_2^2}} .$$

Kent and Tayler (1988) proved the existence and uniqueness of the maximum likelihood estimates and the convergence of the MLE of the wrapped Cauchy algorithm by noting that the wrapped Cauchy distributions are related to the angular central Gaussian distributions on the circle. The authors have written R-subroutine to obtain the maximum likelihood estimates and available upon request.

2.2 Asymptotic Variances of Parameters' Estimates

Due to the difficulty of getting the sampling variance of the parameters, estimates based on the expectation of the second partial derivative of the log likelihood function of the model, bootstrapping methods can be used to obtain the approximated variances (Efron, 1979). The steps of bootstrapping method for obtaining the variances of parameters' estimates can be summarized as follows:

Consider again n pairs of circular observations $(x_1, y_1), ..., (x_n, y_n)$ of two circular variables X and Y with linear relationship. We following steps

***Step 1:* (*Resampling*)**

Randomly select m pairs from the sample, where ($m < n$).

***Step 2:* (*Bootstrap Parameters Estimates*)**

The estimates of α, β and ρ are obtained as described in Subsection 2.1 for the pairs selected in *Step* 1 and labeled as $\hat{\alpha}_{(1)}, \hat{\beta}_{(1)}$ and $\hat{\rho}_{(1)}$, respectively.

***Step 3:* (*Repetition*)**

Repeat *Step* 1 and *Step* 2, *B* times i.e. $\hat{\alpha}_{(1)}, ..., \hat{\alpha}_{(B)}$, $\hat{\beta}_{(1)}, ..., \hat{\beta}_{(B)}$ and $\hat{\rho}_{(1)}, ..., \hat{\rho}_{(B)}$.

***Step 4:* (*Variances of parameters' Estimates*)**

Obtain the variance for each parameters' estimates using the formula $\mathrm{var}\left(\hat{\theta}\right)=\frac{1}{B-1}\sum_{j=1}^{B}\left(\hat{\theta}_j-\overline{\theta}\right)^2$, where $\hat{\theta}_j$ is the estimated parameter (i.e. α,β and ρ) at bootstrap j, and $\overline{\theta}$ is the sample mean of the $\hat{\theta}_j$ for $j=1,\ldots,B$.

2.3 Confidence Intervals

We use the "percentile" or bootstrap-*p* method for constructing the 100(1−*a*)% confidence interval, which is the most widely used one and the simplest bootstrapping method in approximating the confidence intervals. The 100(1−*a*)% bootstrap-*p* confidence intervals for parameters estimates are obtained by following the first three steps mentioned in Subsection 2.1, and then ended by the following step:

***Step 4:* (*Confidence Interval*)**

Arrange the bootstrap estimates in an increasing order, $\hat{\alpha}_{[1]}\le\ldots\le\hat{\alpha}_{[B]}$, $\hat{\beta}_{[1]}\le\ldots\le\hat{\beta}_{[B]}$ and $\hat{\rho}_{[1]}\le\ldots\le\hat{\rho}_{[B]}$. Then the $(1-a)\%$ confidence intervals of α,β and ρ are given by $\left(\hat{\alpha}_{\left(\frac{a}{2}\right)},\hat{\alpha}_{\left(1-\frac{a}{2}\right)}\right)$, $\left(\hat{\beta}_{\left(\frac{a}{2}\right)},\hat{\beta}_{\left(1-\frac{a}{2}\right)}\right)$ and $\left(\hat{\rho}_{\left(\frac{a}{2}\right)},\hat{\rho}_{\left(1-\frac{a}{2}\right)}\right)$, respectively.

The following section investigates the properties of the parameters' estimates numerically via a Monte Carlo simulation study.

3. "SIMULATION"

This section aims to assess the characteristics and quality of the parameters' estimates via simulation study. Without loss of generality, the coefficients of model (3) are fixed at $\alpha=0$ and $\beta=1$, and by considering four different sample sizes, namely $n=30,50,100$ and 150, with various values of concentration parameter, viz. $\rho=0.4$, 0.6, 0.8, 0.85, 0.90, 0.95 and 0.99. The simulation study is conducted as follows:

1) A random sample of size *n* is generated from the wrapped Cauchy with mean 0 and concentration ρ for the error.
2) A random sample of size *n* is generated from the wrapped Cauchy with mean $\pi/3$ and concentration parameter $\rho=0.4$ for the independent variable *X*.
3) The dependent variable *Y* is obtained based on model (3).
4) Estimate the models' parameters using the iterative procedure described in Subsection 2.2.
5) For each combination of sample size *n* and concentration parameter ρ, the process is repeated 1000 times to ensure the convergence.

3.1 Estimators Properties

Three of the accuracy measures are calculated and given in Table 1, which are the circular mean of $\hat{\alpha}$ (Jammalamadaka and SenGupta, 2001) and the arithmetic mean of $\hat{\beta}$ and $\hat{\rho}$ as well as the bias which is the absolute difference between the estimates and true value of the parameter and the Mean Circular Error, $MCE(\alpha)=\frac{1}{1000}\sum_{i=1}^{1000}\left[1-\cos\left(\hat{\alpha}_i-\alpha\right)\right]$ (see Abuzaid, et al., 2013) as well as the mean square error $MSE(\theta)=\frac{1}{1000}\sum_{i=1}^{1000}\left(\hat{\theta}_i-\theta\right)^2$ where $\hat{\theta}$ and θ are the estimated and the true values of the considered parameter, respectively (i. e. β and ρ).

Results in Table 1 reveal the accuracy of the parameters' estimates where the means of estimates are very close to the true values and the bias of $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\rho}$ are less than 0.15, 0.10 and 0.25 respectively, which indicates the unbiasness of the estimators.

The values of *MSE* in all considered cases are also small and less than 0.24. For certain values of the model parameters, there are an inverse relationship between the sample size *n* and the values of the *MSE*, this findings reveal the consistency of the estimators. Furthermore, the results show that the estimates of the parameters approaches their optimum as the concentration parameter, ρ gets closer to one.

3.2 Coverage Probability of Confidence Intervals

Coverage probability is the proportion of number that the confidence interval contains the true value. The simulation study which has been repeated 1000 times in our study have been conducted at 95% of confidence level. Hence, the good indicator must give the coverage probability close to 0.95 which we refer to as target coverage probability. Table 2 presents the coverage probabilities.

From the results obtained and displayed in Table 2, it can be seen that the results are very close to the target value which reveals an excellent coverage of the confidence intervals.

3.3 Robustness of the Parameters Estimators

Robustness is one of the desired properties of any estimator. An estimator is said to be robust if its sampling distribution is not seriously affected by violations of assumptions. Such violations are often due to inconsistent observations called outliers see (Miller, et al. 2003). To investigate the robustness, similar simulation study to that described at the start of this section is used with some modifications, where 5% of the generated data are contaminated start from observation d as follows

$$y^*[d]=y[d]+\lambda\pi \pmod{2\pi},$$

where $y^*[d]$ and $y[d]$ are the contaminated and original values of the dependent variable *Y*, respectively, and λ is the contamination level, where $0\le\lambda\le1$.

Figure 1 presents the *MCE* values of $\hat{\alpha}$ for $\rho = 0.6$ and almost similar behavior observed of other two parameters. The complete results of simulation for $n = 30, 50, 100$ and 150 and $\rho = 0.2, 0.4, 0.6, 0.8, 0.85, 0.90, 0.95$ and 0.99 are obtained and not presented here. In general, for data with highly concentrated error $(\rho > 0.6)$, the bias of $\hat{\alpha}$ is less than 0.2 for small samples $(n \leq 10)$ and less than 0.1 for moderate and large samples $(n \geq 50)$. Regardless the concentration of error or the sample size, the optimum values of bias or *MCE* of $\hat{\alpha}$ are obtained for moderate levels of contamination $\lambda \approx 0.5$. This may be referred to the nature of the circular data, where the shift of 5% of the generated data by more than $\pi/2$ of its original values may lead to get closer to the majority of the data. Almost similar conclusion can be drawn for both of the slop parameter β and concentration parameter ρ.

Without loss of generality, the increase of the contamination level does not affect the estimates seriously, in other words, the estimator of the proposed model parameters are robust.

4. REAL DATA ANALYSIS

This section introduces and analyzes a real data set to illustrate the use of the proposed model to fit of wind direction data in the Gaza Strip-Palestine. The importance of modeling such data arises from the hard circumstances in the Gaza Strip because of the tight siege imposed on Gaza since 2007. There is a high possibility of the damage or disruptions of one or more meteorological stations or some of their instruments. Thus, the ability of predicting metrological measurements in certain location by the means of other locations is highly needed.

4.1 Data Source and Description

A real data set has been randomly selected from a series of calculations conducted to measure the monthly average wind directions in two main cities in the Gaza Strip-Palestine, namely Gaza and Khan Younis. The data were provided by the Palestinian Metrological Authority (2007). A total of 96 measurements for each city are representing the monthly average wind directions every three hours a day viz., mid night, 3:00 am, 6:00 am, 9:00 am, 12:00 noon, 15:00 pm, 18:00 pm and 21:00 pm. Badawi (2013) analyzed the Palestinian wind data in order to establish a wind farm to reach the optimum electricity energy. Figure 2 displays the spoke plot (Zubairi, et al., 2008) where the inner and outer rings represent the measurements of wind direction at Khan Younis and Gaza, respectively. Since almost all the lines do not cross the inner circle, it means that the data are highly correlated. Furthermore, there are only three lines crossing the inner ring which are associated with observation numbers 28, 77 and 85. This indicates that the pairs corresponding to these three observations are inconsistent with the rest of observations. Figure 3 shows the ordinary scatter plot of the monthly average wind directions in Gaza and Khan Younis. There is a sort of linear relationship between the wind directions of Gaza and Khan Younis. Three points seem to be outliers, two of them are above the proposed straight line, while the other one is lower of the line. Since the relationship between the wind directions is reasonably linear; thus the simple angular regression model (3) is suggested to fit the data.

4.2 Modeling of Wind Direction Data

The parameters' estimates are obtained by applying the iterative procedure and after converting the data into radial, where we take 0.3 as initial values of α_1 and α_2. Convergence is occurred after 113 iterations. The parameters estimates, standard errors and the 95% confidence intervals are given in Table 3.

So, the linear relationship between dependent and independent variable is given by

$$y = 0.747 + 0.842\,x + \varepsilon \quad (\mathrm{mod}\, 2\pi),$$

where dependent variable "*Y*" is the wind direction data for Gaza while the wind direction data for Khan Younis is the independent variable "*X*".

The obtained residuals were tested to follow the wrapped Cauchy distribution via the Kolmogorov-Smirnov test, where the value of the test is 0.0401 with p-value = 0.835.

Figure 4 presents the circular error versus the observations numbers. It is obvious that the mean of errors is zero with a constant variance and that is one of the required assumptions of the regression model. There are three observations are suspected to be outliers, namely observation numbers 28, 77 and 85.

The simple circular regression model with von Mises error (Hussin, et al., 2004), which will be denoted by (H) is used to fit the wind direction data. The results of the maximum likelihood estimates are given in Table 4. The standard error of parameters' estimates for the three parameters are very large compared to those obtained by fitting the data by using the simple circular regression with wrapped Cauchy error. Furthermore, the width of their confidence intervals are about three times the one obtained for the proposed model. For further investigation, the three suspected outlier observations with numbers 28, 77 and 85 have been removed and both models have been refitted and their results are given in Table 4. The estimates of α based on the proposed model is close to the difference between the mean direction of wind in the two cities compare to (H) model.

The goodness-of-fit; $A^*\left(\hat{\kappa}\right) = \frac{1}{n}\sum_{i=1}^{n}\cos^2\left(y_i - \hat{y}_i\right)$, (Abuzaid, 2010) regardless the outliers for the proposed model is about two times the (H) model. The removal of suspected three outliers affected the estimation of the concentration parameter κ and decreased its standard error.

These notes indicate that our proposed model fit the data much better than the (H) model and more robust to existence of outliers.

CONCLUSIONS

A regression model when the dependent and independent variables are circular has been proposed with wrapped Cauchy error, and the estimates of parameters were obtained based on maximizing the log likelihood function via iterative procedure. The simulation results have shown that the parameters' estimates have desirable properties such as unbiasness, consistency and robustness.

The comparison of the fitted model of circular regression with von Mises error has shown that the proposed model is more robust and fits the data much better. This refers to the nature of heavily tailed of the wrapped Cauchy distribution which enables the model to adapt most of deviations from the proposed straight line. One of the possible extensions of this model is proposing the functional relationship model with wrapped Cauchy errors, especially with large concentration parameter $\rho > 0.8$.

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Table 1
Mean, Bias and Mean Squared/Circular Errors of Parameter Estimates

n		ρ													
		0.4		0.6		0.8		0.85		0.90		0.95		0.99	
30	$\hat{\alpha}$	-0.063 (0.063)	0.111	-0.024 (0.024)	0.022	-0.012 (0.012)	0.001	0.000 (0.000)	0.000	0.000 (0.000)	0.000	0.000 (0.000)	0.000	0.000 (0.000)	0.000
	$\hat{\beta}$	0.992 (0.008)	0.034	1.013 (0.013)	0.014	1.001 (0.001)	0.000	1.000 (0.000)	0.000	1.000 (0.000)	0.000	1.000 (0.000)	0.000	1.000 (0.000)	0.000
	$\hat{\rho}$	0.611 (0.211)	0.152	0.681 (0.081)	0.012	0.822 (0.022)	0.001	0.861 (0.011)	0.000	0.910 (0.010)	0.000	0.950 (0.000)	0.000	0.990 (0.000)	0.000
50	$\hat{\alpha}$	-0.094 (0.094)	0.094	0.062 (0.062)	0.013	-0.123 (0.123)	0.002	0.000 (0.000)	0.000	0.000 (0.000)	0.000	0.000 (0.000)	0.000	0.000 (0.000)	0.000
	$\hat{\beta}$	1.062 (0.062)	0.004	0.892 (0.108)	0.001	1.030 (0.030)	0.000	1.000 (0.000)	0.000	1.000 (0.000)	0.000	1.000 (0.000)	0.000	1.000 (0.000)	0.000
	$\hat{\rho}$	0.591 (0.191)	0.012	(0.653) (0.053)	0.001	0.86 (0.06)	0.00	0.861 (0.011)	0.000	0.900 (0.000)	0.000	0.950 (0.000)	0.000	0.990 (0.000)	0.000
100	$\hat{\alpha}$	0.110 (0.110)	0.010	-0.194 (0.194)	0.000	0.084 (0.084)	0.001	0.000 (0.000)	0.000	0.000 (0.000)	0.000	0.000 (0.000)	0.000	0.000 (0.000)	0.000
	$\hat{\beta}$	0.951 (0.049)	0.000	1.010 (0.010)	0.000	0.990 (0.010)	0.000	1.000 (0.000)	0.000	1.000 (0.000)	0.000	1.000 (0.000)	0.000	1.000 (0.000)	0.000
	$\hat{\rho}$	0.501 (0.101)	0.000	0.610 (0.010)	0.000	0.812 (0.012)	0.000	0.850 (0.000)	0.000	0.900 (0.000)	0.000	0.950 (0.000)	0.000	0.990 (0.000)	0.000
150	$\hat{\alpha}$	0.080 (0.080)	0.000	0.062 (0.062)	0.000	-0.044 (0.044)	0.000	0.000 (0.000)	0.000	0.000 (0.000)	0.000	0.000 (0.000)	0.000	0.000 (0.000)	0.000
	$\hat{\beta}$	0.971 (0.029)	0.000	0.981 (0.019)	0.000	1.023 (0.023)	0.000	1.000 (0.000)	0.000	1.000 (0.000)	0.000	1.000 (0.000)	0.000	1.000 (0.000)	0.000
	$\hat{\rho}$	0.462 (0.062)	0.000	0.613 (0.013)	0.000	0.794 (0.006)	0.000	0.850 (0.000)	0.000	0.900 (0.000)	0.000	0.950 (0.000)	0.000	0.990 (0.000)	0.000

Table 2
Results for Coverage Percentage of Confidence Interval

ρ	*n*														
	30			50			70			100			150		
	α	β	ρ	α	β	ρ	α	β	ρ	α	β	ρ	α	β	ρ
0.2	95.8	96.3	97.2	96.5	96.2	94.8	94.5	95.3	94.9	94.2	94.5	94.2	96.4	94.0	96.1
0.4	95.0	93.8	95.6	94.9	96.5	94.6	96.2	94.1	84.9	97.3	96.1	94.9	95.0	94.7	95.2
0.6	96.1	96.7	94.2	95.5	94.4	94.9	96.4	97.0	95.9	95.7	92.4	95.1	94.8	95.4	97.1
0.8	94.5	95.2	95.1	93.9	96.9	96.8	94.6	95.7	95.1	96.7	96.2	94.2	94.3	97.1	96.7
0.999	95.8	96.0	99.7	94.7	95.4	97.2	93.3	92.0	99.8	94.8	97.4	97.2	96.1	97.8	96.2

Table 3
Results of Parameters' Estimation for Wind Direction Data

Parameter	Estimate	Standard Error	Confidence Interval
$\hat{\alpha}$	0.747	0.0024	(0.643,0.845)
$\hat{\beta}$	0.842	0.0004	(0.798,0.886)
$\hat{\rho}$	0.823	0.0011	(0.766,0.879)

Table 4
Comparison between Two Models for Full and Reduced Data

Index	Parameters	Model with Wrapped Cauchy Error		Model with Von Mises Error (H)	
		Full data	reduced	Full data	reduced
Estimation	$\hat{\alpha}$	0.7471	0.7566	0.9830	1.0050
	$\hat{\beta}$	0.8425	0.8412	0.7750	0.7610
	concentration	$\hat{\rho}$ =0.8232	$\hat{\rho}$ =0.8311	$\hat{\kappa}$ =4.6200	$\hat{\kappa}$ =6.4700
Standard Error	$\hat{\alpha}$	0.0024	0.0021	0.1648	0.1396
	$\hat{\beta}$	0.0004	0.0003	0.0455	0.0387
	concentration	$\hat{\rho}$ =0.0011	$\hat{\rho}$ =0.0006	$\hat{\kappa}$ =0.6290	$\hat{\kappa}$ =0.9110
Confidence Length	$\hat{\alpha}$	0.2020	0.2192	0.6450	0.5460
	$\hat{\beta}$	0.0880	0.0902	0.1780	0.2172
	concentration	$\hat{\rho}$ =0.1130	$\hat{\rho}$ =0.0913	$\hat{\kappa}$ =2.4680	$\hat{\kappa}$ =3.5730
Goodness-of-Fit	$A^*(\hat{\kappa})$	0.8371	0.8551	0.4840	0.4870

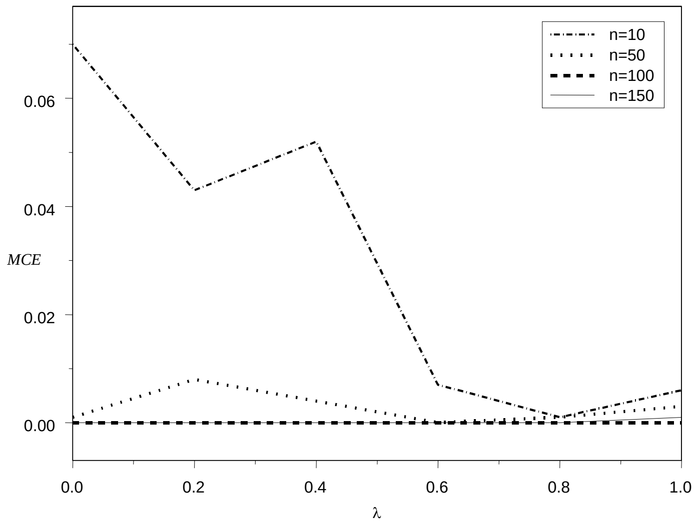

Fig. 1: *MCE* of Intercept Parameter α, at $\rho = 0.6$.

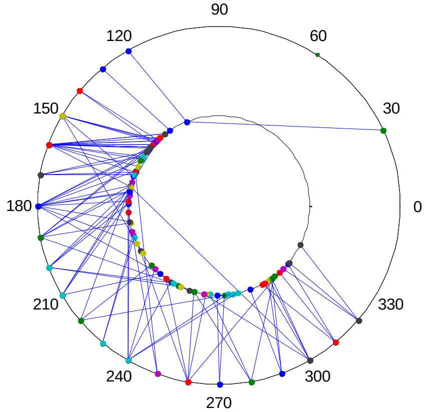

Fig. 2: Spoke Plot of Wind Directions in Gaza and Khan Younis

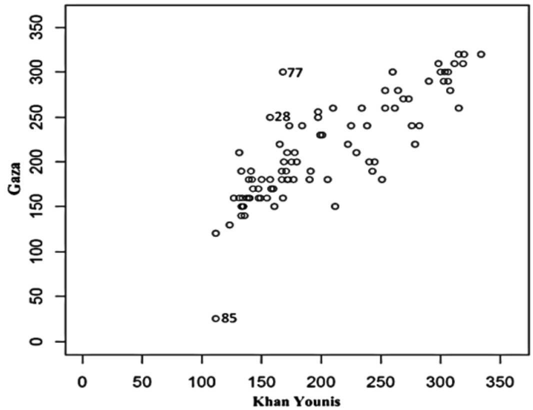

Fig. 3: A Scatter Plot of Wind Directions in Gaza versus Khan Younis

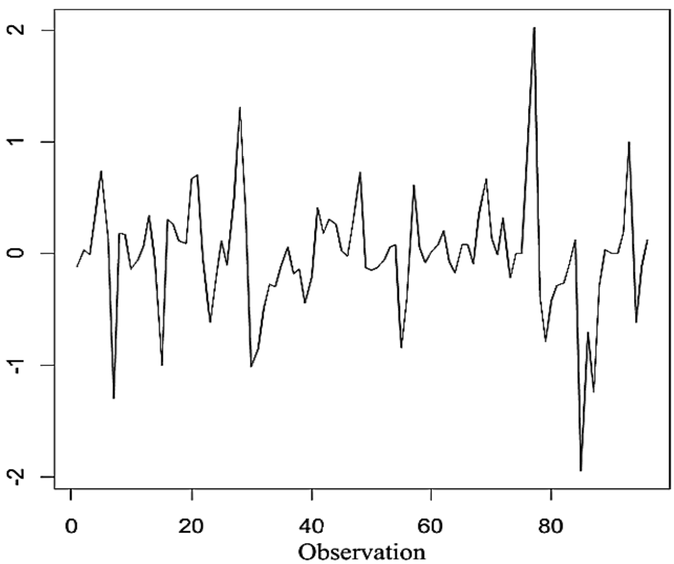

Fig. 4: Error of Wind Direction Data for Gaza and Khan Younis