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THE FIVE PARAMETER LINDLEY DISTRIBUTION

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ABSTRACT

In this paper, a five parameter Lindley Distribution (FPLD) is proposed as a new generalization of the basic Lindley distribution. The structural properties of the new distribution are investigated. These include the compounding representation of the distribution, reliability analysis and statistical measures. Expressions for Lorenz and Bonferroni curves and Renyi entropy as a measure for uncertainty reduction are derived. Maximum likelihood estimation is used to evaluate the parameters. The new model contains twelve lifetime distributions as special cases such as the Lindley, Quasi Lindley, gamma, and exponential distributions, among others. This model has the advantage of being capable of modeling various shapes of aging and failure criteria. Finally, the usefulness of the new model for modeling reliability data is illustrated using a real data set.

KEYWORDS

Lindley distribution; mixture; reliability analysis; moment generating function; order statistics; maximum likelihood estimation.

1. INTRODUCTION

In many applied sciences such as medicine, engineering and finance, amongst others, modeling and analyzing lifetime data are crucial. Several lifetime distributions have been used to model such kinds of data. The quality of the procedures used in a statistical analysis depends heavily on the assumed probability model or distribution along with relevant statistical methodologies. However, there still remain many important problems where the real data does not follow any of the classical or standard probability models.

The one parameter family of distributions with density function

$$f(x;\theta)=\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x};x>0,\theta>0, \tag{1}$$

is used by Lindley (1958) to illustrate the difference between fiducial distribution and posterior distribution. Sankaran (1970) introduced the discrete Poisson-Lindley distribution by combining the Poisson and Lindley distributions. Ghitany et al. (2008, a) have discussed various properties of (1). Another discrete version of this distribution has been suggested by Deniz and Ojeda (2011) with applications in count data related to insurance. Ghitany et al. (2008 b,c) obtained size-biased and zero-truncated version of Poisson-Lindley distribution with various properties and applications.

The aim of this paper is to introduce a new generalization of Lindley (1958) distribution. This generalization is flexible enough to model different types of lifetime data having different forms of failure rate. The new distribution can accommodate both decreasing and increasing failure rates as its antecessors, as well as unimodal and bathtub shaped failure rates. The Lindley distribution is generalized by mixing. Several authors have considered versions from usual density functions by following this idea. For instance, Zakerzadeh and Dolati (2009) considered a generalized Lindley distribution as a generalization of the usual Lindley distribution, Rama and Mishra (2013) studied quasi Lindley distribution, Rama et al. (2013) introduced a new distribution for generalizing the Lindley model which called Janardan distribution, and Elbatal et al. (2013) have suggested a new generalized Lindley distribution.

The introduced model will be named Five Parameter Lindley Distribution (FPLD). The basic idea behind this generalization is to use a unified approach that accommodates all the proceeding generalizations of Lindley distribution abovementioned, and to add new model that could offer a better fit to life time data. The proposed distribution includes nine models as special cases plus three new models (all special cases are shown in section 2). The procedure used here is based on certain mixtures of two gamma distributions with various weights. The research examines various properties of the new distribution. The rest of paper is organized as follows: Section 2 introduces the definition of the probability density function (pdf) of the FPLD including its cumulative distribution function (cdf) and the sub-models of the new suggested model. The reliability analysis including the survival function, the hazard (or failure) rate function, the reversed hazard rate function, the cumulative hazard rate function, and the mean residual lifetime is explored in Section 3. The statistical properties of the new distribution such as the moments, the moment generating function, and the distribution of order statistics are investigated in Section 4, with a proposed algorithm for generating random data from the new distribution in this section. Section 5 introduces Lorenz and Bonferroni curves and Renyi entropy as measures of inequality and uncertainty, respectively. Section 6 discusses the estimation of parameters by using maximum likelihood estimation. Finally, Section 7 provides an application for modeling real data sets to illustrate the performance of the new distribution.

2. GENERALIZATION AND RELATED SUB-MODELS

In this section, we introduce the pdf and the cdf of the five parameter Lindley distribution and then the special cases of the FPLD are mentioned.

2.1 Generalization

Let

$$f_1(x;\alpha,\theta)=\frac{\theta^{\alpha}}{\Gamma(\alpha)}x^{\alpha-1}e^{-\theta x};\ x>0,\theta,\alpha>0, \tag{2}$$

and

$$f_2(x;\beta,\theta)=\frac{\theta^{\beta}}{\Gamma(\beta)}x^{\beta-1}e^{-\theta x};\ x>0,\theta,\beta>0, \tag{3}$$

be two gamma$(\alpha,\theta)$and gamma(β,θ) densities, respectively.

We define a new five parameter Lindley distribution as a mixture of (2) and (3) with probabilities $p=\frac{\theta k}{\eta+\theta k}$ and $(1-p)$, respectively, as follows:

$$f(x;\theta,\alpha,\beta,k,\eta)=p\,f_1(x;\alpha,\theta)+(1-p)f_2(x;\beta,\theta);\ p\in(0,1).$$

Therefore, the pdf of the Five Parameter Lindley Distribution (FPLD), is defined as

$$f(x;\theta,\alpha,\beta,k,\eta)=\frac{\theta^2}{\eta+\theta k}\left[\frac{k(\theta x)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\eta(\theta x)^{\beta-1}}{\theta\Gamma(\beta)}\right]e^{-\theta x};\ x>0. \tag{4}$$

We note that $f(x;\theta,\alpha,\beta,k,\eta)$ incorporates five parameters namely; $\theta>0,\alpha>0$, $\beta>0,k\geq 0$ and $\eta\geq 0$, subject to k and η are not allowed to be simultaneously zeros.

The corresponding cumulative distribution function (cdf) of the FPLD is

$$F(x;\theta,\alpha,\beta,k,\eta)=\frac{1}{\eta+\theta k}\left[\theta k\,\gamma_{\alpha}(\theta x)+\eta\gamma_{\beta}(\theta x)\right]; \quad x>0,\theta,\alpha,\beta>0,k,\eta\geq 0, \tag{5}$$

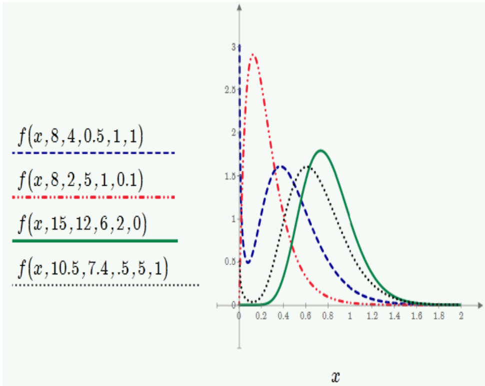
where

$$\gamma_a(b)=\frac{\gamma(a,b)}{\Gamma(a)}=\frac{1}{\Gamma(a)}\int_0^b t^{a-1}e^{-t}dt,$$

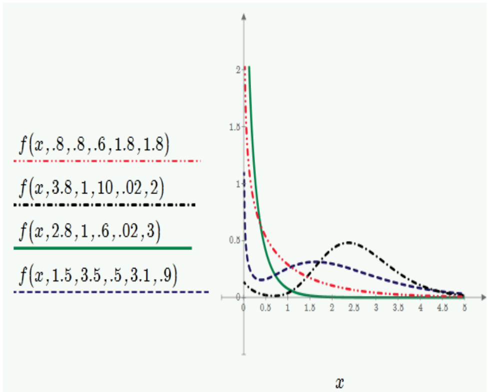
is known as the lower incomplete gamma function ratio. Also the upper incomplete gamma function ratio is given by

$$\Gamma_a(b)=\frac{\Gamma(a,b)}{\Gamma(a)}=\frac{1}{\Gamma(a)}\int_b^{\infty} t^{a-1}e^{-t}dt.$$

Figures (1) and (2) illustrate some of the possible shapes of the pdf and the cdf, respectively, of the FPLD for different values of the parameters θ,α,β,k and η chosen from the ranges specified in Equation (4).

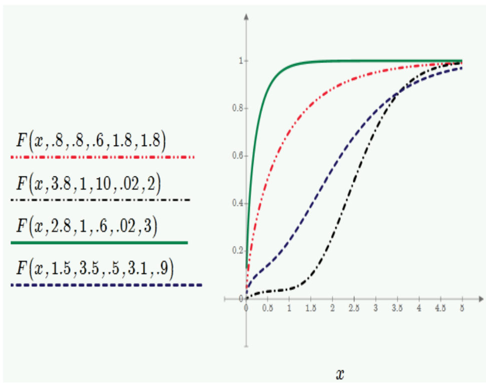


(a)

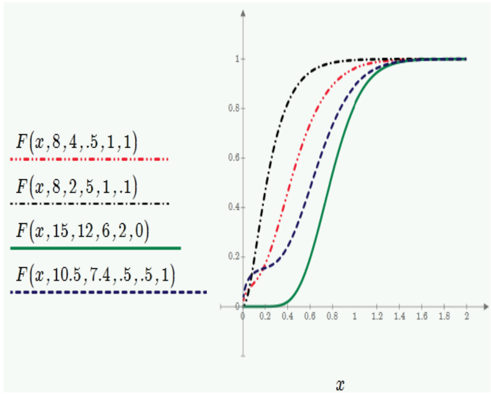


(b)

Figure 1: (a), (b) Different Shapes of the pdf for the FPLD



(a)



(b)

Figure 2: (a), (b) The Distribution Function (cdf) of the FPLD

2.2 Sub-Models of the FPLD

It is clear that the five parameter Lindley distribution is very flexible. Assigning particular numerical values of some subsets of the parameters yields several special generalizations of Lindley distribution. The special cases include nine distributions namely; the new generalized Lindley distribution (NGLD) introduced by Elbatal et al. (2013), generalized Lindley distribution (GLD) introduced by Zakerzadeh and Dolati (2009), quasi Lindley distribution (QLD) introduced by Rama and Mishra (2013), Lindley distribution (LD) by Lindley (1958), Erlang distribution, Janardan distribution introduced by Rama et al. (2013), gamma distribution, the exponential distribution (ED), and Chi-square distribution. In addition to yield all the previous distributions, our generalization model allowed us to create new three distributions namely; 4-parameter Lindley type I (4-p L type I) distribution, 4-parameter Lindley type II (4-p L type II) distribution and 2-parameter Lindley (2-p L) distribution.

Table 1 shows the specific values of the parameters used to generate the abovementioned nine special cases, plus the three new models.

Table 1: The Special Cases of the FPL Distribution

Distribution	Parameters					Author
	θ	α	β	k	η	
Gamma			1	1	0	Brow & Flood (1947)
ED		1	1	1	0	Steffensen (1930)
LD		1	2	1	1	Lindley (1958)
Erlang		$v, v \in \mathbb{N}$	1	1	0	A. K. Erlang (1917)
QLD		1	2		θ	Rama & Mishra (2013)
GLD			$\alpha + 1$	1		Zakerzadeh & Dolati (2009)
Janardan	θ/η	1	2	1		Rama et al. (2013)
NGLD				1	1	Elbatal et al. (2013)
Chi-square	½	$v/2, v \in \mathbb{N}$	1	1	0	Fisher (1924)
4-p L type I				1		New
4-p L type II					1	New
2-p L		1	2		1	New

3. RELIABILITY ANALYSIS

In this section, we present the survival function, the hazard rate function, the reversed hazard rate function, the cumulative hazard rate function and the mean residual lifetime for the five parameter Lindley distribution.

3.1 The Survival Function

The survival function $R(x)$, which is the probability of an item not failing prior to some time t, is defined by $R(x) = 1 - F(x)$. Therefore, the survival function of the FPLD is given by

$$R(x) = 1 - \frac{1}{\eta + \theta k}\left[\theta k\, \gamma_\alpha(\theta x) + \eta \gamma_\beta(\theta x)\right];\ x > 0. \tag{6}$$

Figure (3) illustrates the survival behavior of a FPLD as the values of the parameters are changed.

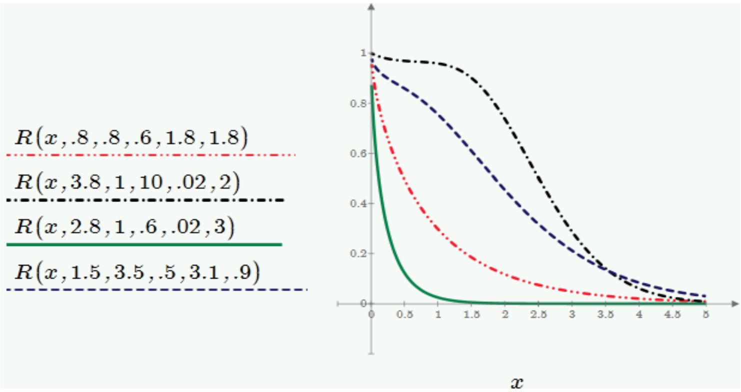


Figure 3: The Reliability (Survival) Function of the FPLD

3.2 The Hazard Rate Function

Let X be the life time of a device (or a component in a system). Suppose a component follow that X has a pdf as in (4). One of the most important characteristics of X is its hazard rate function $h(x)$ defined by

$$h(x) = \lim_{\Delta x \to 0} \frac{Pr(x < X < x + \Delta x | X > x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x\, R(x)} = \frac{f(x)}{R(x)},$$

which provides information about a small interval after time x $(x + \Delta x)$. The hazard rate function for a FPLD can be shown to be

$$h(x) = \frac{\theta^2\left[\frac{k(\theta x)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\eta(\theta x)^{\beta-1}}{\theta\Gamma(\beta)}\right] e^{-\theta x}}{\eta + \theta k - \left[\theta k\, \gamma_\alpha(\theta x) + \eta\gamma_\beta(\theta x)\right]}. \tag{7}$$

We note that $h(x)$ might be constant, increasing, decreasing, or bathtub shaped depending on the values of the parameters involved. For example, if $\eta = 0$ and $\alpha = 1$then $h(x) = \theta$, a constant, while for $\alpha \geq 1$ and $\beta \geq 2$ it will be increasing, and it is going to be decreasing if $\alpha \leq 1, \beta \leq 2$, and $\eta = 0$, and the bathtub-type curve appears for $\alpha < 1, \beta < 2$, and $\eta > 0$.

The next result describes some particular cases for the hazard rate function arising from the five parameter Lindley distribution by assigning relevant values of the parameters.

Theorem 1:

The hazard rate function of the particular cases from the five parameter Lindley distribution are given by

(i) *If α=k=η=1 and β=2 the failure rate is same as the LD(θ).*
(ii) *If α=1, β=2, and η=θ the failure rate is same as the QLD(k,θ).*
(iii) *If α=1, β=2, and η=0 the failure rate is same as the ED(θ).*
(iv) *If α=k=1, θ=(θ/η), and β=2 the failure rate is same as the JD(θ,η).*
(v) *If k=η=1 the failure rate is same as the NGLD(θ,α,β).*
(vi) *If η=0 the failure rate is same as the Gamma(θ,α).*

Proof:

(i) If $\alpha = k = \eta = 1$ and $\beta = 2$ the failure rate is same as the LD(θ)

$$h(x) = \frac{\theta^2(1+x)}{\theta + \theta x + 1}.$$

(ii) If $\alpha = 1, \beta = 2$, and $\eta = \theta$ the failure rate is same as the QLD(k, θ)

$$h(x) = \frac{\theta(k + \theta x)}{k + \theta x + 1}.$$

(iii) If $\alpha = 1, \beta = 2$, and $\eta = 0$ the failure rate is same as the ED(θ)

$$h(x) = \theta.$$

(iv) If $\alpha = k = 1, \theta = (\theta/\eta)$, and $\beta = 2$ the failure rate is same as the JD(θ, η)

$$h(x) = \frac{\theta^2(1 + \eta x)}{\eta(\theta + \eta^2) + \theta\eta^2 x}.$$

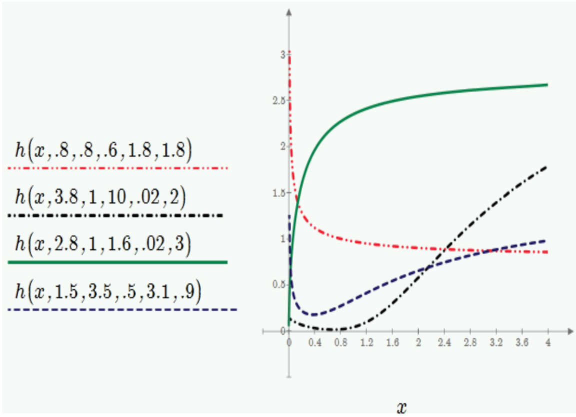
(v) If $k = \eta = 1$ the failure rate is same as the NGLD(θ, α, β)

$$h(x) = \frac{\theta^2\left[\frac{(\theta x)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(\theta x)^{\beta-1}}{\theta\Gamma(\beta)}\right] e^{-\theta x}}{1 + \theta - \left[\theta\gamma_\alpha(\theta x) + \gamma_\beta(\theta x)\right]}.$$

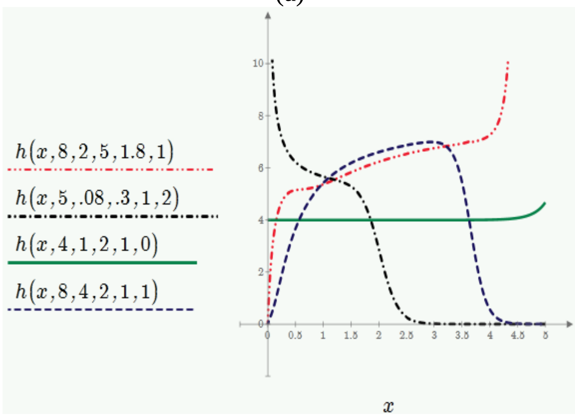
(vi) If $\eta = 0$ the failure rate is same as the Gamma(θ, α)

$$h(x) = \frac{\theta^\alpha x^{\alpha-1} e^{-\theta x}}{\Gamma(\alpha, \theta x)}.$$

Figure 4 illustrates the behavior of the hazard rate function of the new model to illustrate through its different shapes.



(a)



(b)

Figure 4: (a), (b) Increasing, Decreasing, Constant, Bathtub and Upside-Down Shapes for the Hazard Rate Function of the FPLD

3.3 The Reversed Hazard Rate Function

The reversed hazard rate function $r(x)$, is the probability of observing an outcome in a neighborhood of x, conditional on the outcome being no more than x. This is defined as

$$r(x) = \lim_{\Delta x \to 0} \frac{Pr(x < X < x + \Delta x | X \leq x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x\, F(x)} = \frac{f(x)}{F(x)}.$$

Therefore, the reversed hazard rate function of a random variable distributed according to the FPLD$(x; \theta, \alpha, \beta, k, \eta)$ is given by

$$r(x) = \theta^2 \left[\frac{\theta k (\theta x)^{\alpha-1}\Gamma(\beta) + \eta(\theta x)^{\alpha-1}\Gamma(\alpha)}{\theta k\, \gamma(\alpha, \theta x)\Gamma(\beta) + \eta\gamma(\beta, \theta x)\Gamma(\alpha)} \right] e^{-\theta x}. \tag{8}$$

3.4 The Cumulative Hazard Rate Function

Many generalized models have been proposed in reliability literature through the relationship between the reliability function $R(x)$ and its cumulative hazard rate function $H(x)$, given by $H(x) = -\ln R(x)$. The cumulative hazard rate function of the FPLD is given by

$$H(x) = -\ln\left\{1 - \frac{1}{\eta + \theta k}\left[\theta k\, \gamma_\alpha(\theta x) + \eta\gamma_\beta(\theta x)\right]\right\}. \tag{9}$$

where $H(x)$ is the total number of failure or deaths over an interval of time, and $H(x)$ is a non-decreasing function of x satisfying;

(a) $H(0) = 0$, (b) $\lim_{x\to\infty} H(x) = \infty$.

Figure 5 illustrates the behavior of the cumulative hazard rate function of the five parameter Lindley distribution at different values of the parameters.

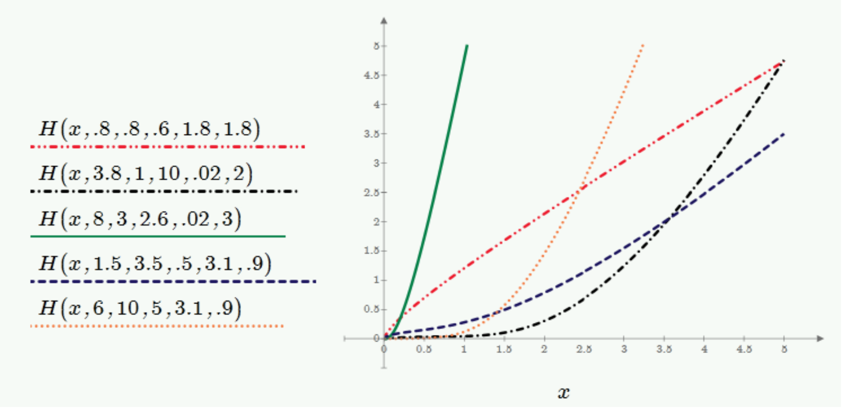


Figure 5: The Cumulative Hazard Rate Function of the FPLD

3.5 The Mean Residual Lifetime

The additional lifetime given that the component has survived up to time x is called the residual life function of the component, then the expectation of the random variable X_x that represent the remaining lifetime is called the mean residual lifetime (MRL) and is given by

$$m(x)=\mathrm{E}(X_x)=\mathrm{E}(X - x|X > x)=\frac{\int_x^\infty R(t)dt}{R(x)},$$

or equivalently

$$m(x) = \frac{\int_x^\infty t\, f(t)dt}{R(x)} - x.$$

While the hazard rate function $h(x)$ provides information about a small interval after time x (just after x), the MRL considers information about the whole interval after x (all after x). The MRL as well as the hazard rate function or the reliability function is very important as each of them can be used to characterize a unique corresponding life time distribution.

The MRL function m(x) for FPL random variable is given by

$$m(x) = \frac{\alpha\theta k\, \Gamma_{\alpha+1}(\theta x) + \beta\eta\Gamma_{\beta+1}(\theta x)}{\theta\{\eta + \theta k - [\theta k\, \gamma_\alpha(\theta x) + \eta\gamma_\beta(\theta x)]\}} - x. \tag{10}$$

The MRL function given in Equation (10) satisfies the following properties;

(i) $m(x) \geq 0$, (ii)$\frac{dm(x)}{dx} \geq -1$, (iii) $\int_0^\infty [m(x)]^{-1} dx = \infty$, and
(iv) $m(0) = \mu_1' = \frac{\alpha\theta k+\beta\eta}{\theta(\eta+\theta k)}$,

where μ_1' is the first non-central moment of the FPLD (the mean of the distribution).

4. STATISTICAL PROPERTIES

This section investigates the statistical properties of the FPLD as the moments (non-central and central), the moment generating function and an algorithm for random number generating.

4.1 The Moment Generating Function

The following theorem gives the moment generating function (mgf) of FPLD$(x; \theta, \alpha, \beta, k, \eta)$.

Theorem 2:

If X has the FPLD$(x; \theta, \alpha, \beta, k, \eta)$, then the mgf of X say $M_X(t)$ is given as follows

$$M_X(t) = \frac{1}{\eta + \theta k}\left\{\sum_{r=0}^{\infty} \frac{(-1)^r}{\theta^r}\left[\theta k \binom{-\alpha}{r} + \eta \binom{-\beta}{r}\right] t^r\right\}; t < \theta. \tag{11}$$

Proof :

$$M_X(t) = \mathrm{E}(e^{tX}) = \int_0^\infty e^{tx} f(x; \theta, \alpha, \beta, k, \eta) dx$$

$$M_X(t) = \frac{\theta^2}{\eta + \theta k}\left[\frac{k\theta^{\alpha-1}}{\Gamma(\alpha)}\int_0^\infty x^{\alpha-1}e^{-(\theta-t)x}dx + \frac{\eta\theta^{\beta-2}}{\Gamma(\beta)}\int_0^\infty x^{\beta-1}e^{-(\theta-t)x}dx\right]$$

$$M_X(t) = \frac{1}{\eta + \theta k}\left[\frac{k\theta^{\alpha+1}}{(\theta - t)^\alpha} + \frac{\eta\theta^\beta}{(\theta - t)^\beta}\right]$$

$$M_X(t) = \frac{1}{\eta + \theta k}\left[\theta k \left(1 - \frac{t}{\theta}\right)^{-\alpha} + \eta \left(1 - \frac{t}{\theta}\right)^{-\beta}\right],$$

using the expansion $(1-Z)^{-d} = \sum_0^\infty (-1)^j \binom{-d}{j} Z^j$,one has

$$M_X(t) = \frac{1}{\eta+\theta k}\left\{\sum_{r=0}^{\infty}\frac{(-1)^r}{\theta^r}\left[\theta k\binom{-\alpha}{r}+\eta\binom{-\beta}{r}\right]t^r\right\}; t<\theta.$$

This completes the proof.

Depending on the previous theorem, we can conclude the basic statistical properties as follows;

(i) The r^{th} non-central moments μ_r' are the coefficients of $\frac{t^r}{r!}$ In Equation (11), for $r = 0,1,2,\ldots$. Therefore, the mean μ_1' and the variance $Var(X)$ of the FPL random variable X are, respectively, given by

$$\mu_1' = \frac{\alpha\theta k+\beta\eta}{\theta(\eta+\theta k)}, \tag{12}$$

and

$$Var(X) = \mu_2' - (\mu_1')^2, \tag{13}$$

where μ_2' is the second non-central moment which is given by

$$\mu_2' = \frac{\alpha\theta k(\alpha+1)+\beta\eta(\beta+1)}{\theta^2(\eta+\theta k)}. \tag{14}$$

(ii) The n^{th} central moments μ_n can be obtained easily from the r^{th} moments through the relation

$$\mu_n = \sum_{r=0}^{n}\binom{n}{r}(-\mu)^{n-r}\mu_r',$$

where $\mu \equiv \mu_1'$.

Then the n^{th} central moments of the FPLD are given by

$$\mu_n = \sum_{r=0}^{n}\binom{n}{r}(-1)^{n-r}\left(\frac{\alpha\theta k+\beta\eta}{\theta(\eta+\theta k)}\right)^{n-r}\mu_r'. \tag{15}$$

(iii) Finally, the coefficient of variation (ρ), the coefficient of skewness (ρ_1), and the coefficient of kurtosis (ρ_2) of FPLD are, respectively, obtained according the following relations

$$\rho = \frac{\sqrt{\mu_2}}{\mu_1'} = \frac{\sqrt{\mu_2' - {\mu_1'}^2}}{\mu_1'}, \tag{16}$$

$$\rho_1 = \frac{\mu_3}{[\mu_2]^{\frac{3}{2}}} = \frac{\mu_3' - 3\mu_2'\mu_1' + 2{\mu_1'}^3}{\left[\mu_2' - {\mu_1'}^2\right]^{\frac{3}{2}}}, \tag{17}$$

$$\rho_2 = \frac{\mu_4}{[\mu_2]^2} = \frac{\mu_4{}' - 4\mu_3{}'\mu_1{}' + 6\mu_2{}'\mu_1{}'^2 - 3\mu_1{}'^2}{\left[\mu_2{}' - \mu_1{}'^2\right]^2}. \tag{18}$$

4.2 Distribution of Order Statistics

Let $X_1, X_2, \dots, X_n$ denote n independent random variables from a distribution function $F_X(x)$ with pdf $f_X(x)$, and then the pdf of $X_{(j)}$ (the j order sample arrangement) is given by

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!\,(n-j)!} f_X(x)[F_X(x)]^{j-1}.[1 - F_X(x)]^{n-j}, j = 1,2, \dots, n. \tag{19}$$

Using Equations (4) and (5) into Equation (19), then the pdf of $X_{(j)}$ according to the FPLD is given by

$$\begin{aligned} f_{X_{(j)}}(x) = {} & \frac{n!}{(j-1)!\ (n-j)!} \left\{ \frac{\theta^2}{\eta + \theta k} \left[\frac{k(\theta x)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\eta(\theta x)^{\beta-1}}{\theta\Gamma(\beta)} \right] e^{-\theta x} \right\} \\ & \times \left\{ \frac{1}{\eta + \theta k} \left[\theta k\, \gamma_\alpha(\theta x) + \eta \gamma_\beta(\theta x) \right] \right\}^{j-1} \\ & \times \left\{ 1 - \frac{1}{\eta + \theta k} \left[\theta k\, \gamma_\alpha(\theta x) + \eta \gamma_\beta(\theta x) \right] \right\}^{n-j}. \end{aligned} \tag{20}$$

Hence, the pdf of the largest order statistic $X_{(n)}$and the smallest order statistic $X_{(1)}$ are, respectively, given by

$$\begin{aligned} f_{X_{(n)}}(x) = {} & n \left\{ \frac{\theta^2}{\eta + \theta k} \left[\frac{k(\theta x)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\eta(\theta x)^{\beta-1}}{\theta\Gamma(\beta)} \right] e^{-\theta x} \right\} \\ & \times \left\{ \frac{1}{\eta + \theta k} \left[\theta k\, \gamma_\alpha(\theta x) + \eta \gamma_\beta(\theta x) \right] \right\}^{n-1}, \end{aligned} \tag{21}$$

and

$$\begin{aligned} f_{X_{(1)}}(x) = {} & n \left\{ \frac{\theta^2}{\eta + \theta k} \left[\frac{k(\theta x)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\eta(\theta x)^{\beta-1}}{\theta\Gamma(\beta)} \right] e^{-\theta x} \right\} \\ & \times \left\{ 1 - \frac{1}{\eta + \theta k} \left[\theta k\, \gamma_\alpha(\theta x) + \eta \gamma_\beta(\theta x) \right] \right\}^{n-1}. \end{aligned} \tag{22}$$

4.3 Random Variates Generation

The probability density function of the FPLD can be expressed in terms of the gamma density function as follows

$$f(x; \theta, \alpha, \beta, k, \eta) = \frac{\theta k}{\eta + \theta k} f_1(x; \alpha, \theta) + \frac{\eta}{\eta + \theta k} f_2(x; \beta, \theta).$$

To generate random variates X_i, for $i = 1, \dots, n$, from FPL$(\theta, \alpha, \beta, k, \eta)$, we can use the following algorithm;

1. GenerateU_i, $i = 1, \dots, n$, from Uniform$(0,1)$ distribution.
2. Generate Y_{1i}, $i = 1, \dots, n$, from Gamma(α, θ).

3. Generate Y_{2i}, $i = 1, \ldots, n$, from Gamma(β, θ).
4. If $U_i \leq \frac{\theta k}{\eta + \theta k}$, then the set of random variates $X_i = Y_{1i}$; otherwise set $= X_i Y_{2i}$, $i = 1, \ldots, n$.

5. MEASURES OF INEQUALITY AND UNCERTAINTY

In this section Lorenz and Bonferroni curves are introduced as measures of inequality. Also, Renyi entropy will be mentioned as an important measure of uncertainty.

5.1 Lorenz and Bonferroni Curves

Lorenz and Bonferroni curves are the most widely used inequality measures in income and wealth distribution (Dagum, 2004).

In fact, Lorenz and Bonferroni curves are depending on the length-biased distribution with pdf $f^*(x)$ defined by

$$f^*(x) = \frac{x . f(x)}{\mathrm{E}(x)}, \tag{23}$$

where $f(x)$ is the pdf of the base distribution with mean $E(x)$.

Accordingly, Lorenz and Bonferroni curves denoted by $L(x)$ and $B(x)$ respectively, defined by

$$L(x) = \frac{F^*(x)}{\mathrm{E}(x)} \; and \; B(x) = \frac{L(x)}{F(x)}, \tag{24}$$

where $F^*(x)$ is the cdf of the length-biased distribution. Now, we shall derive the expressions of $L(x)$ and $B(x)$ based on $f^*(x)$ and$F^*(x)$ for FPLD.

It is easily shown that the pdf of the length-biased distribution can be obtained as follows

$$f^*(x) = \frac{\theta \left[\frac{\alpha\theta k (\theta x)^{\alpha}}{\Gamma(\alpha+1)} + \frac{\beta\eta(\theta x)^{\beta}}{\Gamma(\beta+1)} \right] e^{-\theta x}}{\alpha\theta k + \beta\eta}, \tag{25}$$

with cdf defined by

$$F^*(x) = \frac{\alpha\theta k \, \gamma_{\alpha+1}(\theta x) + \beta\eta\gamma_{\beta+1}(\theta x)}{\alpha\theta k + \beta\eta}. \tag{26}$$

It follows from (12), (24), and (26) that $L(x)$ and $B(x)$ are

$$L(x) = \frac{\theta(\theta k + \eta)\left[\alpha\theta k \, \gamma_{\alpha+1}(\theta x) + \beta\eta\gamma_{\beta+1}(\theta x)\right]}{(\alpha\theta k + \beta\eta)^2}, \tag{27}$$

and

$$B(x) = \frac{\theta(\theta k + \eta)^2\left[\alpha\theta k \, \gamma_{\alpha+1}(\theta x) + \beta\eta\gamma_{\beta+1}(\theta x)\right]}{(\alpha\theta k + \beta\eta)^2\left[\theta k \, \gamma_{\alpha}(\theta x) + \eta\gamma_{\beta}(\theta x)\right]}. \tag{28}$$

5.2 Renyi Entropy

If X is a random variable having an absolutely continuous cdf $F(x)$ and pdf $f(x)$, then the basic uncertainty measure for distribution F (called the entropy of F) is defined as $\Upsilon(x) = \mathrm{E}[-\ln(f(x))]$. Statistical entropy is a probabilistic measure of uncertainty or ignorance about the outcome of a random experiment, and is a measure of a reduction in that uncertainty. Abundant entropy and information indices, among them the Renyi entropy, have been developed and used in various disciplines and contexts. Information theoretic principles and methods have become integral parts of probability and statistics and have been applied in various branches of statistics and related fields.

Renyi entropy is an extension of Shannon entropy. Renyi entropy of the FPLD is defined to be

$$\Upsilon_v\{f(x;\theta,\alpha,\beta,k,\eta)\} = \frac{1}{(1-v)} \times \ln\left(\int_0^\infty f_{FPL}^v(x;\theta,\alpha,\beta,k,\eta)dx\right), \tag{29}$$

where $v > 0$ and $v \neq 1$. Renyi entropy tends to Shannon entropy as $v \to 1$. Now,

$$\int_0^\infty f_{FPL}^v(x;\theta,\alpha,\beta,k,\eta)dx = \left(\frac{\theta^2}{\eta+\theta k}\right)^v \int_0^\infty \left(\frac{k(\theta x)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\eta(\theta x)^{\beta-1}}{\theta\Gamma(\beta)}\right)^v e^{-v\theta x}dx. \tag{30}$$

Using $A = \int_0^\infty \left(\frac{k(\theta x)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\eta(\theta x)^{\beta-1}}{\theta\Gamma(\beta)}\right)^v e^{-v\theta x}dx$. Then one has

$$A = \int_0^\infty \left(1 + \frac{\eta\Gamma(\alpha)\times(\theta x)^{\beta-\alpha}}{\theta k\Gamma(\beta)}\right)^v \left(\frac{k(\theta x)^{\alpha-1}}{\Gamma(\alpha)}\right)^v e^{-v\theta x}dx. \tag{31}$$

Using the expansion: $(1+Z)^n = \sum_{j=0}^{n} \binom{n}{j} Z^j$, one can have

$$\left(1 + \frac{\eta\Gamma(\alpha)\times(\theta x)^{\beta-\alpha}}{\theta k\Gamma(\beta)}\right)^v = \sum_{j=0}^{v} \binom{v}{j} \left(\frac{\eta\Gamma(\alpha)\times(\theta x)^{\beta-\alpha}}{\theta k\Gamma(\beta)}\right)^j. \tag{32}$$

Substitute (32) into (31) and make the necessary simplifications, gives

$$A = \left(\frac{k}{\Gamma(\alpha)}\right)^v \sum_{j=0}^{v} \binom{v}{j} \left(\frac{\eta\Gamma(\alpha)}{\theta k\Gamma(\beta)}\right)^j \int_0^\infty (\theta x)^{v(\alpha-1)+j(\beta-\alpha)} e^{-v\theta x}dx. \tag{33}$$

Using the gamma function to evaluate the integral in(33) and collecting the entire above evaluations then substitute into (29), the Renyi entropy of the FPLD can be written as

$$\Upsilon_v = \frac{1}{1-v}\left[\ln C + \ln\left\{\sum_{j=0}^{v}\binom{v}{j}\left(\frac{\eta\Gamma(\alpha)}{\theta k\Gamma(\beta)}\right)^j \times \frac{\Gamma(v(\alpha-1)+j(\beta-\alpha)+1)}{v^{j(\beta-\alpha)}}\right\}\right], \tag{34}$$

where C is a constant as

$$C = \left[\frac{k\,\theta^{\frac{2v-1}{v}}}{v^{\alpha}(\eta+\theta k)\Gamma(\alpha)}\right]^v.$$

6. ESTIMATION OF THE PARAMETERS

In this section, we use the method of likelihood to estimate the parameters involved and use them to create confidence intervals for the unknown parameters.

Let $X_1, X_2, \ldots, X_n$ be a sample size n from FPL distribution. Then the likelihood function (l) is given by

$$l = \prod_{i=1}^{n} f_i(x) = \left(\frac{\theta^2}{\eta+\theta k}\right)^n \prod_{i=1}^{n}\left[\frac{k\theta^{\alpha-1}x_i^{\alpha-1}}{\Gamma(\alpha)} + \frac{\eta\theta^{\beta-2}x_i^{\beta-1}}{\Gamma(\beta)}\right] e^{-\theta x_i},$$

then,

$$l = \left(\frac{\theta^2}{\eta+\theta k}\right)^n \exp\left[-\theta\sum_{i=1}^{n} x_i\right][\Gamma(\alpha)\Gamma(\beta)]^{-n}\prod_{i=1}^{n}\left[k\Gamma(\beta)\theta^{\alpha-1}x_i^{\alpha-1} + \eta\Gamma(\alpha)\theta^{\beta-2}x_i^{\beta-1}\right]. \tag{35}$$

Hence, the log-likelihood function $\mathcal{L} = \ln l$ becomes

$$\mathcal{L} = 2n\ln\theta - n\ln(\eta+\theta k) - \theta\sum_{i=1}^{n} x_i - n\ln\Gamma(\alpha) - n\ln\Gamma(\beta) + \sum_{i=1}^{n}\left[k\Gamma(\beta)\theta^{\alpha-1}x_i^{\alpha-1} + \eta\Gamma(\alpha)\theta^{\beta-2}x_i^{\beta-1}\right]. \tag{36}$$

Therefore, the maximum likelihood estimators (MLEs) of θ, α, β, k and η are derived from the derivatives of $\mathcal{L}$. They should satisfy the following equations

$$\frac{\partial\mathcal{L}}{\partial\theta} = \frac{2n}{\theta} - \frac{nk}{\eta+\theta k} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\left[\frac{k(\alpha-1)\Gamma(\beta)\theta^{\alpha-2}x_i^{\alpha-1} + \eta(\beta-2)\Gamma(\alpha)\theta^{\beta-3}x_i^{\beta-1}}{k\Gamma(\beta)\theta^{\alpha-1}x_i^{\alpha-1} + \eta\Gamma(\alpha)\theta^{\beta-2}x_i^{\beta-1}}\right] = 0, \tag{37}$$

$$\frac{\partial \mathcal{L}}{\partial \alpha} = -n\psi(\alpha) + \sum_{i=1}^{n}\left[\frac{k\Gamma(\beta)\theta^{\alpha-1}x_i^{\alpha-1}\log(\theta x_i) + \eta\Gamma'(\alpha)\theta^{\beta-2}x_i^{\beta-1}}{k\Gamma(\beta)\theta^{\alpha-1}x_i^{\alpha-1} + \eta\Gamma(\alpha)\theta^{\beta-2}x_i^{\beta-1}}\right] = 0, \quad (38)$$

$$\frac{\partial \mathcal{L}}{\partial \beta} = -n\psi(\beta) + \sum_{i=1}^{n}\left[\frac{k\Gamma'(\beta)\theta^{\alpha-1}x_i^{\alpha-1} + \eta\Gamma(\alpha)\theta^{\beta-2}x_i^{\beta-1}\log(\theta x_i)}{k\Gamma(\beta)\theta^{\alpha-1}x_i^{\alpha-1} + \eta\Gamma(\alpha)\theta^{\beta-2}x_i^{\beta-1}}\right] = 0, \quad (39)$$

$$\frac{\partial \mathcal{L}}{\partial k} = -\frac{n\theta}{\eta + \theta k} + \sum_{i=1}^{n}\left[\frac{\Gamma(\beta)\theta^{\alpha-1}x_i^{\alpha-1}}{k\Gamma(\beta)\theta^{\alpha-1}x_i^{\alpha-1} + \eta\Gamma(\alpha)\theta^{\beta-2}x_i^{\beta-1}}\right] = 0, \quad (40)$$

$$\frac{\partial \mathcal{L}}{\partial \eta} = -\frac{n}{\eta + \theta k} + \sum_{i=1}^{n}\left[\frac{\Gamma(\alpha)\theta^{\alpha-2}x_i^{\alpha-1}}{k\Gamma(\beta)\theta^{\alpha-1}x_i^{\alpha-1} + \eta\Gamma(\alpha)\theta^{\beta-2}x_i^{\beta-1}}\right] = 0. \quad (41)$$

where $\psi(.)$ is the diagamma function, and it is defined as

$$\psi(a) = \frac{d}{da}\ln\Gamma(a) = \frac{\Gamma'(a)}{\Gamma(a)}.$$

To solve the equations (37) through (41), it is usually more convenient to use nonlinear optimization algorithms such as quasi-Newton algorithm to numerically maximize the log-likelihood function. In order to compute the standard errors and asymptotic confidence intervals we use the usual large sample approximation, in which the MLEs can be treated as being approximately trivariate normal.

Hence as $n \to \infty$, the asymptotic distribution of the MLE is given by, see Zaindin et al. (2009):

$$\begin{pmatrix}\hat{\theta}\\ \hat{\alpha}\\ \hat{\beta}\\ \hat{k}\\ \hat{\eta}\end{pmatrix} \sim \left[\begin{pmatrix}\hat{\theta}\\ \hat{\alpha}\\ \hat{\beta}\\ \hat{k}\\ \hat{\eta}\end{pmatrix}, \begin{pmatrix}\hat{V}_{11}\hat{V}_{12}\hat{V}_{13}\hat{V}_{14}\hat{V}_{15}\\ \hat{V}_{21}\hat{V}_{22}\hat{V}_{23}\hat{V}_{24}\hat{V}_{25}\\ \hat{V}_{31}\hat{V}_{32}\hat{V}_{33}\hat{V}_{34}\hat{V}_{35}\\ \hat{V}_{41}\hat{V}_{42}\hat{V}_{43}\hat{V}_{44}\hat{V}_{45}\\ \hat{V}_{51}\hat{V}_{52}\hat{V}_{53}\hat{V}_{54}\hat{V}_{55}\end{pmatrix}\right],$$

where $\left(\hat{V}_{ij} = V_{ij}\middle|\theta = \hat{\theta}\right)$, and

$$\begin{pmatrix}V_{11}V_{12}V_{13}V_{14}V_{15}\\ V_{21}V_{22}V_{23}V_{24}V_{25}\\ V_{31}V_{32}V_{33}V_{34}V_{35}\\ V_{41}V_{42}V_{43}V_{44}V_{45}\\ V_{51}V_{52}V_{53}V_{54}V_{55}\end{pmatrix} = \begin{pmatrix}A_{11}A_{12}A_{13}A_{14}A_{15}\\ A_{21}A_{22}A_{23}A_{24}A_{25}\\ A_{31}A_{32}A_{33}A_{34}A_{35}\\ A_{41}A_{42}A_{43}A_{44}A_{45}\\ A_{51}A_{52}A_{53}A_{54}A_{55}\end{pmatrix},$$

is the approximate variance-covariance matrix with its elements obtained from

$$A_{11}=\frac{\partial^2\mathcal{L}}{\partial\theta^2}\quad A_{12}=\frac{\partial^2\mathcal{L}}{\partial\theta\partial\alpha}\quad A_{13}=\frac{\partial^2\mathcal{L}}{\partial\theta\partial\beta}\quad A_{14}=\frac{\partial^2\mathcal{L}}{\partial\theta\partial k}\quad A_{15}=\frac{\partial^2\mathcal{L}}{\partial\theta\partial\eta}$$

$$A_{22}=\frac{\partial^2\mathcal{L}}{\partial\alpha^2}\quad A_{23}=\frac{\partial^2\mathcal{L}}{\partial\alpha\partial\beta}\quad A_{24}=\frac{\partial^2\mathcal{L}}{\partial\alpha\partial k}\quad A_{25}=\frac{\partial^2\mathcal{L}}{\partial\alpha\partial\eta}$$

$$A_{33}=\frac{\partial^2\mathcal{L}}{\partial\beta^2}\quad A_{34}=\frac{\partial^2\mathcal{L}}{\partial\beta\partial k}\quad A_{35}=\frac{\partial^2\mathcal{L}}{\partial\beta\partial\eta}$$

$$A_{44}=\frac{\partial^2\mathcal{L}}{\partial k^2}\quad A_{45}=\frac{\partial^2\mathcal{L}}{\partial k\partial\eta}$$

$$A_{55}=\frac{\partial^2\mathcal{L}}{\partial\eta^2}$$

By solving this inverse dispersion matrix, these solutions will yield the asymptotic variances and covariances of these MLEs for $\theta,\alpha,\beta,k$and η.

Approximate $100(1-\alpha)\%$ confidence intervals for $\theta,\alpha,\beta,k,$ and η can be determined as

$$\hat{\theta}\pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{11}},\quad \hat{\alpha}\pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{22}},\quad \hat{\beta}\pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{33}},\quad \hat{k}\pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{44}},\quad \hat{\eta}\pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{55}},$$

where $Z_{\frac{\alpha}{2}}$ is the upper α^{th} percentile of the standard normal distribution.

7. APPLICATION

In this section, we use a real data set to compare the fits of the FPL distribution with three sub-models. In each case, the parameters are estimated by maximum likelihood as described in Section 6, using the R software.

The data set consist of uncensored data set from Nichols and Padgett on the breaking stress of carbon fibers (in Gba). The data are given below:

3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 3.56,
4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 1.57,
2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85,
1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03,
1.89, 2.88, 2.82, 2.05, 3.65, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96,
2.35, 2.55, 2.59, 2.03, 1.61, 2.12, 3.15, 1.08, 2.56, 1.80, 2.53.

The summary of the above data is given by

Units	*Minimum*	*1st Qu.*	*Median*	*Mean*	*3rd Qu.*	*Maximum*
66	0.390	2.178	2.835	2.760	3.278	4.900

In order to compare the two distribution models, we consider criteria like KS (Kolmogorov Smirnov),$-2\mathcal{L}$, AIC (Akaike information criterion), AICC (corrected

Akaike information criterion), and BIC (Bayesian information criterion) for the data set. The better distribution corresponds to smaller KS, $-2\mathcal{L}$, AIC and AIC_C values:

$$\text{AIC} = 2k - 2\mathcal{L}, \text{AICC} = \text{AIC} + \frac{2k(k+1)}{n-k-1},$$

$$\text{BIC} = 2\mathcal{L} + k \times \log n \ \textbf{and} \ KS = \max_{1 \le i \le n}\left(F(X_i) - \frac{i-1}{n}, \frac{i}{n} - F(X_i)\right),$$

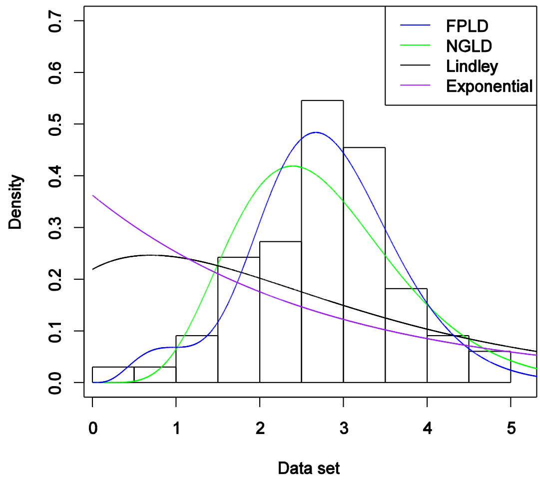
where $\mathcal{L}$ denotes the log-likelihood function evaluated at the maximum likelihood estimates, k is the number of parameters, and n is the sample size.

Also, for calculating the values of KS we use the sample estimates of $\alpha, \beta, \gamma, \theta, \lambda, a$ and b. Table 2 shows the parameter estimation based on the maximum likelihood and least square estimation, and gives the values of the criteria AIC, AICC, BIC, and KS test. The values in Table 2 indicate that the FPL distribution leads to a better fit over all the other models.

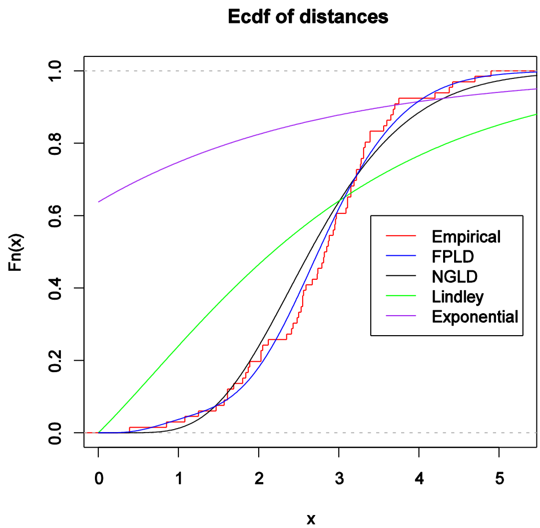
Table 2
MLEs, the Measures AIC, AIC_C and BIC, and KS Test under Considered Models based on Real Data

Model	Parameter Estimates	−Log L	AIC	AIC_C	BIC	KS
FPLD	$\hat{\boldsymbol{\theta}}$**=5.224**	**85.864**	**181.393**	**182.393**	**192.341**	**0.061**
	$\hat{\boldsymbol{\alpha}}$**=6.183**					
	$\hat{\boldsymbol{\beta}}$**=15.266**					
	$\hat{\boldsymbol{k}}$**=0.425*10^{-3}**					
	$\hat{\boldsymbol{\eta}}$**=0.021**					
NGLD	$\hat{\theta}$=2.788	91.107	188.215	188.602	194.784	0.117
	$\hat{\alpha}$=7.412					
	$\hat{\beta}$=8.485					
Lindley	$\hat{\theta}$=0.590	122.384	246.768	246.830	248.957	0.282
Exponential	$\hat{\lambda}$= 0.362	132.994	267.988	268.051	270.178	0.711

A density plot compares the fitted densities of the models with the empirical histogram of the observed data (Figures 6 and 7). The fitted density for the FPL model is closer to the empirical histogram than the fits of the other models.



(a)



(b)

Figure 6: (a) Estimated Densities of the FPL, NGL, Lindley and Exponential Distributions for the Data.
(b) Estimated cdf Function from the Fitted the FPL, NGL, Lindley and Exponential Distributions and the Empirical cdf for the Data.

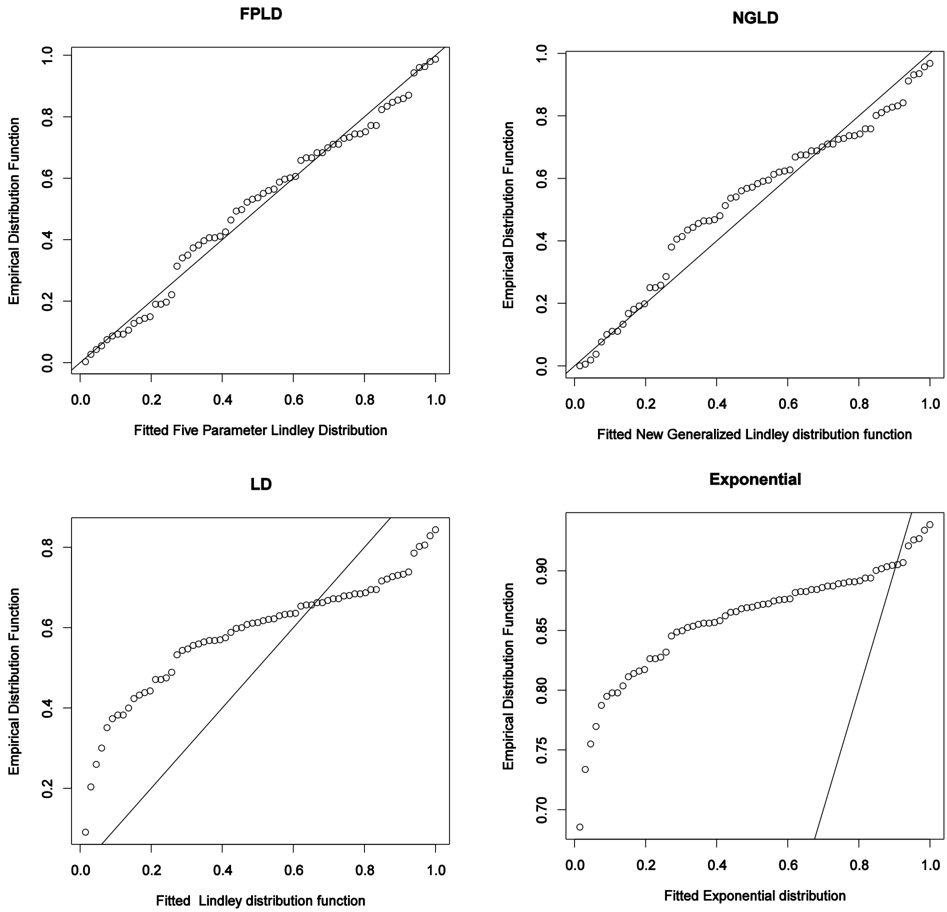


Figure 7: PP plots for the fitted FPLD, NGLD, Lindley distribution and Exponential distribution for the data set.

CONCLUDING REMARKS

There has been a great interest among statisticians and applied researchers in constructing flexible lifetime models to facilitate better modelling of survival data. Consequently, a significant progress has been made towards the generalization of some well-known lifetime models and their successful application to problems in several areas. In this paper, we introduce a new five-parameter distribution obtained using the idea of mixture of distributions. We refer to the new model as the Five Parameter Lindley Distribution (FPLD) and study some of its mathematical and statistical properties. We provide the pdf, the cdf and the hazard rate function of the new model and explicit expressions for the moments. The model parameters are estimated by the method of maximum likelihood. The new model is compared with three lifetime models and provides consistently better fit than them. We hope that the proposed distribution will serve as an alternative model to other models available in the literature for modelling positive real data in many areas such as engineering, survival analysis, hydrology and economics.

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