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PARAMETER ESTIMATION OF SIMULTANEOUS LINEAR FUNCTIONAL RELATIONSHIP MODEL FOR CIRCULAR VARIABLES ASSUMING EQUAL ERROR VARIANCES

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ABSTRACT

In this paper, a new simultaneous model is extended from the simple linear functional relationship model for circular data proposed by Caires and Wyatt (2003) by assuming equal error variances. The Maximum Likelihood Estimator of the parameters in the simultaneous model are obtained and the covariance between the parameters is derived using the Fisher Information Matrix. A simulation study was done to investigate the bias in the parameters. Results from this study suggest that the estimated parameters have small bias. The applicability of the proposed simultaneous model is illustrated using a real data set.

KEY WORDS

Simultaneous linear functional relationship model; Fisher information matrix; parameter estimation; circular variables.

1. INTRODUCTION

Directional data arises quite frequently in many natural and physical sciences (Jammaladaka and Sengupta, 2001). Two examples are:- biologists studying the direction of flight of birds, or geologists studying the direction of the earth's magnetic pole (Batschelet, 1981). The directions may be in two or in three-dimension. Observations on two-dimensional directions can be referred to circular data while the observations on three-dimensional directions can be referred to spherical data.

Circular data maybe represented as angles or as points on the circumference of a unit of a circle (Jammaladaka and Sengupta, 2001). The data refers to a set of observations measured by angles in the intervals of $[0, 2\pi)$ radians or $[0, 360)°$. Circular data cannot escape very far from each other and are not able to hide from view (Fisher, 1993).

The early root of circular data analysis dates back to the mid-18th century in the study of the angular separations between stars with a view to assessing the hypothesis that the directions of the stars were uniformly distributed (Fisher, 1993). Gould (1969) started discussion on the development of circular regression models and predicted the mean direction of a circular response variable from a vector of linear covariates (Ibrahim *et al*., 2013). Notable references on circular variables can be found in Batschelet (1981), Fisher (1993), Mardia and Jupp (2000) and Jammaladaka and Sengupta (2001).

The Von Mises distribution is said to be the most useful distribution on the circle (Mardia and Jupp, 2000). The distribution is analogous to the normal distribution as it has some similar characteristics (Hassan et al., 2012). The von Mises distribution has probability density function

$$g(\theta;\mu,\kappa)=\frac{1}{2\pi I_0(\kappa)}e^{\kappa\cos(\theta-\mu)}, \quad (1)$$

where $I_0(\kappa)$ is the modified Bessel function of the first kind and order zero, which can be defined by:

$$I_0(\kappa)=\frac{1}{2\pi}\int_0^{2\pi} e^{\kappa\cos\theta}d\theta, \quad (2)$$

where μ is the mean direction and κ is the concentration parameter.

The concentration parameter κ influences the Von Mises distribution $VM(\mu,\kappa)$ inversely as σ^2 influences the Normal distribution $N(\mu,\sigma^2)$. Thus, a concentrated Von Mises distribution will have large concentration parameter, and a dispersed Von Mises distribution will have a small concentration parameter (Caires and Wyatt, 2003).

2. SIMULTANEOUS LINEAR FUNCTIONAL RELATIONSHIP MODEL

In this paper, we extend the model proposed by Caires and Wyatt (2003) to become a simultaneous model. This enhancement is done so that the relationship between more than 2 circular variables can be studied statistically. Suppose the variables $Y_{ji}\left(j=1,...,q;\ i=1,...,n\right)$ and $X_i\left(i=1,...,n\right)$ are related by the linear functional relationship of $Y_j=\alpha_j+X(\mathrm{mod}\,2\pi)$. Let the observations be $\left(x_i,y_{ji}\right)$ and these observations correspond to measurement of the true values of $\left(X_i,Y_{ji}\right)$, made with some random errors $\left(\delta_i,\varepsilon_{ji}\right)$. The random errors δ_i and ε_{ji} are assumed to be independently distributed with von Mises distribution with $\delta_i\sim VM(0,\kappa)$ and $\varepsilon_{ji}\sim VM\left(0,\nu_j\right)$, respectively.

The model can then be written as:

$$x_i=X_i+\delta_i \text{ and } y_{ji}=Y_{ji}+\varepsilon_{ji},$$

where $Y_j=\alpha_j+X(\mathrm{mod}\,2\pi)$.

3. PARAMETER ESTIMATION

In this paper we consider the case when $\kappa = \nu$ or equal error variance for all observations on both variables. This means that there are $(n+q+1)$ parameters to be estimated which are $\alpha_1,...,\alpha_q, X_1,...,X_n$ and κ. The log likelihood function of the model is given by equation (3).

$$\log L = -2n\log(2\pi) - (1+q)n\log I_0(\kappa) + \kappa\sum_{i=1}^{n}\cos\left(x_i - X_i\right) + \kappa\sum_{j=1}^{q}\sum_{i=1}^{n}\cos\left(y_{ji} - \alpha_j - X_i\right) \tag{3}$$

Maximum Likelihood Estimation of α_j

Differentiating Equation (3) with respect to α_j we get

$$\frac{\partial \log L}{\partial \alpha_j} = \kappa\sum_{i=1}^{n}\sin\left(y_{ji} - \alpha_j - X_i\right), \tag{4}$$

and by setting $\frac{\partial \log L}{\partial \alpha_j} = 0$, we obtain

$$\kappa\sum_{i=1}^{n}\sin\left(y_{ji} - \alpha_j - X_i\right) = 0, \tag{5}$$

rearranging the terms in (5), we get

$$\kappa\sum_{i=1}^{n}\left[\sin\left(y_{ji} - X_i\right)\cos\left(\alpha_j\right) - \cos\left(y_{ji} - X_i\right)\sin\left(\alpha_j\right)\right] = 0, \tag{6}$$

$$\sum_{i=1}^{n}\sin\left(y_{ji} - X_i\right)\cos\left(\alpha_j\right) = \sum_{i=1}^{n}\cos\left(y_{ji} - X_i\right)\sin\left(\alpha_j\right), \tag{7}$$

$$\frac{\sin\left(\alpha_j\right)}{\cos\left(\alpha_j\right)} = \frac{\sum_{i=1}^{n}\sin\left(y_{ji} - X_i\right)}{\sum_{i=1}^{n}\cos\left(y_{ji} - X_i\right)} = \frac{S}{C}, \tag{8}$$

or

$$\tan\left(\alpha_j\right) = \frac{S}{C}. \tag{9}$$

Therefore,

$$\hat{\alpha}_j = \tan^{-1}\left(\frac{S}{C}\right) \tag{10}$$

where $\tan^{-1}\left(\frac{S}{C}\right)$ is the arc tangent of $\left(\frac{S}{C}\right)$ with

$$\hat{\alpha}_j = \begin{cases} \tan^{-1}\left(\frac{S}{C}\right) \text{ when } S>0,\, C>0 \\ \tan^{-1}\left(\frac{S}{C}\right)+\pi \text{ when } C<0 \\ \tan^{-1}\left(\frac{S}{C}\right)+2\pi \text{ when } S<0,\, C>0 \end{cases}. \tag{11}$$

Maximum Likelihood Estimation of *X*

Differentiating Equation (3) with respect to X_i, we obtain

$$\frac{\partial \log L}{\partial X_i} = \kappa \sin\left(x_i - X_i\right) + \kappa \sum_{j=1}^{q} \sin\left(y_{ji} - \alpha_j - X_i\right) = 0. \tag{12}$$

Setting $\frac{\partial \log L}{\partial X_i} = 0$, we get

$$\kappa \sin\left(x_i - X_i\right) + \kappa \sum_{j=1}^{q} \sin\left(y_{ji} - \alpha_j - X_i\right) = 0. \tag{13}$$

X_i may be solved iteratively with some "initial guess". Suppose $\hat{X}_{i0}$ is an initial estimate of $\hat{X}_i$, then

$$x_i - \hat{X}_i = x_i - \hat{X}_{i0} + \hat{X}_{i0} - \hat{X}_i = \left(x_i - \hat{X}_{i0}\right) + \Delta_i \tag{14}$$

where $\Delta_\text{i} = \hat{X}_{i0} - \hat{X}_i$.

We also have $y_{ji} - \hat{\alpha}_j - \hat{X}_i = \left(y_{ji} - \hat{\alpha}_j - \hat{X}_{i0}\right) + \Delta_i$.

Thus, the partial derivative equation above becomes:

$$\sin\left(x_i - \hat{X}_{i0} + \Delta_i\right) + \sum_{j=1}^{q} \sin\left(y_{ji} - \hat{\alpha}_j - \hat{X}_{i0} + \Delta_i\right) = 0 \tag{15}$$

When Δ_i is small, then $\cos\Delta_i \approx 1$ and $\sin\Delta_i = \Delta_i$. Hence, the equation is simplified (approximately) become:

$$\frac{\sin\left(x_i - \hat{X}_{i0}\right) + \sum_{j=1}^{q} \sin\left(y_{ji} - \hat{\alpha}_j - \hat{X}_{i0}\right)}{\cos\left(x_i - \hat{X}_{i0}\right) + \sum_{j=1}^{q} \cos\left(y_{ji} - \hat{\alpha}_j - \hat{X}_{i0}\right)} + \left(\hat{X}_{i0} - \hat{X}_{i1}\right) = 0 \tag{16}$$

$$\hat{X}_{i1} \approx \hat{X}_{10} + \frac{\sin\left(x_i - \hat{X}_{i0}\right) + \sum_{j=1}^{q} \sin\left(y_{ji} - \hat{\alpha}_j - \hat{X}_{i0}\right)}{\cos\left(x_i - \hat{X}_{i0}\right) + \sum_{j=1}^{q} \cos\left(y_{ji} - \hat{\alpha}_j - \hat{X}_{i0}\right)} \quad (17)$$

where $\hat{X}_{i1}$ is an improvement of $\hat{X}_{i0.}$

Maximum Likelihood Estimation of κ

From the log likelihood function in Equation (3), the equation is differentiated with respect to κ and we get

$$\frac{\partial \log L}{\partial \kappa} = -n(1+q)\frac{I_0'(\kappa)}{I_0(\kappa)} + \sum_{i=1}^{n}\cos\left(x_i - X_i\right) + \sum_{j=1}^{q}\sum_{i=1}^{n}\cos\left(y_{ji} - \alpha_j - X_i\right) \quad (18)$$

Setting $\frac{\partial \log L}{\partial \kappa} = 0$, we obtain

$$-n(1+q)\frac{I_0'(\kappa)}{I_0(\kappa)} + \sum_{i=1}^{n}\cos\left(x_i - X_i\right) + \sum_{j=1}^{q}\sum_{i=1}^{n}\cos\left(y_{ji} - \alpha_j - X_i\right) = 0, \quad (19)$$

Rearranging Equation (19) and get

$$-n(1+q)\frac{I_0'(\kappa)}{I_0(\kappa)} = \sum_{i=1}^{n}\cos\left(x_i - X_i\right) + \sum_{j=1}^{q}\sum_{i=1}^{n}\cos\left(y_{ji} - \alpha_j - X_i\right), \quad (20)$$

$$-n(1+q)A(\kappa) = \sum_{i=1}^{n}\cos\left(x_i - X_i\right) + \sum_{j=1}^{q}\sum_{i=1}^{n}\cos\left(y_{ji} - \alpha_j - X_i\right), \quad (21)$$

$$A(\kappa) = \left[\frac{1}{n(1+q)}\right]\left\{\sum_{i=1}^{n}\cos\left(x_i - X_i\right) + \sum_{j=1}^{q}\sum_{i=1}^{n}\cos\left(y_{ji} - \alpha_j - X_i\right)\right\}, \quad (22)$$

where $A(\kappa) = \frac{I_0'(\kappa)}{I_0(\kappa)} = \frac{I_1(\kappa)}{I_0(\kappa)}$.

Hence,

$$\hat{\kappa} = A^{-1}\left(\left[\frac{1}{n(1+q)}\right]\left\{\sum_{i=1}^{n}\cos\left(x_i - X_i\right) + \sum_{j=1}^{q}\sum_{i=1}^{n}\cos\left(y_{ji} - \alpha_j - X_i\right)\right\}\right). \quad (23)$$

One approximation in the estimation of κ has been given by Fisher (1993) as:

$$A^{-1}(w)=\begin{cases}2w+w^3+\dfrac{5}{6}w^3 & w<0.53\\ -0.4+1.39w+\dfrac{0.43}{(1-w)} & 0.53\le w<0.85,\\ \dfrac{1}{\mathrm{w}^3-4w^2+3w} & w\ge 0.85\end{cases} \tag{24}$$

In circular case, the estimation of a concentration parameter (whose inverse is equivalent of the variance for linear data) needs to be corrected by multiplying it by $\frac{q}{(q+1)}$ where q is the number of equations in the simultaneous relationship (Hussin et al., 2010). In this case, the correction factor becomes $\frac{2}{3}$ when $q=2$. Therefore, $\tilde{\kappa}=\frac{2\hat{\kappa}}{3}$ gives a better approximation to the value of κ.

4. COVARIANCE MATRIX OF PARAMETERS

Collecting our earlier results, we have the log likelihood function of the Von Mises distribution is given by

$$\log L=-2n\log(2\pi)-(1+q)n\log I_0(\kappa)$$
$$+\kappa\sum_{i=1}^{n}\cos\left(x_i-X_i\right)+\kappa\sum_{j=1}^{q}\sum_{i=1}^{n}\cos\left(y_{ji}-\alpha_j-X_i\right).$$

The first derivatives of the log likelihood function with respect to α_j, X_i and κ are as follows

$$\frac{\partial\log L}{\partial\alpha_j}=\kappa\sum_{i=1}^{n}\sin\left(y_{ji}-\alpha_j-X_i\right), \tag{25}$$

$$\frac{\partial\log L}{\partial X_i}=\kappa\sin\left(x_i-X_i\right)+\kappa\sum_{j=1}^{q}\sin\left(y_{ji}-\alpha_j-X_i\right), \tag{26}$$

$$\frac{\partial\log L}{\partial\kappa}=-n(1+q)\frac{I_0'(\kappa)}{I_0(\kappa)}+\sum_{i=1}^{n}\cos\left(x_i-X_i\right)+\sum_{j=1}^{q}\sum_{i=1}^{n}\cos\left(y_{ji}-\alpha_j-X_i\right), \tag{27}$$

The second derivative of equation (25) is then:

$$\frac{\partial^2\log L}{\partial{\alpha_j}^2}=-\kappa\sum_{i=1}^{n}\cos\left(y_{ji}-\alpha_j-X_i\right).$$

Therefore

$$\mathrm{E}\left(-\frac{\partial^2 \log L}{\partial \alpha_j^{\,2}}\right) = \kappa\ \mathrm{E}\left(\sum_{i=1}^{n} \cos\left(y_{ji} - \alpha_j - X_i\right)\right) = \kappa n A(\kappa),$$

where

$$A(\kappa) = \left[\frac{1}{n(1+q)}\right]\left\{\sum_{i=1}^{n} \cos\left(x_i - X_i\right) + \sum_{j=1}^{q}\sum_{i=1}^{n} \cos\left(y_{ji} - \alpha_j - X_i\right)\right\}.$$

$$\frac{\partial^2 \log L}{\partial \alpha_j \partial X_i} = -\kappa \cos\left(y_{ji} - \alpha_j - X_i\right).$$

Therefore,

$$\mathrm{E}\left(-\frac{\partial^2 \log L}{\partial \alpha_j \partial X_i}\right) = \kappa\ \mathrm{E}\left(\cos\left(y_{ji} - \alpha_j - X_i\right)\right) = \kappa A(\kappa).$$

$$\frac{\partial^2 \log L}{\partial \alpha_j \partial \kappa} = \sum_{i=1}^{n} \sin\left(y_{ji} - \alpha_j - X_i\right).$$

Therefore

$$\mathrm{E}\left(-\frac{\partial^2 \log L}{\partial \alpha_j \partial \kappa}\right) = \mathrm{E}\left(-\sum_{i=1}^{n} \sin\left(y_{ji} - \alpha_j - X_i\right)\right) = 0.$$

The second derivative of equation (26) is then:

$$\frac{\partial^2 \log L}{\partial X_i^{\,2}} = -\kappa \cos\left(x_i - X_i\right) - \kappa \sum_{j=1}^{q} \cos\left(y_{ji} - \alpha_j - X_i\right).$$

Therefore

$$\mathrm{E}\left(-\frac{\partial^2 \log L}{\partial X_i^{\,2}}\right) = \kappa\ \mathrm{E}\left(\cos\left(x_i - X_i\right)\right) + \kappa\ \mathrm{E}\left(\sum_{j=1}^{q} \cos\left(y_{ji} - \alpha_j - X_i\right)\right),$$

$$= \kappa A(\kappa) + \kappa q A(\kappa) = (1+q)\kappa A(\kappa).$$

$\dfrac{\partial^2 \log L}{\partial X_i\ \partial X_j} = 0,$ for i ≠ j. Therefore, $\mathrm{E}\left(-\dfrac{\partial^2 \log L}{\partial X_i\ \partial X_j}\right) = 0.$

$$\frac{\partial^2 \log L}{\partial X_i\ \partial \kappa} = \sin\left(x_i - X_i\right) + \sum_{j=1}^{q} \sin\left(y_i - \alpha - X_i\right).$$

Therefore

$$E\left(-\frac{\partial^2 \log L}{\partial X_i\ \partial X_j}\right) = E\left(-\sin\left(x_i - X_i\right) + \sum_{j=1}^{q} \sin\left(y_i - \alpha - X_i\right)\right) = 0 ,$$

Since

$$E\left[\sin\left(x_i - X_i\right)\right] = 0.$$

The second derivative of equation (27) is then:

$$\frac{\partial^2 \log L}{\partial \kappa^2} = -n(1+q)A'(\kappa).$$

Therefore,

$$E\left(-\frac{\partial^2 \log L}{\partial \kappa^2}\right) = n(1+q)A'(\kappa).$$

The estimated Fisher information matrix, F, for $\hat{X}_{1,}\ \hat{X}_2, ..., \hat{X}_n,\ \tilde{\kappa}$ and $\hat{\alpha}_j$ is given by:-

$$F = \begin{bmatrix} R & 0 & W \\ 0 & S & 0 \\ W^T & 0 & U \end{bmatrix},$$

where in this study, R is an $n \times n$ matrix of $E\left(-\frac{\partial^2 \log L}{\partial X_i^{\ 2}}\right)$ and $E\left(-\frac{\partial^2 \log L}{\partial X_i\ \partial X_j}\right)$ which is:-

$$R = \begin{bmatrix} (1+q)\tilde{\kappa}A(\tilde{\kappa}) & & 0 \\ & \ddots & \\ 0 & & (1+q)\tilde{\kappa}A(\tilde{\kappa}) \end{bmatrix},$$

W is an $n \times 1$ matrix of $E\left(-\frac{\partial^2 \log L}{\partial X_i\ \partial \alpha_j}\right)$ which is :-

$$W = \begin{bmatrix} \tilde{\kappa}A(\tilde{\kappa}) \\ \vdots \\ \tilde{\kappa}A(\tilde{\kappa}) \end{bmatrix},$$

U is a 1×1 matrix of $E\left(-\frac{\partial^2 \log L}{\partial \alpha_j^{\ 2}}\right)$ which is :-

$$U = \left[n\tilde{\kappa} A(\tilde{\kappa}) \right],$$

and S is a 1×1 matrix of $\mathrm{E}\left(-\dfrac{\partial^2 \log L}{\partial \kappa^2}\right)$ which is:

$$S = \left[n(1+q)A'(\tilde{\kappa}) \right].$$

From the F matrix, we are interested in the bottom right minor of the inverse of F, where it is in the order of 2×2, where the asymptotic covariance matrix of $\hat{\alpha}_j$ and $\tilde{\kappa}$ is formed. From the partitioned matrix, the covariance matrix of $\hat{\alpha}_j$ and $\tilde{\kappa}$ will be of the form

$$\mathrm{cov}\begin{bmatrix} \hat{\alpha}_j \\ \tilde{\kappa} \end{bmatrix} = \begin{bmatrix} S^{-1} & 0 \\ 0 & \left(U - W^T R^{-1} W\right)^{-1} \end{bmatrix},$$

We note that

$$U^{-1} \text{ is } \left(U - W^T R^{-1} W\right)^{-1},$$

where $\left(U - W^T R^{-1} W\right)^{-1} = \left(\dfrac{1+q}{q}\right)\left(\dfrac{1}{n\tilde{\kappa}A(\tilde{\kappa})}\right) = \dfrac{1+q}{qn\tilde{\kappa}A(\tilde{\kappa})}$

and $S^{-1} = \dfrac{1}{n(1+q)A'(\tilde{\kappa})}$.

Thus, the covariance matrix for the model is:

$$\mathrm{cov}\begin{bmatrix} \hat{\alpha}_j \\ \tilde{\kappa} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{n(1+q)A'(\tilde{\kappa})} & 0 \\ 0 & \dfrac{1+q}{qn\tilde{\kappa}A(\tilde{\kappa})} \end{bmatrix}. \tag{28}$$

5. SIMULATION STUDY

A simulation study was carried to assess the accuracy and the bias of the parameters for the proposed model in the case of $q = 2$. The number of simulations is set to be $s = 5000$, throughout, while the values of n and κ for the error terms have been varied. In this model, the value of α_j are circular while κ values are continuous. The following are the measures used to assess the quality of estimation of $\hat{\alpha}_1, \hat{\alpha}_2$ and $\tilde{\kappa}$.

5.1 Estimation of $\hat{\alpha}_j$

Three measurements are used to measure the performance in estimating $\hat{\alpha}_j$ namely the mean of the circular parameter, the circular distance from the true value and the mean resultant length. If the value of the mean resultant length is close to 1, that shows a good accuracy of the circular estimator. The following are the formulations of the measures.

(a) Mean of circular parameter $\bar{\hat{\alpha}}_j$:

$$\bar{\hat{\alpha}}_j = \begin{cases} \tan^{-1}\left(\frac{S}{C}\right) & \text{when } S>0,\ C>0 \\ \tan^{-1}\left(\frac{S}{C}\right)+\pi & \text{when } C<0 \\ \tan^{-1}\left(\frac{S}{C}\right)+2\pi & \text{when } S<0,\ C>0 \end{cases}$$

where $C=\sum_{p=1}^{s}\cos\left(\hat{\alpha}_{jp}\right)$ and $S=\sum_{p=1}^{s}\sin\left(\hat{\alpha}_{jp}\right);\ j=1,\ldots,q;\ p=1,\ldots,s.$

(b) Circular distance, $d=\pi-\left|\pi-\left|\bar{\hat{\alpha}}_j-\alpha_j\right|\right|$

(c) Mean resultant length, $R=\frac{1}{s}\sqrt{\left(\sum_{p=1}^{s}\cos\left(\hat{\alpha}_{jp}\right)\right)^2+\left(\sum_{p=1}^{s}\sin\left(\hat{\alpha}_{jp}\right)\right)^2}$

5.2 Estimation of $\tilde{\kappa}$

The performance measurements for $\tilde{\kappa}$ are mean, estimated bias and estimated root mean square error. The following are the statistics.

(a) Mean of $\tilde{\kappa}$, $\bar{\tilde{\kappa}}=\frac{1}{s}\sum_{p=1}^{s}\tilde{\kappa}_p$

(b) Estimated bias, $EB=\bar{\tilde{\kappa}}-\kappa$

(c) Estimated Root Mean Square Error defined as;-

$$ERMSE=\sqrt{\frac{1}{s}\sum_{p=1}^{s}\left(\tilde{\kappa}_p-\kappa\right)^2}$$

In this study, the value of s is set to be 5000 for each simulation and $q=2$. The values of X have been generated from the Von Mises distribution of $VM\left(\frac{\pi}{4},3\right)$ and the true values of α_1 and α_2 are $\frac{\pi}{4}\approx 0.7854$. The true values of α_1 and α_2 can also be different but in this case we have chosen the values to be equal to make the task easier. In the simulation, the values considered for the concentration parameters of the error term are $\kappa = 5,\ 8,\ 10,\ 15,\ 20,\ 30,\ 50$ and for each value of κ, the sample size are $n = 30$, 70, 100, 150 and 300. Results of the simulation study are presented in Tables 1 to 3.

From Table 1 and Table 2, generally for any fixed *n*, the circular distance for $\hat{\alpha}_1$ and $\hat{\alpha}_2$ gets smaller as we increase the κ value. Similarly, based on mean resultant length, the estimation gets better with increasing κ for a fixed n.

As for $\tilde{\kappa}$, from Table 3, as κ increases, the bias measure increases slowly for any fixed *n*. The same can be said of ERMSE. As n increases, estimated bias shows a modest increase for any fixed κ. For a fixed κ, as n increases, the ERMSE decreases. In summary, the estimation seems to be adequate for almost all values of κ and n considered. These results suggest the adequacy of the parameter estimates of the model.

6. APPLICATION TO REAL DATA

Most weather stations coastal record wave and wind data. Wave buoy data are commonly used to obtain information of the condition of the sea. Also, information on local winds is important in determining whether conditions of coastal waves are safe for surfers. The HF radar, a more expansive and informative method, is also used in harnessing wave data as it gives good temporal coverage. In this paper, we illustrate the applicability of the model using the wind and wave direction data collected from the Humberside coast of the North Sea, United Kingdom. With a sample size of 49, the data of the wind direction (which was measured by HF radar system, developed by UK Rutherford Appleton Laboratories), is addressed as the variable x. Variable y_1 is the data measured by anchored wave buoy and the variable y_2 is for the wave direction data measured by the HF radar. In this illustration, the data is assumed to have equal error variances.

The relationship between the variables x, y_1 and y_2 is given by $y_1 = 0.0925 + x \pmod{2\pi}$ and $y_2 = 6.0749 + x \pmod{2\pi}$ where $\mathrm{var}\left(\hat{\alpha}_j\right) = 2.4484$, $\tilde{\kappa} = 13.6888$ and $\mathrm{var}(\tilde{\kappa}) = 0.0023$. Both variance of $\hat{\alpha}$ and $\tilde{\kappa}$ are rather small, and these indicate good estimation for α_1, α_2 and κ.

7. CONCLUSION

This study proposes Maximum Likelihood Estimation for the parameters in a simultaneous linear functional model for circular data and derives the covariance matrix based on the Fisher information matrix. A simulation study suggests that the parameter estimation is good because the bias becomes smaller as we increase the sample size and the concentration parameter. An application of the method is also given in this paper.

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Table 1
Simulation Result for $\hat{\alpha}_1$ (True Value $\alpha_1 = 0.7854$)

	n	**Circular Mean**	**Circular Distance**	**Mean Resultant Length**
κ =5	30	0.7843	0.0011	0.9919
	70	0.7871	0.0017	0.9965
	100	0.7852	0.0002	0.9976
	150	0.7856	0.0002	0.9984
	300	0.7854	0.0000	0.9992
κ =8	30	0.7870	0.0016	0.9953
	70	0.7838	0.0016	0.9981
	100	0.7859	0.0005	0.9986
	150	0.7847	0.0007	0.9991
	300	0.7852	0.0002	0.9995
κ =10	30	0.7850	0.0004	0.9965
	70	0.7847	0.0007	0.9985
	100	0.7855	0.0001	0.9989
	150	0.7856	0.0002	0.9993
	300	0.7852	0.0002	0.9996
κ =15	30	0.7858	0.0004	0.9976
	70	0.7845	0.0009	0.9990
	100	0.7851	0.0003	0.9993
	150	0.7845	0.0009	0.9995
	300	0.7857	0.0003	0.9998
κ =20	30	0.7842	0.0012	0.9983
	70	0.7858	0.0004	0.9992
	100	0.7860	0.0006	0.9995
	150	0.7853	0.0001	0.9997
	300	0.7855	0.0001	0.9998
κ =30	30	0.7854	0.0000	0.9989
	70	0.7856	0.0002	0.9995
	100	0.7857	0.0003	0.9997
	150	0.7855	0.0001	0.9998
	300	0.7855	0.0001	0.9999
κ =50	30	0.7852	0.0002	0.9993
	70	0.7858	0.0004	0.9997
	100	0.7853	0.0001	0.9998
	150	0.7857	0.0003	0.9999
	300	0.7854	0.0000	0.9999

Table 2
Simulation Result for $\hat{\alpha}_2$ (True Value $\alpha_2 = 0.7854$)

	n	**Circular Mean**	**Circular Distance**	**Mean Resultant Length**
κ =5	30	0.7861	0.0007	0.9920
	70	0.7886	0.0032	0.9966
	100	0.7862	0.0008	0.9976
	150	0.7852	0.0002	0.9984
	300	0.7853	0.0001	0.9992
κ =8	30	0.7856	0.0002	0.9953
	70	0.7840	0.0014	0.9980
	100	0.7861	0.0007	0.9987
	150	0.7854	0.0000	0.9991
	300	0.7852	0.0002	0.9995
κ =10	30	0.7854	0.0000	0.9965
	70	0.7842	0.0012	0.9984
	100	0.7850	0.0004	0.9990
	150	0.7851	0.0003	0.9993
	300	0.7853	0.0001	0.9997
κ =15	30	0.7859	0.0005	0.9977
	70	0.7837	0.0017	0.9990
	100	0.7844	0.0010	0.9993
	150	0.7845	0.0008	0.9996
	300	0.7857	0.0003	0.9998
κ =20	30	0.7851	0.0003	0.9983
	70	0.7850	0.0004	0.9993
	100	0.7855	0.0001	0.9995
	150	0.7854	0.0000	0.9996
	300	0.7857	0.0003	0.9998
κ =30	30	0.7857	0.0003	0.9989
	70	0.7857	0.0003	0.9995
	100	0.7855	0.0001	0.9997
	150	0.7852	0.0002	0.9998
	300	0.7855	0.0001	0.9999
κ =50	30	0.7855	0.0001	0.9993
	70	0.7854	0.0000	0.9997
	100	0.7853	0.0001	0.9998
	150	0.7856	0.0002	0.9999
	300	0.7855	0.0001	0.9999

Table 3
Simulation Result for $\tilde{\kappa}$

	n	**Mean**	**Estimate Bias**	**ERMSE**
κ =5	30	4.9868	-0.0132	1.0972
	70	4.7099	-0.2901	0.7986
	100	4.6642	-0.3358	0.7217
	150	4.6181	-0.3819	0.6477
	300	4.5685	-0.4315	0.5656
κ =8	30	8.2941	0.2941	1.6272
	70	7.9817	-0.0183	1.0262
	100	7.9346	-0.0654	0.8455
	150	7.8564	-0.1436	0.7050
	300	7.8179	-0.1821	0.5225
κ =10	30	10.4789	0.4789	2.0103
	70	10.0778	0.0778	1.2523
	100	10.0058	0.0058	1.0326
	150	9.9340	-0.0660	0.8218
	300	9.8498	-0.1502	0.5864
κ =15	30	15.9079	0.9079	3.1910
	70	15.2226	0.2226	1.8476
	100	15.1279	0.1279	1.5468
	150	14.9858	-0.0142	1.2180
	300	14.9296	-0.0704	0.8576
κ =20	30	21.1870	1.1870	4.2632
	70	20.3943	0.3943	2.4750
	100	20.2111	0.2111	2.0520
	150	20.0639	0.0639	1.6401
	300	19.9512	-0.0488	1.1604
κ =30	30	32.1068	2.1068	6.6107
	70	30.7936	0.7936	3.7440
	100	30.4434	0.4434	3.1461
	150	30.2615	0.2615	2.4566
	300	30.0441	0.0441	1.7087
κ =50	30	53.3213	3.3213	10.8182
	70	51.3679	1.3679	6.3802
	100	50.9333	0.9333	5.1943
	150	50.4800	0.4800	4.1802
	300	50.1248	0.1248	2.8952