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THE TRANSMUTED EXPONENTIAL–WEIBULL DISTRIBUTION WITH APPLICATIONS

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ABSTRACT

A new lifetime distribution is being introduced in this paper. The new distribution is quite flexible for analyzing positive data and has a bathtub-shaped hazard rate function. Some basic statistical functions associated with the proposed distribution are obtained. The parameters of the proposed distribution can be estimated by making use of the maximum likelihood approach. This distribution is fitted to model two lifetime data sets. The proposed distribution is shown to provide a better fit than related distributions as measured by two well-known goodness–of–fit statistics. The proposed distribution may serve as a viable alternative to other distributions available in the literature for modeling positive data arising in various fields of scientific investigation such as the physical and biological sciences, reliability theory, hydrology, medicine, meteorology, survival analysis and engineering.

KEYWORDS

Weibull distribution; Transmuted distribution; Exponential–Weibull distribution; Goodness–of–fit statistics; Lifetime data.

1. INTRODUCTION

The Weibull distribution is a popular life time distribution model in reliability engineering. Since this distribution does not have a bathtub or upside–down bathtub–shaped hazard rate function it cannot be utilized to model the life time of certain systems. To overcome this shortcoming, several generalizations of the classical Weibull distribution have been discussed by different authors in recent years. Many authors introduced flexible distributions for modeling complex data and obtaining a better fit. Extensions of the Weibull distribution arise in different areas of research as discussed for instance in Ghitany *et al.* (2005), Nichol and Padgett (2006), Carrasco *et al.* (2007), Barreto *et al.* (2010), Cordeiro *et al.* (2010), Silva *et al.* (2010), Aryal and Tsokos (2011), Provost *et al.* (2011), Pinho *et al.* (2012), Singla *et al.* (2012), Badmus *et al.* (2013), Cordeiro *et al.* (2013a), Cordeiro *et al.* (2013b), Cordeiro *et al.* (2013c), Cordeiro and Lemonte (2013), Cordeiro *et al.* (2014a), Cordeiro *et al.* (2014b), Cordeiro *et al.* (2014c), Peng and Yan (2014), Saboor *et al.* (2014), Tojeiro *et al.* (2014), Saboor and Pogány (2015) and Saboor *et al.* (2015). Many extended Weibull models have an upside--down bathtub shaped hazard rate, which is the case of the extensions discussed by Jiang and Murthy (1998), Carrasco *et al.* (2008), Nadarajah *et al.* (2011) and Singla *et al.* (2012), among others.

Adding new shape parameters to expand a model into a larger family of distributions to provide significantly skewed and heavy-tails plays a fundamental role in distribution theory. More recently, there has been an increased interest in defining new univariate continuous distributions by introducing additional shape parameters to the baseline model. There has been an increased interest in defining new generators for univariate continuous families of distributions by introducing one or more additional shape parameter(s) to the baseline distribution. This induction of parameter(s) has been proved useful in exploring tail properties and also for improving the goodness-of-fit of the proposed generator family.

In this article, we defined a new family of transmuted exponential Weibull distribution. The main feature of this model is that a transmuted parameter is introduced in the subject distribution which provides greater flexibility in the form of new distributions. Using the quadratic rank transmutation map studied by Shaw and Buckley (2007), we develop the four parameter transmuted exponential Weibull. We provide a comprehensive description of mathematical properties of the subject distribution with the hope that it will attract wider applications in reliability, engineering and in other areas of research. If the baseline distribution has the cumulative density function (cdf) $G(x)$ and pdf $g(x)$, the *transmuted extended* distribution is defined by the cdf and probability density function (pdf) (for $|\alpha| \le 1$).

$$F(x) = (\alpha+1)G(x) - \alpha G(x)^2, \qquad |\alpha| \le 1 . \tag{1}$$

Cordeiro *et al.* (2013a) introduced an exponential-Weibull distribution. The cdf and pdf of their distribution are defined as follow:

$$G(x) = 1 - e^{-\lambda x - \beta x^k} \, 1_{\mathbb{R}_+}(x), \; \lambda > 0, \beta > 0 \, , 0 < k > \infty \, , \tag{2}$$

and

$$g(x) = \left(\lambda + \beta k \, x^{k-1}\right) \mathrm{e}^{-\lambda x - \beta x^k} \, 1_{\mathbb{R}_+}(x) \, , \tag{3}$$

here and in what follows $1_A(x)$ denotes the indicator function of the set *A*, that is $1_A(x) = 1$ when $x \epsilon A$ and equals 0 else.

We further generalize their model by applying the transmuted technique to equations (2) and (3), which defines the so-called transmuted exponential-Weibull (TEW) distribution.

The cdf and pdf of the transmuted exponential–Weibull distribution, for which $G(x)$ is the baseline cdf, are given by

$$F(x) = (\alpha+1)\left(1-e^{-\lambda x-\beta x^k}\right) - \alpha\left(1-e^{-\lambda x-\beta x^k}\right)^2 \, 1_{\mathbb{R}_+}(x)\,, \tag{4}$$

and

$$f(x) = \frac{\left(\lambda x+\beta k x^k\right)\left(e^{\lambda x+\beta x^k}\left(1-\alpha\right)+2\alpha\right)e^{-2\left(\lambda x+\beta x^k\right)}}{x} \, 1_{\mathbb{R}_+}(x)\,, \tag{5}$$

here $\lambda > 0, \beta > 0, k > 0, |\alpha| \le 1$. Accordingly, the four–parameter distribution of the random Variable (rv) X having cdf in the form (4) will signify this correspondence as $X \sim TEW_\alpha\left(\lambda, \beta, k\right)$. In short, we use TEW as a Transmuted variant of the exponential–Weibull distribution. We note that

$$\lim_{x\to 0} f(x) = \infty \text{ and } \lim_{x\to\infty} f(x) = 0$$

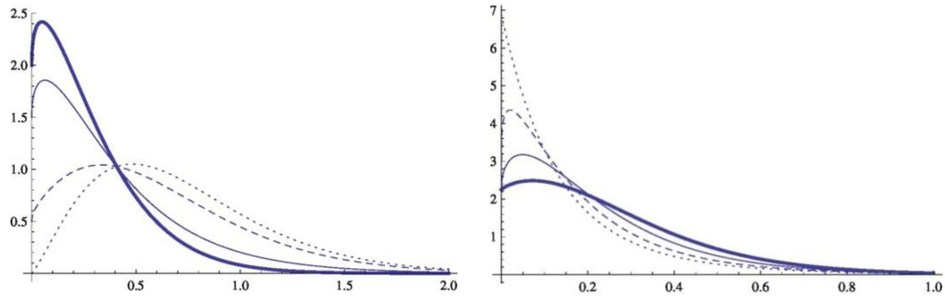

Fig. 1: The TEW pdf.

Left Panel: $\lambda = 1$, $\beta = 1.1$, $k = 1.5$ and $\alpha = -1$ (dotted line), $\alpha = -0.5$ (dashed line), $\alpha = 0.5$ (Solid line), $\alpha = 1$ (thick line).

Right Panel: $\lambda = 1.5$, $\beta = 3.1$, $\alpha = 0.5$, $k = 1$ (dotted line), $k = 1.2$ (dashed line), $k = 1.5$ (Solid line), $k = 1.9$ (thick line).

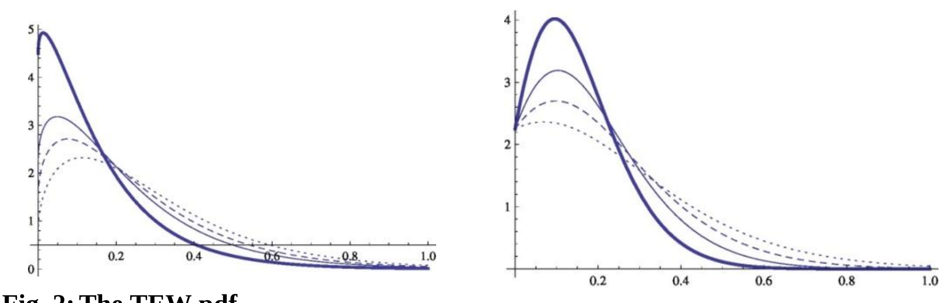

Fig. 2: The TEW pdf.

Left Panel: $\alpha = 0.5$, $\beta = 3.1$, $k = 1.5$ and $\lambda = 0.5$ (dotted line), $\lambda = 1$ (dashed line), $\lambda = 1.5$ (Solid line), $\lambda = 3$ (thick line).

Right Panel: $\lambda = 1.5, k = 2$, $\alpha = 0.5$, $\beta = 3$ (dotted line), $\beta = 5$ (dashed line), $\beta = 8$ (Solid line), $\beta = 14$ (thick line).

The left and right panels of Figure 1 and 2 illustrate that the parameters $\alpha, k,\ \lambda$ and β effect the TEW distribution. As seen from left panel of Figure 1, left and right panel of Figure 2, if the pdf increases α, λ and β increase respectively. As seen from right panel of Figure 1, the pdf decreases when k increases.

The structure of the density function (5) can be motivated as it provides more flexible distribution than the standard two parameter Weibull and many other generalized Weibull distributions. Representations of certain statistical functions are provided in Section 2. The parameter estimation technique described in Section 3 is utilized in connection with the modeling of two actual data sets originating from the engineering and biological sciences in Section 4, where the new model is compared with several related distributions.

2. STATISTICAL FUNCTIONS OF TEW DISTRIBUTION

In this section, we derive computable representations of some statistical functions associated with the TEW distribution whose probability density function is specified by (5). The resulting expressions can be evaluated exactly or numerically with symbolic computational packages such as *Mathematica*, *MATLAB* or *Maple*. In numerical applications, infinite sum can be truncated whenever convergence is observed. We now derive closed form representations of the positive, negative and factorial moments of a TEW random variable. Let us begin with the following *Lemma* (Provost *et al.*, 2011, Saboor *et al.*, 2012).

Lemma 1.

For all $\Re(\eta), \Re(\theta), \Re(s) > 0$ and k is rational number such that $k = p/q$, where p and $q \neq 0$ are integers, we have the following computational representation

$$\int_0^\infty x^{\eta-1} e^{-\theta x^k} e^{-sx} dx = \frac{(2\pi)^{1-(q+p)/2} q^{1/2} p^{r+1/2}}{(s)^\eta} \times G_{p,q}^{q,p}\left(\left(\frac{p}{s}\right)^p \left(\frac{\theta}{q}\right)^q \middle| \begin{matrix} 1-\frac{i+\eta}{p}, & i = 0,1,\ldots,p-1 \\ j/p & j = 0,1,\ldots,p-1 \end{matrix}\right), \tag{6}$$

where the symbol $G_{p,q}^{m,n}(.|.)$ denotes Meijer's G-function (Meijer, 1946). For the definition of the Meijer's G-function, see Appendix A.

Proof:

Now consider the integral on L.H.S of equation (6)

$$\int_0^\infty x^{\eta-1} e^{-\theta x^k} e^{-sx} dx. \tag{7}$$

First, we shall show that (7) is proportional to $h_1\left(\theta^{1/k}\right)$ where $h_1(.)$ denotes the pdf of the ratio of the random variables X_1 and X_2 whose pdf's are

$$g_1\left(x_1\right)=c_1 e^{-x_1^k}, x_1>0$$

and

$$g_2\left(x_2\right)=c_2 e^{s x_2} x_2^{\eta-2}, x_2>0,$$

respectively, c_1 and c_2 being normalizing constants. Let $u=x_1 / x_2$ and $v=x_2$ so that $x_1=uv$ and $x_2=v$, the absolute value of the Jacobian of the inverse transformation being ν. Thus, the joint pdf of the random variables U and V is $v\, g_1(u\, v)\ \ g_2(v)$ and the marginal pdf of $U=X_1 / X_2$ is

$$h_1(u)=\int_0^{\infty} v\, g_1(uv)\, g_2(v)\, dv,$$

that is

$$h_1(u)=c_1 c_2 \int_0^{\infty} e^{-(uv)^k} v\, v^{\eta-2} e^{-sv} dv,$$

which on letting $u=\theta^{1/k}$ and $v=x$, becomes

$$h_1\left(\theta^{1/k}\right)=c_1\, c_2 \int_0^{\infty} e^{-sx}\, x^{\eta-1}\, e^{-\theta x^k}\, dx. \tag{8}$$

Alternatively, the pdf of X_1 / X_2 can be obtained by means of the inverse Mellin transform technique. The required moments of X_1 and X_2 are given below

$$E\left(X_1^{t-1}\right)=c_1 \int_0^{\infty} x_1^{t-1} e^{-x_1^k} dx_1=\frac{c_1}{k} \Gamma(t / k)$$

and

$$E\left(X_2^{1-t}\right)=c_2 \int_0^{\infty} x_2^{\eta-t-1} e^{-s x_2} dx_2=c_2\left(\frac{1}{s}\right)^{\eta-t} \Gamma(\eta-t),$$

provided $\Re(s)>0$ and $\Re(\eta)>0$. The inverse Mellin transform of $U=X_1 / X_2$ is then

$$h_1(u)=\frac{c_1 c_2}{k(s)^\eta} \frac{1}{2 \pi i} \int_c(u / s)^{-t}\, \Gamma(t / k) \Gamma(\eta-t)\, dt, \tag{9}$$

where C denotes an appropriate Bromwich path (Meijer, 1946). Thus, (9) can be expressed as follows in terms of an H–function (Mathai and Sexena, 1978):

$$h_1(u)=\frac{c_1 c_2}{k(s)^\eta} H_{1,1}^{1,1}\left(\frac{u}{s} \middle| \begin{matrix}(1-\eta, 1) \\ (0,1 / k)\end{matrix}\right). \tag{10}$$

Since, (8) is equal to (10) when $u=\theta^{1/k}$, then one has

$$\int_0^{\infty} x^{\eta-1} e^{-\theta x^k} e^{-sx} dx=\frac{1}{k(s)^\eta} H_{1,1}^{1,1}\left(\frac{\theta^{1/k}}{s} \middle| \begin{matrix}(1-\eta, 1) \\ (0,1 / k)\end{matrix}\right). \tag{11}$$

When $k = p/q,\ q \neq 0$, the ratio of two positive integers, the integral on the right–hand side of (9) can be expressed as a Meijer's G-function. Considering $z = t/p$ and making use of the Guass-Legender multiplication formula,

$$\Gamma\left(r+q\,z\right)=\left(2\pi\right)^{\frac{1-q}{2}}\,q^{r+qz-\frac{1}{2}}\prod_{k=0}^{q-1}\Gamma\left(\frac{k+r}{q}+z\right), \tag{12}$$

one has

$$\begin{aligned} h_1\left(u\right) &= \frac{c_1\,c_2\,q}{\left(s\right)^{\eta}}\,\frac{1}{2\pi i}\int_c\left(u/s\right)^{-pz}\,\Gamma\left(q\,z\right)\Gamma\left(\eta-p\,z\right)dz \\ &= \frac{c_1 c_2 q}{\left(s\right)^{\eta}}\,\frac{i}{2\pi i}\int_c\left(u/s\right)^{-pz}\left(2\pi\right)^{\frac{1-q}{2}+\frac{1-p}{2}} \\ &\quad \times q^{q\,z-1/2}\,p^{\eta-pz-1/2}\left\{\prod_{j=0}^{q-1}\Gamma\left(\frac{j}{q}+z\right)\right\}\left\{\prod_{i=0}^{p-1}\Gamma\left(\frac{i+\eta}{p}+z\right)\right\}, \end{aligned}$$

that is

$$\begin{aligned} h_1\left(u\right) &= \frac{c_1 c_2\left(2\pi\right)^{1-\left(p+q\right)/2}\,q^{1/2}\,p^{\eta-1/2}}{\left(s\right)^{\eta}} \\ &\quad \times G^{q,p}_{p,q}\left(\left(\frac{up}{s}\right)^{p} q^{-q}\,\middle|\, \begin{matrix} 1-\frac{i+\eta}{p},\ i=0,1,\dots,p-1 \\ j/q,\ \ j=0,1,\dots,q-1 \end{matrix}\right). \end{aligned} \tag{13}$$

which on considering $u = \theta^{p/q}\left(=\theta^{1/k}\right)$, yields

$$h_1\left(\theta^{1/k}\right)=\frac{c_1 c_2\left(2\pi\right)^{1-\left(p+q\right)/2}\,q^{1/2}\,p^{\eta-1/2}}{\left(s\right)^{\eta}}\;G^{q,p}_{p,q}\left(\left(\frac{p}{s}\right)^{p}\left(\frac{\theta}{q}\right)^{q}\,\middle|\,\begin{matrix} 1-\frac{i+\eta}{p},\ i=0,1,\dots,p-1 \\ j/q,\ \ j=0,1,\dots,q-1 \end{matrix}\right). \tag{14}$$

Since the expressions in (14) and (8) are equal when $k = p/q$ which gives us (6).

Theorem 1.

Let $X \sim TEW_{\alpha}\left(\lambda,\beta,k\right)$. For all $\lambda,\beta,k,\left|\alpha\right|<1$ we have the computational representation real r^{th} order moment of the TEW distribution whose density function is specified by (5) is

$$\begin{aligned} E\left(X^{r}\right) &= 2\alpha\lambda\,\frac{\left(2\pi\right)^{1-\left(p+q\right)/2}\,q^{1/2}\,p^{r+1/2}}{\left(2\lambda\right)^{r+1}} \\ &\quad \times G^{q,p}_{p,q}\left(\left(\frac{p}{2\lambda}\right)^{p}\left(\frac{2\beta}{q}\right)^{q}\,\middle|\,\begin{matrix} 1-\frac{i+r+1}{p},\ i=0,1,\dots,p-1 \\ j/q,\ \ j=0,1,\dots,q-1 \end{matrix}\right) \end{aligned}$$

$$+\lambda(1-\alpha)\frac{(2\pi)^{1-(p+q)/2}\, q^{1/2}\, p^{r+1/2}}{(\lambda)^{r+1}}$$

$$\times G_{p,q}^{q,p}\left(\left(\frac{p}{\lambda}\right)^{p}\left(\frac{\beta}{q}\right)^{q}\middle|\begin{matrix}1-\frac{i+r+1}{p}, i=0,1,\ldots,p-1\\ j/q, \quad j=0,1,\ldots,q-1\end{matrix}\right)$$

$$+k\beta(1-\alpha)\frac{(2\pi)^{1-(p+q)/2}\, q^{1/2}\, p^{r+k-1/2}}{(\lambda)^{r+k}}$$

$$\times G_{p,q}^{q,p}\left(\left(\frac{p}{\lambda}\right)^{p}\left(\frac{\beta}{q}\right)^{q}\middle|\begin{matrix}1-\frac{i+r+k}{p}, i=0,1,\ldots,p-1\\ j/q, \quad j=0,1,\ldots,q-1\end{matrix}\right)$$

$$+2k\alpha\beta\frac{(2\pi)^{1-(p+q)/2}\, q^{1/2}\, p^{r+k-1/2}}{(\lambda)^{r+k}}$$

$$\times G_{p,q}^{q,p}\left(\left(\frac{p}{2\lambda}\right)^{p}\left(\frac{2\beta}{q}\right)^{q}\middle|\begin{matrix}1-\frac{i+r+k}{p}, i=0,1,\ldots,p-1\\ j/q, \quad j=0,1,\ldots,q-1\end{matrix}\right). \tag{15}$$

Proof:

Applying standard formula of real r[th] order moment on (5), we have

$$E\left(X^{r}\right)=\int_{0}^{\infty} x^{r-1}\left(\lambda x+\beta k x^{k}\right)\left(e^{\lambda x+\beta x^{k}}(1-\alpha)+2\alpha\right)e^{-2\left(\lambda x+\beta x^{k}\right)}dx$$

$$=2\alpha\lambda\int_{0}^{\infty} x^{r}e^{-2\beta x^{k}}e^{-2\lambda x}dx+(1-\alpha)\lambda\int_{0}^{\infty} x^{r}e^{-\beta x^{k}}e^{-\lambda x}dx$$

$$+k\beta(1-\alpha)\int_{0}^{\infty} x^{k+r+1}e^{-\beta x^{k}}e^{-\lambda x}dx+2k\beta\alpha\int_{0}^{\infty} x^{k+r+1}e^{-2\beta x^{k}}e^{-2\lambda x}dx. \tag{16}$$

Using *Lemma* 1 and replacing η with $r+1$, s with 2λ and θ with 2β in the first integrand of first integral, η with $r+1$, s with λ and θ with β in the second integrand, η with $r+k$, s with λ and θ with β in the third integrand and η with $r+k$, s with 2λ and θ with 2β in the fourth integrand of fourth integral on the R.H.S of Equation (16), one obtains (15), which finishes the proof.

Consequently by (15), being $|\alpha|\leq 1$, we conclude

$$EX^{0}\ \Big|_{r=0}=1.$$

So the TEW distribution is well defined.

Remark 1:

The h^{th} order negative moment of a rv X can readily be determined by replacing r with $-h$ in (15).

Remark 2:

The factorial moments of a rv X of the positive integer order $\boldsymbol{N} \in \mathbb{N}$ is

$$EX\left(X-1\right)\left(X-2\right)...\left(X-N+1\right)=\sum_{m=0}^{N-1}\phi_m(-1)^j E\left(X^{N-m}\right).$$

Theorem 2:

Let $X \sim TEW_\alpha\left(\lambda,\beta,k\right)$. For all $\lambda,\beta,\left|\alpha\right|\le 1$ and k is rational number such $k=p/q$, p and $q\ne 0$ are integers, we have the computational representation the moment generating function of the TEW distribution whose density function is specified by (5) is

$$\begin{aligned} M(t) &= 2\alpha\lambda\frac{\left(2\pi\right)^{1-\left(p+q\right)/2}q^{1/2}p^{1/2}}{2\lambda-t} \\ &\times G_{p,q}^{q,p}\left(\left(\frac{p}{2\lambda-t}\right)^p\left(\frac{2\beta}{q}\right)^q\middle|\begin{matrix}1-\frac{i+2}{p}, i=0,1,...,p-1\\ j/q,\quad j=0,1,...,q-1\end{matrix}\right) \\ &+\lambda(1-\alpha)\frac{\left(2\pi\right)^{1-\left(p+q\right)/2}q^{1/2}p^{1/2}}{\lambda-t} \\ &\times G_{p,q}^{q,p}\left(\left(\frac{p}{\lambda-t}\right)^p\left(\frac{\beta}{q}\right)^q\middle|\begin{matrix}1-\frac{i+2}{p}, i=0,1,...,p-1\\ j/q,\quad j=0,1,...,q-1\end{matrix}\right) \\ &+2\lambda\beta k\frac{\left(2\pi\right)^{1-\left(p+q\right)/2}q^{1/2}p^{k-1/2}}{(2\lambda-t)^k} \\ &\times G_{p,q}^{q,p}\left(\left(\frac{p}{2\lambda-t}\right)^p\left(\frac{2\beta}{q}\right)^q\middle|\begin{matrix}1-\frac{i+k}{p}, i=0,1,...,p-1\\ j/q,\quad j=0,1,...,q-1\end{matrix}\right) \\ &+k\beta(1-\alpha)\frac{\left(2\pi\right)^{1-\left(p+q\right)/2}q^{1/2}p^{k-1/2}}{(\lambda-t)^k} \\ &\times G_{p,q}^{q,p}\left(\left(\frac{p}{\lambda-t}\right)^p\left(\frac{\beta}{q}\right)^q\middle|\begin{matrix}1-\frac{i+k}{p}, i=0,1,...,p-1\\ j/q,\quad j=0,1,...,q-1\end{matrix}\right). \end{aligned} \tag{17}$$

Proof:

Bearing in mind the formula of *the moment generating function*, one has

$$\mathrm{M}(t)=\int_0^\infty e^{tx}x^{-1}\left(\lambda x+\beta k x^k\right)\left(e^{\lambda x+\beta x^k}(1-\alpha)+2\alpha\right)e^{-2\left(\lambda x+\beta x^k\right)}dx$$

$$= \int_0^\infty e^{tx}\left(2\lambda\alpha e^{-2(\beta x^k+\lambda x)} + (1-\alpha)\lambda\right)e^{-\beta x^k-\lambda x}$$
$$+2\alpha\beta k\, x^{k-1}e^{-2(\beta x^k+\lambda x)} + \beta k(1-\alpha)x^{k-1}e^{-\beta x^k-\lambda x}dx$$
$$= 2\alpha\lambda\int_0^\infty e^{-2\beta x^k}e^{-(2\lambda-t)x}dx + (1-\alpha)\lambda\int_0^\infty e^{-\beta x^k}e^{-(\lambda-t)x}dx$$
$$+2k\beta\alpha\int_0^\infty x^{k-1}e^{-2\beta x^k}e^{-(2\lambda-t)x}dx + k\beta(1-\alpha)\int_0^\infty x^{k-1}e^{-\beta x^k}e^{-(\lambda-t)x}dx. \qquad (18)$$

On replacing η with 1, s with $2\lambda - t$ and θ with 2β in the integrand of first integral, η with 1, s with λ-t and θ with β in the integrand of second integral, η with k, s with $2\lambda - t$ and θ with 2β in the integrand of third integral and η with k, s with λ and θ with β in the integrand of fourth integral and making use of (15), when $k = p/q$, which gives (17).

Here, we will discuss some other significant statistical properties corresponding to (5) i.e. *the mean residual life function, survival function, hazard rate function, mean deviation* and *quantile function.*

Central role is playing in the reliability theory by the quotient of the probability density function and the survival function. The survival function of the TEW distribution whose density function is specified by (5) is

$$S(x) = 1-(\alpha+1)\left(1-e^{-\lambda x-\beta x^k}\right) - \alpha\left(1-e^{-\lambda x-\beta x^k}\right)^2 1_{\mathbb{R}_+}(x). \qquad (19)$$

The *hazard function* (or also frequently called failure rate function) of the TEW distribution whose density function is specified by (5) is

$$h(x) = \frac{f(x)}{1-F(x)} = \frac{\left(\lambda+\beta k x^{k-1}\right)\left(e^{\lambda x+\beta x^k}(1-\alpha)+2\alpha\right)e^{-2\left(\lambda x+\beta x^k\right)}}{x\left(\lambda+e^{\lambda x+\beta x^k}\left(-1-3\alpha+2e^{\lambda x+\beta x^k}(1+\alpha)\right)\right)}1_{\mathbb{R}_+}(x). \qquad (20)$$

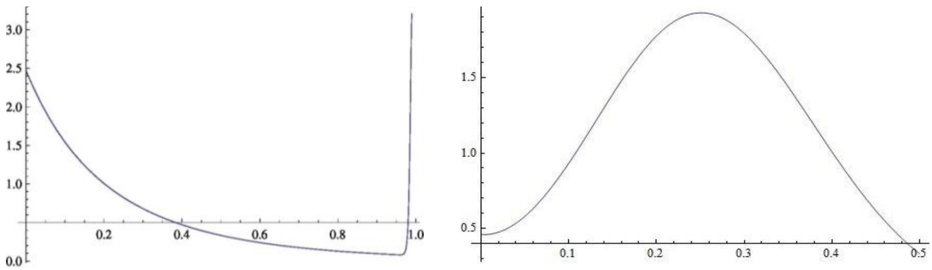

Fig. 3: The TEW Hazard Rate Function.
Left Panel: λ= 1:3, β= 10, k = 300, α= 0:9.
Right Panel: $\lambda = 0.23, \beta = 10, k = 3, \alpha = 1.$

Figure 3 represent the bathtub-shaped and upside-down bathtub-shaped hazard rate functions.

The *mean residual life function* is defined as

$$K(x)=\frac{1}{S(x)}\int_x^\infty (y-x)f(y)dy=\frac{1}{S(x)}\int_x^\infty yf(y)dy-x$$
$$=\frac{1}{S(x)}\left[E(Y)-\int_0^x yf(y)dy\right]-x,$$

where $S(x)$, $f(y)$ and $E(Y)$ are given in (19), (5) and (15), respectively and

$$\int_0^x yf(y)dy=\int_0^x y\frac{\left(\lambda y+\beta k y^{k-1}\right)\left(e^{\lambda y+\beta y^k}(1-\alpha)+2\alpha\right)e^{-2\left(\lambda y+\beta y^k\right)}}{y}dy$$
$$=\int_0^x y\left(\lambda+\beta k y^{k-1}\right)\left(e^{\lambda y+\beta y^k}(1-\alpha)+2\alpha\right)e^{-2\left(\lambda y+\beta y^k\right)}dy$$
$$=\int_0^x y\left[\left(\lambda+\beta k y^{k-1}\right)e^{\lambda y+\beta y^k}(1-\alpha)\right]dy+\int_0^x y\left[\left(\lambda+\beta k y^{k-1}\right)2\alpha e^{-2\left(\lambda y+\beta y^k\right)}\right]dy.$$

By expending exponential in the last expression, one has the following expression

$$\int_0^x yf(y)dy=(1-\alpha)\sum_{j=0}^{\infty}\frac{(-1)^j\lambda^j}{j!}\int_0^x y\left(\lambda+\beta k y^{k-1}\right)e^{-\beta y^k}dy$$
$$+2\alpha\sum_{j=0}^{\infty}\frac{(-1)^j(2\lambda)^j}{j!}\int_0^x y\left(\lambda+\beta k y^{k-1}\right)e^{-2\beta y^k}dy$$
$$=(1-\alpha)\sum_{j=0}^{\infty}\frac{(-1)^j\lambda^j}{j!}\lambda\int_0^x yG_{0,1}^{1,0}\left(\beta y^{p/q}\Big|_{0}^{-}\right)dy$$
$$+\beta\frac{p}{q}\int_0^x y^{p/q}G_{0,1}^{1,0}\left(\beta y^{p/q}\Big|_{0}^{-}\right)dy$$
$$+2\alpha\sum_{j=0}^{\infty}\frac{(-1)^j(2\lambda)^j}{j!}\lambda\int_0^x G_{0,1}^{1,0}\left(2\beta y^{p/q}\Big|_{0}^{-}\right)dy$$
$$+\beta\frac{p}{q}\int_0^x y^{p/q}G_{0,1}^{1,0}\left(2\beta y^{p/q}\Big|_{0}^{-}\right)dy, \tag{21}$$

where $e^{-g(x)}=G_{0,1}^{1,0}\left(g(x)\Big|_{0}^{-}\right)$, $k=p/q$, $p\ge 0$, $q\ge 0$ are natural co-prime numbers and

$$\int_0^x y^t G_{0,1}^{1,0}\left((\mathrm{n}+1)\beta y^{p/q}\Big|_{0}^{-}\right)dy$$

$$= \frac{qx^{p(t+1)}}{p(2\pi)^{(q-1)/2}} G_{p,p+q}^{q,p}\left(\left(\frac{((n+1)\beta)^{q} x^{p}}{q^{q}}\right)\Bigg|_{0, \frac{-t-1}{p}, \frac{-t}{p}, \ldots \frac{p-t-2}{p}}^{\frac{-t}{p}, \frac{1-t}{p}, \ldots \frac{p-t-1}{p} \quad -}\right). \qquad (22)$$

Equation (21) is obtained by making use of Equation (13) of Cordeiro *et al.* (2014a).

The *mean deviation* about the mean is defined by

$$\delta(x) = \int_0^\infty |x - E(X)| f(x)dx = \int_0^{E(X)} (E(X) - x) f(x)dx = \int_{E(X)}^\infty (x - E(X)) f(x)dx. \qquad (23)$$

where $E(X)$ can be evaluated by letting $r = 1$ in (15). The mean deviation can easily be evaluated by numerical integration.

The *quantile function* is very useful to obtain various mathematical properties of distributions and it is in widespread use in general statistics. For some cases, it is possible to invert the cdf as that one given by (4). However, for some other distributions, the solution is not possible. Power series methods are at the heart of many aspects of applied mathematics and statistics. To obtain the quantile function of X

$$Q_X(p) = \inf\{x \in \mathrm{R} : p \le F(x)\}, \; p \in (0,1),$$

we have to invert the equation $F(x) = p$ for some $p \in (0,1)$ with respect to x setting

$$A = 1 - e^{-\lambda x - \beta x^k},$$

the problem reduces to the quadratic equation $\alpha A^2 - (1+\alpha)A + p = 0$. Thus,

$$A_{1,2} = \frac{1+\alpha \pm \sqrt{(1+\alpha)^2 - 4p\alpha}}{2\alpha},$$

that is, we look for an explicit solution $x = Q_X(p)$ from the nonlinear equation

$$1 - e^{-\lambda x - \beta x^k} = A_{1,2}.$$

However, since the left-hand side of this equation is less than one, not both solutions $A_{1,2}$ are satisfactory in this model. Actually, we have the restriction $A < 1$ on the whole range of parameters $\min(\lambda, \beta, k) > 0$ in conjunction with $p \in (0,1)$. Since

$$A_1 - 1 = \frac{1-\eta+\sqrt{(1+\alpha)^2 - 4p\alpha}}{2\alpha} < \frac{1-\alpha+1+\alpha}{2\alpha} = \frac{1}{\alpha}, \; \alpha > 0,$$

and

$$A_2 - 1 = \frac{1-\alpha+\sqrt{(1+\alpha)^2 - 4p\alpha}}{2\alpha} < \frac{1-\alpha-(1-\alpha)}{2\alpha} = 0, \; \alpha > 0,$$

we obtain

$$1-e^{-\lambda x-\beta x^{k}} = A = \frac{1-\alpha-\sqrt{(1+\alpha)^{2}-4p\alpha}}{2\alpha},$$

where x is the solution of $\lambda x+\beta x^{k}+\ln(1-A)=0$. Let $\mu=$ of $\lambda x+\beta x^{k}$. By using Taylor series expansion, one gets $\sum_{j=0}^{\infty}(k)_{j}(x-1)^{j}/j! = \sum_{h=0}^{\infty}a_{h}x^{h}$ where $a_{h}=\sum_{h=0}^{\infty}(-1)^{j-h}j!(k)_{j}\big/\left[h!(j-h)!j!\right]$ and $(k)_{j}=k(k-1)(k-2)\ldots(k-j+1)$ is the descending factorial. Hence,

$$\mu=\sum_{h=0}^{\infty}b_{h}x^{h},$$

where $b_{0}=\beta a_{0}$, $b_{1}=\beta a_{1}-+\lambda$ and $b_{h}=a_{h}$ for $h\geq 2$. If $b_{1}\neq 0$, we can invert the last power series and obtain [7, Equation 25.2.60] after elementary, but tedious rearrangements,

$$x=Q(p)=\sum_{h=1}^{\infty}c_{h}z^{h}$$

where $c_{1}=b_{1}^{-1}$, $c_{2}=-b_{2}b_{1}^{-3}$, $c_{3}=\left(2b_{2}^{2}-b_{1}b_{3}\right)b_{1}^{-5}$, $c_{4}=\left(5b_{1}b_{2}b_{3}-b_{1}^{2}b_{2}-5b_{2}^{3}\right)b_{1}^{-7}$ and so on.

3. PARAMETER ESTIMATION

In this section, we will make use of the two parameter gamma (Gamma), two parameter Weibull (Weibull), the gamma exponentiated exponential (GEE) (Ristić and Balakrishnan, 2012), exponential-Weibull (EW) (Cordeiro *et al.* 2014a), extended Weibull (ExtW) (Peng and Yan, 2014), Kumaraswamy modified Weibull (Cordeiro *et al.* 2014c) (KwMW) (2014) and the TEW distributions to model two well–known real data sets, namely the Carbon fibres (2006) and the Cancer patients (2003) data sets. The parameters of the TEW distribution can be estimated from the maximum loglikelihood estimation method of the sample in conjunction with the *N Maximize* command in the symbolic computational package *Mathematica*. Additionally, two goodness-of-fit measures are proposed to compare the density estimates.

3.1 Maximum Likelihood Estimation

In order to estimate the parameters of the proposed TEW distribution as specified by the density function appearing in Equation (5), the loglikelihood of the sample is maximized with respect to the parameters by making use of the *NMaximize* command in the symbolic computational package *Mathematica*. Given the data $x_{i}, i=1,2,\ldots,n$ the loglikelihood function is given by

$$\ell\left(\lambda,\beta,k,\alpha\right)=\sum_{i=1}^{n}\log\left(2\alpha+(1-\alpha)\,e^{\lambda x_i+\beta x_i^{k}}\right)+\sum_{i=1}^{n}\log e^{-2\left(\lambda x_i+\beta x_i^{k}\right)}$$
$$+\sum_{i=1}^{n}\log\left(\lambda x_i+\beta k x_i^{k}\right)+\sum_{i=1}^{n}\log\left(x_i^{-1}\right). \qquad (24)$$

where $f\left(x\right)$ is given in (5). The associated nonlinear likelihood function $\frac{\partial\ell(\theta)}{\partial\theta}=0$ for MLE estimator derivation reads as follow:

$$\frac{\partial\ell(\theta)}{\partial\lambda}=\sum_{i=1}^{n}-2x_i+\sum_{i=1}^{n}\frac{e^{\lambda x_i+\beta x_i^{k}}\left(\alpha-1\right)x_i}{e^{\lambda x_i+\beta x_i^{k}}\left(\alpha-1\right)+2\alpha}+\sum_{i=1}^{n}\frac{x_i}{\lambda x_i+k\beta x_i^{k}}=0$$

$$\frac{\partial\ell(\theta)}{\partial\beta}=\sum_{i=1}^{n}-2x_i^{k}+\sum_{i=1}^{n}\frac{e^{\lambda x_i+\beta x_i^{k}}\left(\alpha-1\right)x_i^{k}}{e^{\lambda x_i+\beta x_i^{k}}\left(\alpha-1\right)+2\alpha}+\sum_{i=1}^{n}\frac{kx_i^{k}}{\lambda x_i+k\beta x_i^{k}}=0$$

$$\frac{\partial\ell(\theta)}{\partial k}=\sum_{i=1}^{n}-2\beta x_i^{k}\log\left(x_i\right)+\sum_{i=1}^{n}-\frac{\left(\alpha-1\right)\beta x_i^{k}\log\left(x_i\right)e^{\beta x_i^{k}+\lambda x_i}}{2\alpha-\left(\alpha-1\right)e^{\beta x_i^{k}+\lambda x_i}}+\sum_{i=1}^{n}\frac{\beta x_i^{k}+k\beta x_i^{k}\log\left(x_i\right)}{k\beta x_i^{k}+\lambda x_i}=0$$

$$\frac{\partial\ell(\theta)}{\partial\alpha}=\sum_{i=1}^{n}\frac{\left(2-e^{\lambda x_i+\beta x_i^{k}}\right)}{\left(e^{\lambda x_i+\beta x_i^{k}}\left(\alpha-1\right)+2\alpha\right)}=0\,.$$

Solving the above equations simultaneously produce the maximum likelihood estimates of the four parameters. For estimating the parameters, one can use the numerical iterative techniques. The global maxima of the log-likelihood can be investigated by setting different starting values for the parameters. The information matrix will be required for interval estimation. The 4×4 total observed information matrix along with elements $J(\theta)=Jrs(\theta)$ for $r;s=\lambda,\beta,k,\alpha$ are given in Appendix B. Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}\left(\hat{\theta}-\theta\right)$ is $N_4\left(O,K(\theta)^{-1}\right)$, where $K(\theta)=E\left\{J(\theta)\right\}$ is the expected information matrix. The approximate multivariate normal $N_4\left(O,K(\theta)^{-1}\right)$ distribution, where $J(\theta)^{-1}$ is the observed information matrix evaluated at $\theta=\hat{\theta}$, can be used to construct approximate confidence intervals and confidence regions for the individual parameters and for the hazard and survival functions. An asymptotic confidence interval (ACI) with significance level γ for each parameter θ_r is given by

$$ACI\left(\theta_r,100(1-\gamma)\%\right)=\left(\hat{\theta}_r-z_{\gamma/2}\sqrt{\hat{\kappa}^{\theta_r,\theta_r}},\hat{\theta}_r+z_{\gamma/2}\sqrt{\hat{\kappa}^{\theta_r,\theta_r}}\right),$$

Where $\hat{\kappa}^{\theta_r,\theta_r}$ is the rth diagonal element of $J(\theta)^{-1}$ estimated at $\hat{\theta}$ and $z_{\gamma/2}$ is the quantile $1-\gamma/2$ of the standard normal distribution.

3.2 Goodness-of-Fit Statistics

The Anderson-Darling test statistics (Anderson and Darling, 1952) and Cramér-von-Mises test statistics (Cramér, 1928 and Von-Mises, 1928) are widely utilized to determine how closely a specific distribution whose associated cumulative distribution function denoted by cdf (·) fits the empirical distribution associated with a given data set. The smaller these statistics are, the better the fit. Upper tail percentiles of the asymptotic distributions of these goodness–of–fit statistics were tabulated in Stephens (1976).

4. APPLICATION

In this section, we present two applications where the TEW model is compared with other related models. We make use of two data sets: first, the Carbon fibres data set (Nichols and Padgett, 2006) and, secondly, the Cancer patient's data set (Lee and Wang, 2003).

- The classical gamma (Gamma) distribution with density function

$$f(x)=\frac{x^{\xi-1}e^{-x/\phi}}{\phi^{\xi}\Gamma(\xi)},\ x>0,\xi>0,\phi>0$$

- The classical Weibull (Weibull) distribution with density function

$$f(x)=\frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}e^{-(x/\lambda)^k},\ x>0,k>0,\lambda>0\ .$$

- The gamma exponentiated exponential (GEE) distribution with density function

$$f(x)=\frac{\lambda\alpha^{\delta}\left(1-\mathrm{e}^{-\lambda x}\right)^{\alpha-1}\left(-\log\left(1-\mathrm{e}^{-\lambda x}\right)\right)^{\delta-1}}{\Gamma(\delta)},\ x,\lambda,\alpha,\delta>0\ .$$

- The exponential-Weibull (EW) distribution with density function

$$f(x)=\left(\lambda+\beta kx^{k-1}\right)\mathrm{e}^{-\lambda x-\beta x^k},\ x,\lambda,\beta,k>0\ .$$

- The extended Weibull (ExtW) distribution with density function

$$f(x)=a\left(\mathrm{c}+\mathrm{b}\,x\right)x^{b-2}e^{-c/x-ax^{b}e^{-c/x}},\ x,a,b>0,c\geq 0.$$

- The Kumarswamy modified Weibull (KwMW) distribution with density function

$$f(x)=ab\alpha x^{\gamma-1}\left(\gamma+\lambda x\right)\exp\left(\lambda x-\alpha x^{\gamma}\,\mathrm{e}^{\lambda x}\right)\left(1-\exp\left(-\alpha x^{\gamma}\,\mathrm{e}^{\lambda x}\right)\right)^{\alpha-1}$$
$$\times\left(1-\left(1-\exp\left(-\alpha x^{\gamma}\,\mathrm{e}^{\lambda x}\right)\right)^{\alpha}\right)^{b-1},\ x,a,b,\gamma,\alpha>0,\lambda\geq 0\ .$$

4.1. The Carbon Fibres Data Set

We shall consider the uncensored real data set on the breaking stress of carbon fibres (in Gba) as reported in Nichols and Padgett (2006).

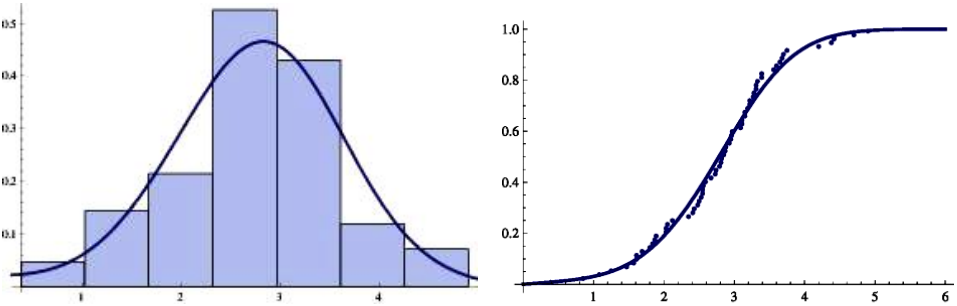

Fig. 4: Left Panel: The TEW density estimates superimposed on the histogram for Carbon fibres data.
Right Panel: The TEW cdf estimates and empirical cdf.

Table 1
Estimates of Parameters (Standard Errors in Parenthesis) and Goodness-of-Fit Statistics for Carbon Fibres Data

Distributions	Estimates					A_0^2	W_0^2
Gamma (ξ, ϕ)	7.48803 (1.27552)	0.368528 (0.0649272)	--	--	--	1.32674	0.248153
Weibull (k, λ)	3.4412 (0.330936)	47.0505 (20.1189)	--	--	--	0.49167	0.084301
GEE $(\lambda, \alpha, \delta)$	0.26555 (0.216206)	10.0365 (2.59504)	7.23658 (7.05288)	--	--	1.43415	0.266823
EW (k, λ, β)	3.73666 (0.445755)	0.0170948 (0.0213386)	0.0140172 (0.00845419)	--	--	0.40364	0.06479
ExtW (a, b, c)	16.1979 (25.7118)	1×10^{-7} (0.938764)	8.05671 (1.65309)	--	--	2.26745	0.416152
KwMW $(\alpha, \gamma, \lambda, a, b)$	0.14981 (0.326517)	1.7994 (2.40813)	0.49987 (0.616749)	0.64975 (1.13328)	0.17111 (0.529126)	1.29338	0.213215
TEW $(\lambda, \beta, k, \alpha)$	0.012974 (0.0137694)	0.005819 (0.00397957)	4.111803 (0.506705)	0.672444 (0.371294)	--	**0.33372**	**0.05325**

4.2. The Cancer Patients Data Set

The second data set represents the remission times (in months) of a random sample of 128 bladder cancer patients as reported in Lee and Wang (2003).

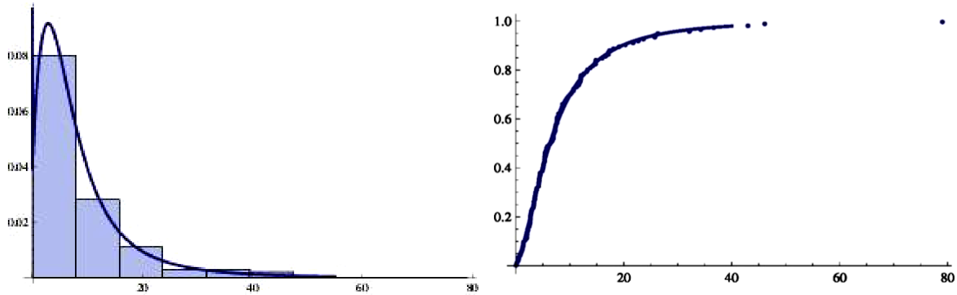

Fig 5: **Left Panel:** The TEW density estimates superimposed on the histogram for Cancer patient's data.
Right Panel: The TEW cdf estimates and empirical cdf.

Table 2
Estimates of Parameters (Standard Errors in Parenthesis) and Goodness-of-Fit Statistics for Bladder Cancer Patients Data

Distributions	Estimates					A_0^2	W_0^2
Gamma (ξ, ϕ)	1.17251 (0.245079)	7.98766 (0.895546)	--	--	--	0.77625	0.136063
Weibull (k, λ)	1.04783 (0.0675775)	10.651 (2.16445)	--	--	--	0.96345	0.154303
GEE $(\lambda, \alpha, \delta)$	0.121167 (0.106783)	1.21795 (0.187678)	1.00156 (0.865881)	--	--	0.71819	0.128403
EW (k, λ, β)	1.04783 (0.314243)	1 x 10^{-7} (0.301314)	0.0938877 (0.117931)	--	--	0.96345	0.154303
ExtW (a,b,c)	1.9621 (0.708999)	1 x 10^{-21} (0.138443)	3.74383 (0.389542)	--	--	13.3317	2.49818
KwMW $(\alpha, \gamma, \lambda, a, b)$	0.639622 (0.116828)	0.381865 (0.064379)	0.029602 (0.00458728)	0.322842 (0.0763509)	0.37499 (0.0594068)	18.8864	3.68568
TEW $(\lambda, \beta, k, \alpha)$	1.087x10^{-10} (0.0784429)	0.047836 (0.0721666)	1.133310 (0.144136)	0.744922 (0.202475)	--	**0.56339**	**0.08825**

5. DISCUSSION

There has been a growing interest among statisticians and applied researchers in constructing flexible lifetime models in order to improve the modeling of survival data. As a result, significant progress has been made towards the generalization of some well–known lifetime models, which have been successfully applied to problems arising in several areas of research. In particular, several authors proposed new distributions that are based on the traditional Weibull model. In this paper, we introduce a four–parameter distribution which is obtained by applying the transmuted technique to the exponential–Weibull model. Interestingly, our proposed model has bathtub-shaped hazard rate function. We studied some of its mathematical and statistical properties. We also provided a computable representation of the positive and negative moments, the factorial moments, the moment generating function, the mean residue life function and the mean deviation. The proposed distribution was applied to two data sets and shown to provide a better fit than other related models. The distributional results developed in this article should find numerous applications in the physical and biological sciences, reliability theory, hydrology, medicine, meteorology and engineering and survival analysis.

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APPENDIX A

Meijer G-Function

The symbol $G_{p,q}^{m,n}(.|.)$ denotes Meijer's *G*-function (Meijer, 1946) defined in terms of the Mellin–Barnes integral as

$$G_{\mathrm{p,q}}^{\mathrm{m,n}}\left(z\left|\begin{matrix}a_1,\ldots,\mathrm{a_p}\\ b_1,\ldots,\mathrm{b_p}\end{matrix}\right.\right)=\frac{1}{2\pi i}\int_c\frac{\prod_{j=1}^{m}\Gamma\left(\mathrm{b_j}-\mathrm{s}\right)\prod_{j=1}^{n}\Gamma\left(1-\mathrm{a_j}+\mathrm{s}\right)}{\prod_{j=m+1}^{q}\Gamma\left(1-\mathrm{b_j}-\mathrm{s}\right)\prod_{j=n+1}^{p}\Gamma\left(\mathrm{a_j}-\mathrm{s}\right)}z^s ds,$$

where $0<\mathrm{m}<\mathrm{q}, 0<\mathrm{n}<\mathrm{p}$ and the poles a_j, b_j are such poles $\Gamma\left(b_j-s\right).j=\overline{1.m}$ coincide with the $\Gamma\left(1-a_j-s\right).j=\overline{1.n}$ i.e. $a_k-b_j\notin\mathbb{Z}$ while $z\neq 0$. c is a suitable integration contour which start at $-\mathrm{i}\infty$ and goes to $i\infty$ and separate the poles $\Gamma\left(b_j-s\right).j=\overline{1.m}$ which lie to the right of the contour, from all poles of $\Gamma\left(1-a_j-s\right).j=\overline{1.n}$ which lie to the left of c. The integral converges if $\delta=m+n-\frac{1}{2}(\mathrm{p}+\mathrm{q})>0$ and $\left|\arg(z)\right|<\delta\pi$, see (Luke, 1969 p. 143) and Meijer (1946).

The G function's Mathematica code reads.

$$MeijerG\left[\left\{\left\{a_1,\ldots,a_n\right\}\left\{a_{n+1},\ldots,a_p\right\}\left\{b_1,\ldots,b_m\right\}\left\{b_{m+1},\ldots,b_q\right\}\right\},z\right].$$

APPENDIX B

The 4×4 total observed information matrix along with the elements are given below

$$J(\theta)=\begin{bmatrix} J_{\lambda\lambda}(\theta) & J_{\lambda\beta}(\theta) & J_{\lambda k}(\theta) & J_{\lambda\alpha}(\theta) \\ J_{\beta\lambda}(\theta) & J_{\beta\beta}(\theta) & J_{\beta k}(\theta) & J_{\beta\alpha}(\theta) \\ J_{k\lambda}(\theta) & J_{k\beta}(\theta) & J_{kk}(\theta) & J_{k\alpha}(\theta) \\ J_{\alpha\lambda}(\theta) & J_{\alpha\beta}(\theta) & J_{\alpha k}(\theta) & J_{\alpha\alpha}(\theta) \end{bmatrix}$$

$$J_{\lambda\lambda}(\theta)=\sum_{i=1}^{n}\left(-\frac{e^{2\lambda x_i+2\beta x_i^k}\left(-1+\alpha\right)^2 x_i^2}{\left(-e^{\lambda x_i+\beta x_i^k}\left(-1+\alpha\right)+2\alpha\right)^2}+\frac{e^{\lambda x_i+\beta x_i^k}\left(-1+\alpha\right)x_i^2}{-e^{\lambda x_i+\beta x_i^k}\left(-1+\alpha\right)+2\alpha}\right)$$

$$+\sum_{i=1}^{n}-\frac{x_i^2}{\left(\lambda x_i+k\beta x_i^k\right)^2},$$

$$J_{\lambda\beta}(\theta)=\sum_{i=1}^{n}-\frac{kx_i^{1+k}}{\left(\lambda x_i+k\beta x_i^k\right)^2}$$

$$+\sum_{i=1}^{n}\left(-\frac{e^{2\lambda x_i+2\beta x_i^k}\left(-1+\alpha\right)^2 x_i^{1+k}}{\left(-e^{\lambda x_i+\beta x_i^k}\left(-1+\alpha\right)+2\alpha\right)^2}-\frac{e^{\lambda x_i+\beta x_i^k}\left(-1+\alpha\right)x_i^{1+k}}{-e^{\lambda x_i+\beta x_i^k}\left(-1+\alpha\right)+2\alpha}\right),$$

$$J_{\lambda k}(\theta)=\sum_{i=1}^{n}\left(-\frac{e^{2\lambda x_i+2\beta x_i^k}\left(-1+\alpha\right)^2\beta\log(x_i)x_i^{1+k}}{\left(-e^{\lambda x_i+\beta x_i^k}\left(-1+\alpha\right)+2\alpha\right)^2}-\frac{e^{\lambda x_i+\beta x_i^k}\left(-1+\alpha\right)\beta\log(x_i)x_i^{1+k}}{-e^{\lambda x_i+\beta x_i^k}\left(-1+\alpha\right)+2\alpha}\right)$$

$$\times\sum_{i=1}^{n}-\frac{x_i\left(\beta x_i^k+k\beta\log\left[x_i\right]x_i^k\right)}{\left(\lambda x_i+k\beta x_i^k\right)^2},$$

$$J_{\lambda\alpha}(\theta)=\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i+\beta x_i^k}\left(2-e^{\lambda x_i+\beta x_i^k}\right)\left(-1+\alpha\right)x_i}{\left(-e^{\lambda x_i+\beta x_i^k}\left(-1+\alpha\right)+2\alpha\right)^2}-\frac{e^{\lambda x_i+\beta x_i^k}x_i}{-e^{\lambda x_i+\beta x_i^k}\left(-1+\alpha\right)+2\alpha}\right),$$

$$J_{\beta\beta}(\theta)=\sum_{i=1}^{n}-\frac{k^2x_i^{2k}}{\left(\lambda x_i+k\beta x_i^k\right)^2}+\sum_{i=1}^{n}\left(-\frac{e^{2\lambda x_i+2\beta x_i^k}\left(-1+\alpha\right)^2x_i^{2k}}{\left(-e^{\lambda x_i+\beta x_i^k}\left(-1+\alpha\right)+2\alpha\right)^2}-\frac{e^{\lambda x_i+\beta x_i^k}\left(-1+\alpha\right)x_i^{2k}}{-e^{\lambda x_i+\beta x_i^k}\left(-1+\alpha\right)+2\alpha}\right),$$

$$J_{\beta k}(\theta)=\sum_{i=1}^{n}\left(-\frac{e^{\lambda x_i+\beta x_i^k}\left(-1+\alpha\right)\log\left[x_i\right]x_i^k}{-e^{\lambda x_i+\beta x_i^k}\left(-1+\alpha\right)+2\alpha}-\frac{e^{2\lambda x_i+2\beta x_i^k}\left(-1+\alpha\right)^2\beta\log\left[x_i\right]x_i^{2k}}{\left(-e^{\lambda x_i+\beta x_i^k}\left(-1+\alpha\right)+2\alpha\right)^2}\right)$$

$$+\sum_{i=1}^{n}\left(\frac{x_i^k}{\lambda x_i+k\beta x_i^k}+\frac{k\log\left[x_i\right]x_i^k}{\lambda x_i+k\beta x_i^k}-\frac{kx_i^k\left(\beta x_i^k+k\beta\log(\mathrm{v})x_i\right)}{\left(\lambda x_i+k\beta x_i^k\right)^2}\right)$$

$$+\sum_{i=1}^{n}-2\log(x_i)x_i^k+\sum_{i=1}^{n}\left(-\frac{e^{\lambda x_i+\beta x_i^k}\left(-1+\alpha\right)\beta\log\left[x_i\right]x_i^{2k}}{-e^{\lambda x_i+\beta x_i^k}\left(-1+\alpha\right)+2\alpha}\right),$$

$$J_{kk}(\theta)=\sum_{i=1}^{n}\left(-\frac{e^{\lambda x_i+\beta x_i^k}\log\left(-1+\alpha\right)\beta\log\left[x_i\right]x_i^{1+k}}{-e^{\lambda x_i+\beta x_i^k}\left(-1+\alpha\right)+2\alpha}-\frac{e^{2\lambda x_i+2\beta x_i^k}\log\left(-1+\alpha\right)^2\beta^2\log\left[x_i\right]x_i^{1+2k}}{\left(-e^{\lambda x_i+\beta x_i^k}\left(-1+\alpha\right)+2\alpha\right)^2}\right)$$

$$+\sum_{i=1}^{n}\left(-\frac{e^{\lambda x_i+\beta x_i^k}\log\left(-1+\alpha\right)\beta^2\log\left[x_i\right]x_i^{1+2k}}{-e^{\lambda x_i+\beta x_i^k}\left(-1+\alpha\right)+2\alpha}\right)+\sum_{i=1}^{n}-2\log\beta\log(x_i)x_i^{1+k}$$

$$J_{k\alpha}(\theta)=\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i+\beta x_i^k}\left(2-e^{\lambda x_i+\beta x_i^k}\right)\log\left(-1+\alpha\right)\beta x_i^{1+k}}{\left(-e^{\lambda x_i+\beta x_i^k}\left(-1+\alpha\right)+2\alpha\right)^2}-\frac{e^{\lambda x_i+\beta x_i^k}\log\beta x_i^{1+k}}{-e^{\lambda x_i+\beta x_i^k}\left(-1+\alpha\right)+2\alpha}\right),$$

$$J_{\alpha\alpha}(\theta)=\sum_{i=1}^{n}-\frac{\left(2-e^{\lambda x_i+\beta x_i^k}\right)^2}{\left(-e^{\lambda x_i+\beta x_i^k}\left(-1+\alpha\right)+2\alpha\right)^2}.$$