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SOME PROPERTIES OF GENERALIZED LOG PEARSON TYPE VII DISTRIBUTION

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ABSTRACT

In this paper a generalized log Pearson (GLP) type VII distribution is proposed. Some properties of the new distribution are investigated. Characterization of GLP type VII distribution is presented through conditional expectation. Finally, some compound scale mixtures for GLP type VII distribution are discussed and its application on real data is shown.

1. INTRODUCTION

Pearson (1895, 1905, 1916) introduced a system of probability distributions useful in engineering and biological sciences, econometrics, survey sampling and life-testing. Pearson (1895) used the fact that the limiting case of the hypergeometric distribution leads to the Pearson system of continuous distribution functions. Pearson models are used in financial markets and capture the stochastic nature of the volatility of rates and stocks.

Gumbel and Keeney (1950), Seshadri (1965) and Ahmad (1985) proposed a class of inverted distributions and presented some of its properties. Cobb (1980) discussed a differential equation of the form $\frac{df}{dx}=\frac{g(y)}{h(y)}f(y)dy$, $h(y)>0$ where $g(y)$ and $h(y)$ are polynomials such that the degree of $h(y)$ is one higher than the degree of $g(y)$. Ahmad (1985) used $\frac{d}{dx}\left[\ln g(x)\right]=-\frac{\left(a_2x^2+a_1x+a_0\right)}{x\left(B_0x^2+B_1x+B_2\right)}$ where the coefficients a_0, a_1, a_2 are given by $a_0=2B_2-1$, $a_1=2B_2-a$, $a_2=2B_0$ and generated the inverted Pearson system of probability distributions. This class includes the inverted normal as well as the inverted I, II, III, V distribution. See also Ahmad and Sheikh (1986), Ali and Ahmad (1985), Ahmad and Kazi (1987) and Ahmad (2007).

The *t*-distribution is symmetric and bell-shaped, like the normal distribution, but has heavier tails, meaning that it is more prone to producing values that fall far from its mean. This makes it useful for understanding the statistical behavior of certain types of ratios of random quantities, in which variation in the denominator is amplified and may produce outlying values when the denominator of the ratio falls close to zero.

Habibullah and Ahmad (2008), Habibullah et al. (2008) discussed a differential equation which generated distributions closed under reciprocation (CUR). She considered the following differential equation:

$$\frac{d}{dy}\left[\ln g\left(y\right)\right]=\frac{b_n y^n+b_{n-1}y^{n-1}+......+b_0}{a_n y^n+a_{n-1}y^{n-1}+......+a_0}\,. \tag{1.1}$$

Then $g\left(y\right)$ is CUR provided that the following conditions hold:

a) $a_i \neq 0$ and $b_j \neq 0$ for some $i,\ j$ where $0 < i,\ j < n$ and

b) $\sum_{i=0}^{2j}\left(-1\right)^i a_{2j-i}b_i = 0,\ \ j=0,1,2,....,m\,,$

$$\sum_{i=0}^{2j}\left(-1\right)^i a_{n-i}b_{n-2j+i} = 0,\ \ j=0,1,2,....,m \tag{1.2}$$

where m is $\frac{n}{2}$ or $\frac{n-1}{2}$ according as n is an even or odd non-negative integer.

Habibullah et al. (2008) generated log Pearson type VII distribution from (1.2) as

$$f(x)=\frac{\alpha^{1/2}}{B\left(\nu-1/2,1/2\right)x}\left(1+\alpha\left(\ln x\right)^2\right)^{-\nu},\ x>0,\alpha>0,\nu>\frac{1}{2}\,, \tag{1.3}$$

$B(a,b)$ is the beta function. The distribution at (1.3) is Closed Under Reciprocation (CUR) derived from (1.1) and satisfied (1.2) for some a_i and b_i. Habibullah et al. (2008) discussed its properties and characterized it through hazard rate function.

In this paper, we propose a generalized log Pearson (GLP) type VII distribution as

$$f(x)=\frac{p\ \alpha^{1/2p}}{B\left(\nu-1/2p,1/2p\right)x}\left(1+\alpha\left(\ln x\right)^{2p}\right)^{-\nu},\ x>0,\alpha>0,\ p\geq 1,\nu>\frac{1}{2p} \tag{1.4}$$

where $B(a,b)$ is the beta function. The function at (1.4) is CUR and is more flexible than the log Pearson of type VII distribution in data fitting and produces many distributions under various reciprocal transformations. If we replace

$$n=2p,b_i=0,i=0,1,2....2p.,b_i\neq 0,i=2p-1,$$
$$a_j=0,\ j=1,2,....,2p-1,a_0\neq 0,a_{2p}\neq 0$$

in (1.1) and under the transformation $y=\ln\left(x\right)$, we have GLP type VII distribution.

The GLP type VII distribution is useful in (1) measuring of size of living tissue (length, skin area, weight), (2) certain physiological measurements, such as blood pressure of adult humans, and in (3) analyzing extreme values of such variables as monthly and annual maximum values of daily rainfall and river discharge volumes.

Graphs for GLP type VII distribution are shown in Section 1. In Section 2, cumulative distribution function and its graph are presented. Hazard rate function and survival function with graphs are discussed in Section 3. Shannon entropy of GLP type VII distribution is found in Section 4. In Section 5 characterization of GLP type VII distribution is presented in terms of conditional expectation. Generalized lognormal distribution is defined in Section 6. In Section 7 some compound scale mixtures of generalized lognormal distribution with gamma distribution and fractional moment exponential distribution are found. An empirical study has been conducted and discussed in Section 8. Finally, concluding remarks are provided in Section 9.

The graphs are shown for different values of parameters.

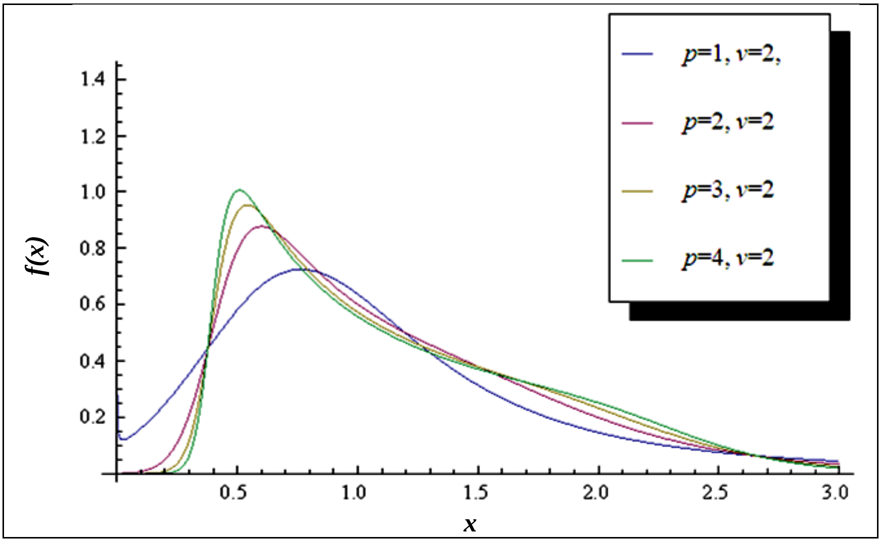

Fig. 1.1: *pdf* of GLP Type VII Distribution for the Indicated Values of p **and** $\nu = 2, \alpha = 1.$

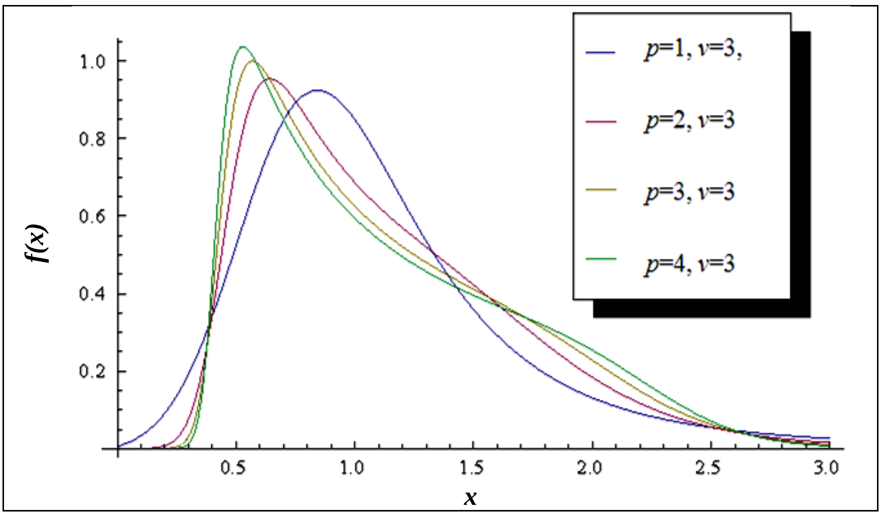

Fig. 1.2: *pdf* of GLP Type VII Distribution for the Indicated Values of p **and** $\nu = 3, \alpha = 1.$

2. DISTRIBUTION FUNCTION OF GLP TYPE VII DISTRIBUTION

The distribution function of GLP type VII distribution is

$$F(x)=\begin{cases}\frac{1}{2}I_{U_x}\left(\nu-\frac{1}{2p},\frac{1}{2p}\right), & if\ \ 0< x\le 1, \alpha>0, p\in Z^{+}, \nu>\frac{1}{2p},\\ 1-\frac{1}{2}I_{U_x}\left(\nu-\frac{1}{2p},\frac{1}{2p}\right), & if\ \ x>1, \alpha>0, p\in Z^{+}, \nu>\frac{1}{2p},\end{cases} \qquad (2.1)$$

where $U_x=\left(1+\alpha\left(\ln x\right)^{2p}\right)^{-1}$ and $I_x\left(a,b\right)=\frac{1}{B\left(a,b\right)}\int_0^x t^{a-1}\left(1-t\right)^{b-1}dt$.

Figure 2.1 shows the graphs of distribution function for various values of p and ν .

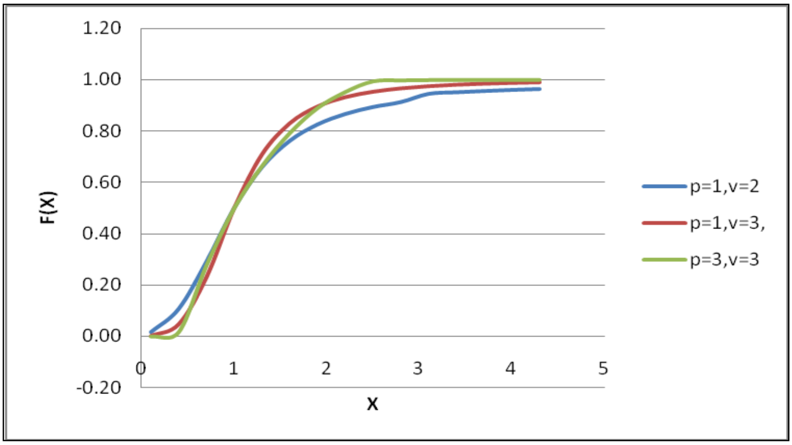

Fig. 2.1: Distribution Function of the GLP Type VII Distribution for the Indicated Values of p and ν

3. THE HAZARD RATE OF GLP TYPE VII DISTRIBUTION

Hazard rate function arises in the situation of the analysis of the time to the event and it describes the current chance of failure for the population that has not yet failed. This function plays a pivotal role in reliability analysis, survival analysis, actuarial sciences and demography, in extreme value theory and in duration analysis in economics and sociology. This is also important for researchers and practitioners working in areas like engineering statistics and biomedical sciences. Hazard rate function is very useful in defining and formulating a model when dealing with lifetime data.

The survival function of the GLP type VII distribution is

$$S(x)=\begin{cases}1-\dfrac{1}{2}I_{U_x}\left(\nu-\dfrac{1}{2p},\dfrac{1}{2p}\right), & if\ \ 0< x\le 1,\alpha>0,\ p\in Z^{+},\nu>\dfrac{1}{2p},\\ \dfrac{1}{2}I_{U_x}\left(\nu-\dfrac{1}{2p},\dfrac{1}{2p}\right), & if\ \ x>1,\alpha>0,\ p\in Z^{+},\nu>\dfrac{1}{2p}.\end{cases} \tag{3.1}$$

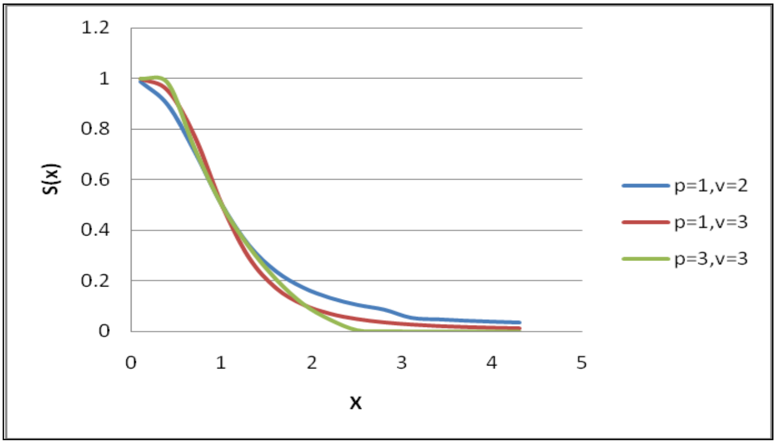

Fig. 3.1: Survival Function of GLP Type VII Distribution for the Indicated Values of p and ν

The corresponding hazard rate function of the GLP type VII distribution is

$$h(x)=\begin{cases}\dfrac{2\,k\left(1+\alpha\left(\ln x\right)^{2p}\right)^{-\nu}}{2x-xI_{U_x}\left(\nu-\dfrac{1}{2p},\dfrac{1}{2p}\right)}, if\ 0< x\le 1,\alpha>0,\ p\in Z^{+},\nu>\dfrac{1}{2p}\\ \dfrac{2\,k\,p\left(1+\alpha\left(\ln x\right)^{2p}\right)^{-\nu}}{x\ I_{U_x}\left(\nu-\dfrac{1}{2p},\dfrac{1}{2p}\right)}, if\ \ x>1,\alpha>0,\ p\in Z^{+},\nu>\dfrac{1}{2p}\end{cases} \tag{3.2}$$

where k and U_x are defined in (1.2) and (3.1) respectively.

Figure 3.2 shows the graphs of the hazard rate function defined in (3.2) for various values of p and v.

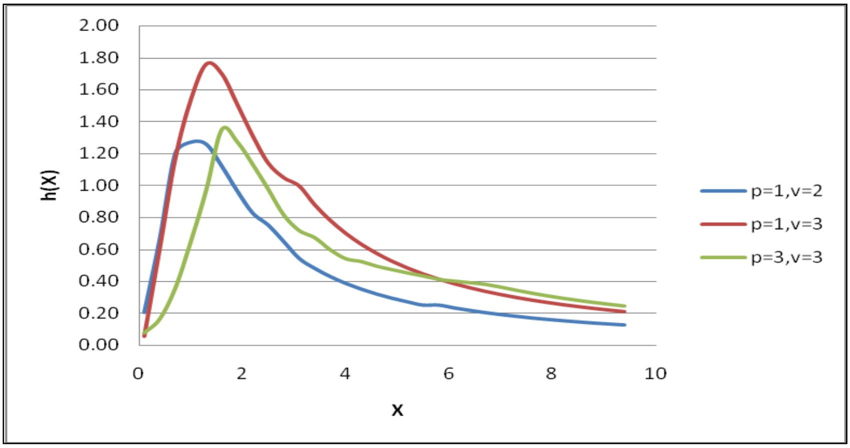

Fig. 3.2: Hazard Rate Function of GLP Type VII Distribution for the Indicated Values of p and ν

Fig 3.2 shows that the failure rate function is upside-down bathtub shaped. A function is termed as upside-down bathtub shaped if it is first increasing and then decreasing, with a single maximum.

The cumulative hazard rate function is

$$H_g(x)=\begin{cases}-\ln\left(1-\frac{1}{2}I_{U_x}\left(\nu-\frac{1}{2p},\frac{1}{2p}\right)\right), & if\ \ 0<x\le 1, \alpha>0, p\in Z^+, \nu>\frac{1}{2p},\\ -\ln\left(\frac{1}{2}I_{U_x}\left(\nu-\frac{1}{2p},\frac{1}{2p}\right)\right), & if\ \ x>1, \alpha>0, p\in Z^+, \nu>\frac{1}{2p}.\end{cases} \tag{3.3}$$

The mean residual function of the GLP type VII distribution is

$$\mu(x)=\begin{cases}\frac{1}{\overline{F}(t)}\int_t^\infty\left(1-\frac{1}{2}I_{U_x}\left(\nu-\frac{1}{2p},\frac{1}{2p}\right)\right)dx, & if\ \ 0<x,t\le 1, \alpha>0, p\in Z^+, \nu>\frac{1}{2p},\\ \frac{1}{\overline{F}(t)}\int_t^\infty\left(\frac{1}{2}I_{U_x}\left(\nu-\frac{1}{2p},\frac{1}{2p}\right)\right)dx, & if\ \ x,t>1, \alpha>0, p\in Z^+, \nu>\frac{1}{2p}.\end{cases} \tag{3.4}$$

The mean residual function gives an interpretable measure of how much more time to be expected to survive for an individual, given that one already reached the time point X. This can be readily estimated from right censored data in which, even the mean, cannot be estimated without additional assumptions.

4. SHANNON ENTROPY OF GLP TYPE VII DISTRIBUTION

Statistical entropy is a measure of uncertainty in probability term or random experiment outcome's ignorance. Since Shannon's (1948) pioneering work on the mathematical theory of communication, entropy has been used as a major tool in information theory and in almost every branch of science and engineering. Information theoretic principles and methods have become integral parts of probability and statistics and have been applied in various branches of statistics and related fields.

The Shannon Entropy $u\left(f\left(X\right)\right)$ of a continuous random variable X with a *pdf* $f\left(x\right)$ is defined as

$$u\left(f\left(X\right)\right)=-\int_{S} f\left(x\right)\ \ln f\left(x\right)dx, \tag{4.1}$$

where S is the support set of random variable. Thus $u\left(f\left(X\right)\right)$ of GLP type VII distribution is

$$u\left(f\left(X\right)\right)=-\ln k+\nu\, k\int_{0}^{\infty}\left(\ln\left(1+\alpha\left(\ln x\right)^{2p}\right)\right)\frac{\left(1+\alpha\left(\ln x\right)^{2p}\right)^{-\nu}}{x}dx$$

Applying transformation $\left(1+\alpha\left(\ln x\right)^{2p}\right)=\frac{1}{z}$, we obtain

$$=-\ln k+\frac{\nu k}{p\alpha^{\frac{1}{2p}}}\int_{0}^{1}\ln z\ \ z^{\nu-\frac{1}{2p}-1}\left(1-z\right)^{\frac{1}{2p}-1}dt$$

Using the following formula from table of integral, series and products

$$\int_{0}^{1}t^{\mu-1}\left(1-t\right)^{\nu-1}\ln t\ dt=\mathrm{B}\left(\mu,\nu\right)\left[\psi\left(\mu\right)-\psi\left(\mu+\nu\right)\right].$$

(See 4.253 Gradshteyn and Ryzhik, 2007)

$$u\left(f\left(X\right)\right)=-\ln k+\frac{\nu k}{2p\alpha^{\frac{1}{2p}}}\mathrm{B}\left(\nu-\frac{1}{2p},\frac{1}{2p}\right)\left[\psi\left(\nu-\frac{1}{2p}\right)-\psi\left(\nu\right)\right]$$

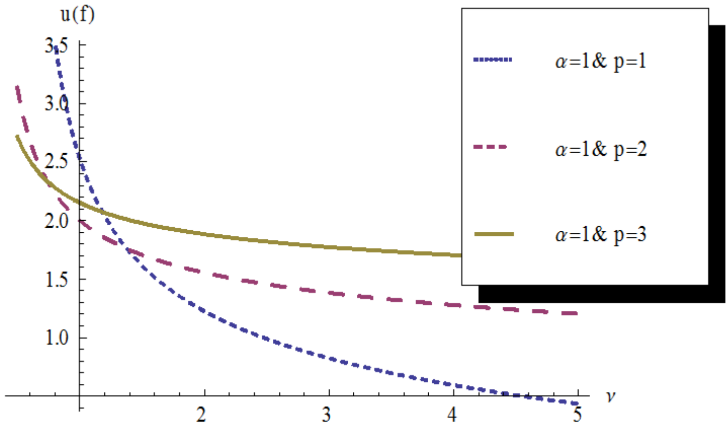

Fig. 4.1: Shannon Entropy of GLP Type VII Distribution for the Indicated Values of α, p **and** $0<\nu<5$

It is clear that the value of $u(f)$ changes as v and p change and it measures the average uncertainty or disorderness in the random variable, and value of $u(f)$ decreases when v increases with fixed p and α.

5. CHARACTERIZATION OF GLP TYPE VII DISTRIBUTION

Characterizations of probability distributions have a position of prominence in statistical theory. A number of methods have been developed by researchers for characterizing both discrete and continuous probability distributions. [See Shanbagh (1970), Kotlarski (1972), Gupta (1975), Glanzel et al. (1984), Telcs et al. (1985), Glanzel (1987), Osaki and Li (1988), Ruiz et al. (1993), Ruiz and Navarro (1994), Franco and Ruitz (1996), Ruiz and Navarro (1996), Wu and Ouyang (1996), Su and Huang (2000), Gupta and Ahsanullah (2004), Gupta and Kirmani (2004) and Sunoj et al. (2009)].

Habibullah and Ahmad (2008) characterized the CUR distributions through conditional expectation. In the next theorem, we characterize GLP type VII distribution.

Theorem 5.1

Let X be a random variable (r.v) with differentiable pdf on its support $(0,\infty)$. Then, the following conditions are equivalent for characterizing GLP type VII distributions:

1) $$\mu_p(y)=\frac{k(y)r(y)}{2p\alpha(v-1)}, \tag{5.1}$$

2) $$\frac{f'(y)}{f(y)}=\frac{-k'(y)-2p(\nu-1)\alpha(\ln y)^{2p-1}}{k(y)},$$

where $\mu_p(y)=E\left((\ln X)^{2p-1}\mid X>y\right)$,

$$k(y) = y\left(1+\alpha(\ln y)^{2p}\right)$$

and $\int_0^\infty f(x)\,dx = 1.$

Proof:

Suppose X follows the GLP type VII distribution. Then,

$$\mu_p(y) = E\left((\ln X)^{2p-1} \mid X > y\right)$$

$$= \frac{C}{\bar{F}(y)} \int_y^\infty \frac{(\ln x)^{2p-1}}{x} \left(1+\alpha(\ln x)^{2p}\right)^{-\nu} dx.$$

Let

$$z = \left[1+\alpha(\ln x)^{2p}\right]^{-1},$$

we have

$$\mu_p(y) = \frac{C\left(1+\alpha(\ln y)^{2p}\right)^{1-\nu}}{2p\alpha\bar{F}(y)(\nu-1)}$$

since

$$yf(y) = C\left(1+\alpha(\ln y)^{2p}\right)^{-\nu},$$

and hence

$$E\left((\ln X)^{2p-1} \mid X \geq y\right) = \frac{y\left(1+\alpha(\ln y)^{2p}\right)}{2p\alpha(\nu-1)} \frac{f(y)}{\bar{F}(y)} = \frac{y\left(1+\alpha(\ln y)^{2p}\right)}{2p\alpha(\nu-1)} r_X(y)$$

and therefore

$$\mu_p(y) = \frac{k(y)}{2p\alpha(\nu-1)} r_X(y),$$

where

$$k(y) = y\left(1+\alpha(\ln y)^{2p}\right).$$

Now, let $f(x)$ be an unknown pdf of a random variable X defined on $(0,\infty)$ such that (6.1) hold. Equation (6.1) implies that

$$\int_y^\infty (\ln x)^{2p-1} f(x)\,dx = \frac{1}{2p\alpha(\nu-1)}\left[y\left(1+\alpha(\ln y)^{2p}\right) f(y)\right],$$

Differentiating both sides with respect to y, we have

$$-(\ln y)^{2p-1}\, f(y) = \frac{1}{2p\alpha(\nu-1)}$$

$$\left[\left(1+\alpha(\ln y)^{2p}\right)\left(yf'(y)+f(y)\right)+2p\alpha(\ln y)^{2p-1}\, f(y)\right],$$

so that

$$\frac{-\left[1+\alpha(\ln y)^{2p}\right]-2p\nu\alpha(\ln y)^{2p-1}}{y\left(1+\alpha(\ln y)^{2p}\right)} = \frac{f'(y)}{f(y)}.$$

Hence,

$$\frac{f'(y)}{f(y)} = \frac{-k'(y)-2p(\nu-1)\alpha(\ln y)^{2p-1}}{k(y)}$$

and

$$\frac{d}{dy}[\ln f(y)] = \frac{-2p\alpha\nu(\ln y)^{2p-1}}{y\left(1+\alpha(\ln y)^{2p}\right)} - \frac{1}{y}$$

on integrating both sides with respect to y, we have

$$\ln f(y) = -\nu\left[\ln\left(1+\alpha\ln^{2p} y\right)\right]-\ln y+\ln k.$$

Then,

$$f(y) = \frac{k}{y}\left(1+\alpha(\ln y)^{2p}\right)^{-\nu},\ y>0, \alpha>0, p\in Z^{+}, \nu>\frac{1}{2p},$$

and for k we use the $\int_0^\infty f(y)\,dy = 1.$

Plugging $f(y)$ and then the normalizing factor as, $k = \dfrac{p\,\alpha^{\frac{1}{2p}}}{B\left(\nu-\frac{1}{2p},\frac{1}{2p}\right)}.$

Thus,

$$f(x) = \frac{p\,\alpha^{\frac{1}{2p}}}{B\left(\nu-\frac{1}{2p},\frac{1}{2p}\right)y}\left(1+\alpha(\ln y)^{2p}\right)^{-\nu},\ \ 0<y<\infty,$$

which is the GLP type VII distribution, where $r_X(y) = f(y)/\bar{F}(y)$, $\bar{F}(y) = 1-F(y)$ and GLP type VII distribution is closed under transformation $1/Y$.

6. GENERALIZED LOG NORMAL DISTRIBUTION

Iqbal (2013) defined a generalized log normal (GLN) distribution with *pdf*

$$f(x) = \frac{p\ \exp\left(-(\ln x)^{2p}/2\right)}{2^{1/2p}\Gamma(1/2p)\ x},\ x > 0 \tag{6.1}$$

They (2012) proved that GLN distribution is the limiting case of GLP type VII distribution.

7. COMPOUND SCALE MIXTURES OF GLP TYPE VII DISTRIBUTION

Scale mixtures of normal distributions are an important class of elliptical distributions. They share good properties with normal distributions. Chu (1973) developed different scale mixtures of normal distributions. Andrews and Mallows (1974) studied conditions for a unidimensional symmetrical distribution to be a scale mixture of normal distributions.

Some results on scale mixtures of normal distributions and their relationship with elliptical distributions can be found in Eaton (1981), Azzalini (1985), Little (1988), Fang et al. (1990), Azzalini and Capitanio (1999, 2003), Arnold and Beaver (2002), Gupta et al. (2004), Arslan (2005), Tyler *et* al. (2009) and Azzalini and Genton (2008).

7.1 Compound Scale Mixture of GLN Distribution with Gamma Distribution

We use Andrew and Mellows (1972) technique of scale mixing of the gamma distribution with the fractional moment exponential distribution to obtain distribution of the GLN.

Theorem (7.1)

Let $f(x)$ be the pdfs of the GLN distribution (6.1) with scale parameter β and of the gamma distribution given by $g(u \mid a,b) = b^a u^{a-1}\exp(-bu)/\Gamma a,\ u > 0$. Then, the GLP type VII distribution (1.4) is the scale mixture of the GLN distribution with the gamma distribution where $a = v - 1/(2p),\ b - 1/2$.

Proof:

The scale mixture of the GLN distribution with the gamma distribution is

$$\begin{aligned} p(x) &= \int_0^\infty f(x \mid u)\, g(u \mid a,b)\, du \\ &= \int_0^\infty f(x \mid u)\, g\left(u \mid v - \frac{1}{2p}, \frac{1}{2}\right) du \end{aligned}$$

$$= \int_0^\infty \frac{p \exp\left(-\frac{(\ln x)^{2p} u}{2}\right) u^{\frac{1}{2p}}}{2^{\frac{1}{2p}} \Gamma\left(\frac{1}{2p}\right) x} \frac{u^{v-\frac{1}{2p}-1} \exp\left(-\frac{u}{2}\right)}{2^{v-\frac{1}{2p}} \Gamma\left(v-\frac{1}{2p}\right)} du$$

$$= \frac{p}{B\left(v-\frac{1}{2p}, \frac{1}{2p}\right) x} \left(1+(\ln x)^{2p}\right)^{-v},$$

which is a GLP type VII distribution.

7.2 Compound Scale Mixture of GLN Distribution with Fractional Moment Exponential Distribution

Theorem (7.2)

Let $f(x)$ be the *pdf* of the GLN distribution (6.1) with scale parameter β and $g(u \mid \delta, \alpha)$ be the fractional moment exponential distribution given by

$$g(u \mid \delta, \alpha) = \frac{u^{\delta} \exp\left(-\frac{u}{\alpha}\right)}{\alpha^{1+\delta} \Gamma(1+\delta)}, \; u > 0, \delta > 0,$$

(see Dara, 2012).

Then, the GLP type VII distribution (1.4) is the scale mixture of the GLN distribution with fractional moment exponential distribution where $\delta = v - 1/(2p) - 1, \alpha = 2$.

Proof:

The scale mixture of the GLN distribution with the fractional moment exponential distribution is

$$p(x) = \int_0^\infty f(x \mid u) g\left(u \mid v - \frac{1}{2p} - 1, 2\right) du$$

$$= \int_0^\infty \frac{p \exp\left(-\frac{(\ln x)^{2p} u}{2}\right) u^{\frac{1}{2p}}}{2^{\frac{1}{2p}} \Gamma\left(\frac{1}{2p}\right) x} \frac{u^{v-\frac{1}{2p}-1} \exp\left(-\frac{u}{2}\right)}{2^{v-\frac{1}{2p}} \Gamma\left(v-\frac{1}{2p}\right)} du$$

$$= \frac{p}{B\left(v-\frac{1}{2p}, \frac{1}{2p}\right) x} \left(1+(\ln x)^{2p}\right)^{-v},$$

which is a GLP type VII distribution.

8. THE BUFFALO SNOWFALL DATA SET

Consider the set of 63 values of annual snowfall precipitation in Buffalo, (in inches), for the winters 1910/11 to 1972/73. This data set has been extensively analyzed in the statistical literature; see for example Silverman (1986). The ordered observations are

Table 8.1
Snowfall Data in Buffalo (in inches)

25.0	39.8	39.9	40.1	46.7	49.1	49.6	51.2	51.6
53.5	54.7	55.5	55.9	58.0	60.3	63.6	65.4	66.1
69.3	70.9	71.4	71.5	71.8	72.9	74.4	76.2	77.8
78.2	78.4	79.0	79.3	79.6	80.7	82.4	82.4	83.0
83.6	83.6	84.8	85.5	87.4	88.7	89.6	89.8	89.9
90.9	97.0	98.3	101.4	102.4	103.9	104.5	105.2	110.0
110.5	110.5	113.7	114.5	115.6	120.5	120.7	124.7	126.4

Source: Silverman (1986)

Singh and Singh (1988), Pilon and Adamowski (1993), Ali (2011) and Izinyon et al. (2011) compared normal, Gumbel, generalized extreme value distribution, lognormal and log Pearson type III distributions in hydrological frequency analysis. Here we compare GLP type VII distribution with Gumbel, generalized extreme value distribution, lognormal and log Pearson type III distribution.

Table 8.2
Maximum Likelihood Estimates of Parameters and Goodness-of-Fit Statistics for the Snowfall Data

Parameter / Distribution	$\hat{k}$	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\nu}$	p	A_0^2	W_0^2
Gumbel	-	69.624	18.493	-	-	-	-	1.419	0.4867
Lognormal	-	4.3368	0.32695	-	-	-	-	0.77524	0.2532
Log Pearson type III	-	-	-	0.025	173.15	-	-	0.97045	0.3832
GLP type VII	-	2.2134	-	-	3.1965	1.1893	1	0.4011	0.1215
Gen. Extreme Value	-0.279	71.782	23.98	-	-	-	-	0.24868	0.0926
GLP type VII	-	1.9217	-	-	3.2418	0.8233	2	0.1923	0.0521

Note: A_0^2 is the Anderson–Darling test statistic and W_0^2 is the Cramér–von Mises test statistic

Table (8.2) shows values of Anderson-Darling statistic at 5% and 1% are 2.5018 and 3.9074 respectively whereas table values of Cramer-von Mises statistic at 5% and 1% are 0.5018 and 0.5374 respectively. The study shows the goodness of fit of GLP type VII over Gumbel, generalized extreme value distribution, lognormal, log Pearson type III distribution. Moreover, this study also shows the effectiveness of $p=2$ on $p=1$ in GLP type VII distribution since the values of A_0^2 and W_0^2 are less when $p=2$ than the values of A_0^2 and W_0^2 when $p=1$.

9. CONCLUDING REMARKS

In this work we have proposed a GLP type VII distribution and developed its various properties including certain characterization of the distribution. Compound scale mixtures of limiting distribution of the GLP type VII distribution (i.e GLN distribution) with gamma distribution and fractional moment exponential distribution are obtained. Since this distribution has a particular distribution such as log Pearson type VII distribution as sub-model, we hope that GLP type VII distribution will useful in different areas of related research in Mathematics, Statistics and Probability as well as related fields.

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