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ON CHARACTERIZATION OF PROBABILITY DISTRIBUTIONS THROUGH CONDITIONAL EXPECTATION OF GENERALIZED AND DUAL GENERALIZED ORDER STATISTICS

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ABSTRACT

In this paper, some general form of continuous probability distributions are characterized through conditional expectation of function of generalized order statistics and dual generalized order statistics. Further, some deductions and related results are also discussed.

KEYWORDS

Generalized order statistics; dual generalized order statistics; order statistics; record values; conditional expectation and characterization of distributions.

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1. INTRODUCTION

Kamps (1995) introduced the concept of generalized order statistics (gos) as follows: Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed (iid) random variables (rv) with absolutely continuous distribution function (df) $F(x)$ and the probability density function (pdf) $f(x)$, $x \in (\alpha, \beta)$. Let $n \in \mathrm{N}$, $(n \geq 2)$, $k \geq 1$, $\tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \Re^{n-1}$, $M_r = \sum_{j=r}^{n-1} m_j$, $1 \leq r \leq n-1$ be the parameters such that $\gamma_r = k + n - r + M_r \geq 1$, for all $r \in \{1, 2, \ldots, n-1\}$. Then $X(r, n, \tilde{m}, k)$ are called gos if their joint pdf is given by

$$f_{X(1,n,\tilde{m},k),\ldots,X(n,n,\tilde{m},k)}(x_1, x_2, \ldots, x_n)$$
$$= k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}\left[\bar{F}(x_i)\right]^{m_i} f(x_i)\right)\left[\bar{F}(x_n)\right]^{k-1} f(x_n) \tag{1.1}$$

on the cone $F^{-1}(0) < x_1 \leq \ldots \leq x_n < F^{-1}(1)$, where $\bar{F}(x) = 1 - F(x)$.

In view of (1.1), when $m_i = m; i = 1, 2, \ldots, n-1$, the pdf of $r-th$ gos $X(r, n, m, k)$ is given as

$$f_{X(r,n,m,k)}(x)=\frac{C_{r-1}}{(r-1)!}\left[\bar{F}(x)\right]^{\gamma_r-1}f(x)g_m^{r-1}\left(F(x)\right),\quad \alpha\le x\le\beta \tag{1.2}$$

and joint *pdf* of $X(r,n,m,k)$ and $X(s,n,m,k)$, $1\le r<s\le n$ is

$$f_{X(r,n,m,k),X(s,n,m,k)}(x,y)=\frac{C_{s-1}}{(r-1)!(s-r-1)!}\left[\bar{F}(x)\right]^{m}f(x)\,g_m^{r-1}\left(F(x)\right)$$
$$\times\left[h_m\left(F(y)\right)-h_m\left(F(x)\right)\right]^{s-r-1}\left[\bar{F}(y)\right]^{\gamma_s-1}f(y),\ \alpha\le x<y\le\beta \tag{1.3}$$

where,

$$C_{r-1}=\prod_{i=1}^{r}\gamma_i\ ,\ \gamma_i=k+(n-i)(m+1),$$

$$h_m(x)=\begin{cases}-\dfrac{1}{m+1}(1-x)^{m+1}, & m\ne-1\\ -\log(1-x), & m=-1\end{cases}$$

and

$$g_m(x)=\int_0^x(1-t)^m dt=h_m(x)-h_m(0),\quad x\in(0,1).$$

The conditional *pdf* of $X(s,n,m,k)$ given $X(r,n,m,k)=x$, $1\le r<s\le n$ in view of (1.2) and (1.3) is given as

$$\frac{C_{s-1}}{C_{r-1}(s-r-1)!(m+1)^{s-r-1}}\frac{\left[\bar{F}(y)\right]^{\gamma_s-1}\left[\left(\bar{F}(x)\right)^{m+1}-\left(\bar{F}(y)\right)^{m+1}\right]^{s-r-1}}{\left[\bar{F}(x)\right]^{\gamma_{r+1}}}f(y) \tag{1.4}$$

and the conditional *pdf* of $X(r,n,m,k)$ given $X(s,n,m,k)=y$, $1\le r<s\le n$ is

$$\frac{(s-1)!(m+1)}{(r-1)!(s-r-1)!}\frac{\left[\bar{F}(x)\right]^{m}\left[1-\left(\bar{F}(x)\right)^{m+1}\right]^{r-1}\left[\left(\bar{F}(x)\right)^{m+1}-\left(\bar{F}(y)\right)^{m+1}\right]^{s-r-1}}{\left[1-\left(\bar{F}(y)\right)^{m+1}\right]^{s-1}}f(x),\ x<y. \tag{1.5}$$

Several models of ordered random variables such as order statistics and record values can be seen as special cases of generalized order statistics. If $m=0$ and $k=1$, then $X(r,n,m,k)$ reduces to the $r-th$ order statistic $X_{r:n}$ (David, 2003). If $m=-1$ and $k=1$, then $X(r,n,m,k)$ is the $r-th$ record value from an infinite sequence of *iidrvs* (Ahsanullah, 1995). Other special cases are $k-th$ record values ($m=-1$, Dziubdziela and Kopociński, 1976) and order statistics with non-integral sample size $(m=0, k=\alpha-n+1, \alpha=n$, Stigler (1977), Rohatgi and Saleh (1988)).

Generalized order statistics can be easily applicable in practical problems except that when $F()$ is so called inverse distribution function. So the concept of dual (lower) generalized order statistics is needed. The concept of lower generalized order statistics $(lgos)$ was first introduced by Pawlas and Szynal (2001) to enable a common approach to descending ordered random variables like reverse order statistics and lower record values. Further, the concept of dual generalized order statistics $(dgos)$ was extensively studied by Burkshat *et al.* (2003).

Let $X_d(r,n,\tilde{m},k),\ r=1,2,\ldots,n$ be the $r-th$ $dgos$ and their joint density function is of the form

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}\left[F\left(x_i\right)\right]^m f\left(x_i\right)\right)\left[F\left(x_n\right)\right]^{k-1} f\left(x_n\right) \tag{1.6}$$

for $F^{-1}(1) > x_1 \ge x_2 \ge \ldots \ge x_n > F^{-1}(0)$.

It may be noted that the joint density (1.6) is obtained by replacing $\left[1-F(x)\right]$ with $F(x)$ in (1.1).

For the case $m_i = m; i = 1,2,\ldots,n-1$, the pdf of $r-th$ $dgos$ $X_d(r,n,m,k)$ is given

$$f_{X_d(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}\left[F(x)\right]^{\gamma_r-1} f(x)\, g_m'^{r-1}\left(F(x)\right) \tag{1.7}$$

and the joint pdf of $X_d(r,n,m,k)$ and $X_d(s,n,m,k)$ is

$$f_{X_d(r,n,m,k),X_d(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!}\left[F(x)\right]^m f(x)\, g_m'^{r-1}\left(F(x)\right)$$

$$\times\left[h_m'\left(F(y)\right)-h_m'\left(F(x)\right)\right]^{s-r-1} f(y)\left[F(y)\right]^{\gamma_s-1},\quad x>y, \tag{1.8}$$

where,

$$h_m'(x) = \begin{cases} -\dfrac{1}{m+1}x^{m+1}, & m \ne -1 \\ -\log x, & m=-1 \end{cases}$$

and

$$g_m'(x) = h_m'(x) - h_m'(1),\ x\in(0,1).$$

Further, the conditional pdf of $X_d(s,n,m,k)$ given $X_d(r,n,m,k)=x$, $1\le r<s\le n$ is

$$\frac{C_{s-1}}{C_{r-1}(s-r-1)!(m+1)^{s-r-1}}\frac{\left[F(y)\right]^{\gamma_{s-1}}}{\left[F(x)\right]^{\gamma_{r+1}}}\left[\left(F(x)\right)^{m+1}-\left(F(y)\right)^{m+1}\right]^{s-r-1} f(y),\ x>y \tag{1.9}$$

and the conditional pdf of $X_d(r,n,m,k)$ given $X_d(s,n,m,k)=y$, $1\le r<s\le n$, is

$$\frac{(s-1)!(m+1)}{(r-1)!(s-r-1)!}\frac{[F(x)]^m\left[1-(F(x))^{m+1}\right]^{r-1}\left[(F(x))^{m+1}-(F(y))^{m+1}\right]^{s-r-1}}{\left[1-(F(y))^{m+1}\right]^{s-1}}f(x), x>y. \tag{1.10}$$

The conditional expectation of order statistics are extensively used in characterizing probability distributions. Khan and Abu-Salih (1989) characterized some general forms of distributions through conditional expectation of order statistics fixing adjacent order statistics. Khan and Abouammoh (2000) extended the result of Khan and Abu-Salih (1989) and characterized the distributions when the conditioning is not adjacent. Khan *et al.* (2006) established characterizing relationships for the distributions through *gos* and characterized several distributions through conditional expectation of function of *gos* whereas Khan *et al.* (2007) established characterizing results for order statistics when conditioning is on a pair of order statistics. Further, Samuel (2008) characterized the distributions considered by Khan and Abu-Salih (1989) for *gos*. Khan *et al.* (2010) characterized several distributions through conditional expectation of function of *dgos*.

For more detailed survey on characterization one may refer to Franco and Ruiz (1995, 1997), López-Blázquez and Moreno-Rebollo (1997), Dembińska and Wesolowski (1998, 2000), Keseling (1999), Wu and Ouyang (1996), Khan and Athar (2004), Athar *et al.* (2003) and references therein.

In this paper, we have extended the result of Samuel (2008) and characterized several continuous distributions when conditioning is not adjacent. In Section 2 characterization theorems based on *gos* and *dgos* are presented while in Section 3 several examples are listed.

2. CHARACTERIZATION THEOREMS

Theorem 2.1

Let X be a random variable with absolutely continuous df $F(x)$ and pdf $f(x)$ over the support (α,β), and $h(x)$ be a monotonic, continuous and differentiable function of x, then for two consecutive values r and $r+1$, $2 \le r+1 \le s \le n$,

$$E\left[h\{X(s,n,m,k)\} \mid X(l,n,m,k)=x\right] = g_{s|l}(x) = h(x) + \frac{1}{a}\sum_{j=l}^{s-1}\frac{1}{\gamma_{j+1}},$$

$$l = r, r+1,\ a \ne 0 \text{ and } \gamma_{j+1} \ne 0 \tag{2.1}$$

if and only if

$$\bar{F}(x) = e^{-ah(x)}. \tag{2.2}$$

Proof:

To prove necessary part, for $s \ge r+1$,

$$E\left[h\{X(s,n,m,k)\}\mid X(r,n,m,k)=x\right]$$
$$=\frac{C_{s-1}}{C_{r-1}(s-r-1)!(m+1)^{s-r-1}}$$
$$\times\int_x^\beta h(y)\left[1-\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{m+1}\right]^{s-r-1}\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_s-1}\frac{f(y)}{\bar{F}(x)}\mathrm{d}y. \qquad (2.3)$$

Set

$$u=\frac{\bar{F}(y)}{\bar{F}(x)}=\frac{e^{-ah(y)}}{e^{-ah(x)}},\ \text{which implies}\ h(y)=h(x)-\frac{1}{a}\log u.$$

Then the right hand side of (2.3) reduces to

$$=\frac{C_{s-1}}{C_{r-1}(s-r-1)!(m+1)^{s-r-1}}\int_0^1\left(h(x)-\frac{1}{a}\ln u\right)\left(1-u^{m+1}\right)^{s-r-1}u^{\gamma_s-1}\mathrm{d}u\,.$$

Let $u^{m+1}=t$, then we get

$$E\left[h\{X(s,n,m,k)\}\mid X(r,n,m,k)=x\right]$$
$$=h(x)-\frac{\prod_{j=r+1}^{s}\gamma_j}{(s-r-1)!(m+1)^{s-r+1}}\frac{1}{a}\int_0^1\log t\, t^{\frac{\gamma_s}{m+1}-1}(1-t)^{s-r-1}\,dt$$
$$=h(x)-\frac{\prod_{j=r+1}^{s}\gamma_j}{(s-r-1)!(m+1)^{s-r+1}}\frac{1}{a}B\left(\frac{\gamma_s}{m+1},s-r\right)\left[\psi\left(\frac{\gamma_s}{m+1}\right)-\psi\left(\frac{\gamma_r}{m+1}\right)\right],$$

where $\psi(x)=\frac{d}{dx}\ln\Gamma(x)$,

$$\psi(x-n)-\psi(x)=-\sum_{k=1}^{n}\frac{1}{x-k} \qquad (2.4)$$

[*c.f.* Gradshteyn and Ryzhik, 2007, pp-540, 905]

and $\mathrm{B}(a,b)$ is the complete beta function.

Therefore,

$$E\left[h\{X(s,n,m,k)\}\mid X(r,n,m,k)=x\right]=h(x)+\frac{1}{a}\sum_{j=r}^{s-1}\frac{1}{\gamma_{j+1}}.$$

To prove sufficiency part, let

$$E\left[h\{X(s,n,m,k)\}\mid X(r,n,m,k)=x\right]=g_{s|r}(x).$$

Therefore,

$$\frac{C_{s-1}}{C_{r-1}(s-r-1)!(m+1)^{s-r-1}}\int_x^\beta h(y)\left[\bar{F}(x)^{m+1}-\bar{F}(y)^{m+1}\right]^{s-r-1}\left[\bar{F}(y)\right]^{\gamma_s-1} f(y)\mathrm{d}y$$

$$= g_{s|r}(x)\left[\bar{F}(x)\right]^{\gamma_{r+1}}.$$

Differentiating both sides with respect to x and adjusting the terms, we get

$$\frac{f(x)}{\bar{F}(x)} = -\frac{1}{\gamma_{r+1}}\frac{g'_{s|r}(x)}{\left[g_{s|r+1}(x)-g_{s|r}(x)\right]} \qquad \text{[Khan } et\ al.\text{, 2006]}$$

$$= \frac{1}{\gamma_{r+1}}\frac{h'(x)}{\left[h(x)+\frac{1}{a}\sum_{j=r+1}^{s-1}\frac{1}{\gamma_{j+1}}-h(x)-\frac{1}{a}\sum_{j=r}^{s-1}\frac{1}{\gamma_{j+1}}\right]}$$

$$= ah'(x).$$

Implying that $\bar{F}(x) = e^{-ah(x)}$.

Remark 2.1:

At $s = r+1$ and $l = r$ in (2.2), we have

$$E\left[h\left\{X(r+1,n,m,k)\right\} \mid X(r,n,m,k) = x\right] = h(x)+\frac{1}{a\,\gamma_{r+1}}$$

as obtained by Samuel (2008).

Remark 2.2:

For order statistics (at $m=0, k=1$), characterization result is given by Khan and Abouammoh (2000) and Wu and Ouyang (1996).

$$E\left[h\left(X_{s:n}\right) \mid X_{r:n} = x\right] = h(x)+\frac{1}{a}\sum_{j=r}^{s-1}\frac{1}{n-j}$$

Further, at $s = r+1$, we get

$$E\left[h\left(X_{r+1:n}\right)\middle| X_{r:n} = x\right] = h(x)+\frac{1}{a}\frac{1}{n-r}$$

as obtained by Khan and Abu-Salih (1989).

Theorem 2.2

Let X be a random variable with absolutely continuous df $F(x)$ and pdf $f(x)$ over the support (α,β), and $h(x)$ be a monotonic continuous and differentiable function of x, then for two consecutive values $s-1$ and s,

$$E\left[h\{X(r,n,m,k)\} \mid X(l,n,m,k)=y\right]=g_{r|l}(y)=h(y)+\frac{1}{a}\sum_{j=r}^{l-1}\frac{1}{j},$$

$$l=s-1,s, \quad a\neq 0 \;\; and \;\; j\neq 0 \tag{2.5}$$

if and only if

$$1-\left[\bar{F}(x)\right]^{m+1}=e^{-ah(x)}, \quad m\neq -1 \tag{2.6}$$

and

$$\bar{F}(x)=\exp\left[-\exp\left(\frac{h(q)-h(x)}{\delta}\right)\right], \quad m=-1, \delta=\frac{1}{a}\neq 0. \tag{2.7}$$

Proof:

To prove (2.6) implies (2.5), let

$$E\left[h\left\{X(r,n,m,k)\right\} \mid X(s,n,m,k)=y\right]=g_{r|s}(y),$$

or

$$g_{r|s}(y)=\frac{(s-1)!(m+1)}{(r-1)!(s-r-1)!}\int_{\alpha}^{y}h(x)\left[\frac{1-\left(\bar{F}(x)\right)^{m+1}}{1-\left(\bar{F}(y)\right)^{m+1}}\right]^{r-1}$$

$$\times\left[1-\frac{1-\left(\bar{F}(x)\right)^{m+1}}{1-\left(\bar{F}(y)\right)^{m+1}}\right]^{s-r-1}\frac{\left[\bar{F}(x)\right]^{m}}{\left[1-\left(\bar{F}(y)\right)^{m+1}\right]}f(x)\,dx.$$

Let

$$u=\frac{1-\left[\bar{F}(x)\right]^{m+1}}{1-\left[\bar{F}(y)\right]^{m+1}}=\frac{e^{-ah(x)}}{e^{-ah(y)}}, \text{ then}$$

$$g_{r|s}(y)=\frac{(s-1)!}{(r-1)!(s-r-1)!}\int_{0}^{1}\left(h(y)-\frac{1}{a}\ln u\right)u^{r-1}(1-u)^{s-r-1}\,du$$

$$=h(y)-\frac{1}{a}\frac{(s-1)!}{(r-1)!(s-r-1)!}\int_{0}^{1}\log u\, u^{r-1}(1-u)^{s-r-1}\,du.$$

Now in view of (2.4), we have

$$g_{r|s}(y)=h(y)+\frac{1}{a}\sum_{j=r}^{s-1}\frac{1}{j}.$$

Now to prove (2.5) implies (2.6), we have

$$E\left[h\left\{X(r,n,m,k)\right\} \mid X(s,n,m,k)=y\right]=g_{r|s}(y)$$

or

$$\frac{(s-1)!(m+1)}{(r-1)!(s-r-1)!}\int_\alpha^y h(x)\left[\bar{F}(x)\right]^m\left[1-\left(\bar{F}(x)\right)^{m+1}\right]^{r-1}$$

$$\times\left[\left(\bar{F}(x)\right)^{m+1}-\left(\bar{F}(y)\right)^{m+1}\right]^{s-r-1} f(x)\mathrm{d}x$$

$$= g_{r|s}(y)\left[1-\left(\bar{F}(y)\right)^{m+1}\right]^{s-1}.$$

Differentiating both the sides with respect to y and rearranging the terms, we get

$$\frac{(m+1)\left[\bar{F}(y)\right]^m f(y)}{1-\left[\bar{F}(y)\right]^{m+1}}=\frac{g'_{r|s}(y)}{(s-1)\left[g_{r|s-1}(y)-g_{r|s}(y)\right]} \qquad \text{[Khan } et\ al.\text{, 2006]}$$

$$=\frac{h'(y)}{(s-1)\left[h(y)+\frac{1}{a}\sum_{j=r}^{s-2}\frac{1}{j}-h(y)-\frac{1}{a}\sum_{j=r}^{s-1}\frac{1}{j}\right]}.$$

That is,

$$\frac{(m+1)\left[\bar{F}(y)\right]^m f(y)}{1-\left[\bar{F}(y)\right]^{m+1}}=-ah'(y). \tag{2.8}$$

Implying that $1-\left[\bar{F}(x)\right]^{m+1}=e^{-ah(x)}$

and hence the (2.6).

Now to prove (2.7), taking the limit as $m\to-1$ in the LHS of (2.8), we get

$$\frac{f(y)}{\bar{F}(y)}\frac{1}{-\log\bar{F}(y)}=-ah'(y)$$

or

$$\log\left[-\log\bar{F}(x)\right]=-\int_q^x ah'(y)dy,\ \ q\in(\alpha,\beta),$$

which gives,

$$\bar{F}(x)=\exp\left[-\exp\left(\frac{h(q)-h(x)}{\delta}\right)\right],\ \ m=-1, \delta=\frac{1}{a}\neq 0.$$

Remark 2.3:

For order statistics (at $m=0, k=1$), characterizing results based on linear regression were obtained by Khan and Abu-Salih (1989), Khan and Abouammoh (2000) and for record values ($m=-1, k=1$), similar characterization result is given by Franco and Ruiz (1997).

Theorem 2.3

Let X *be a random variable with absolutely continuous df* $F(x)$ *and pdf* $f(x)$ *over the support* (α,β), *and let* $h(x)$ *be a monotonic, continuous and differentiable function of* x, *then for two consecutive values of* r *and* $r+1$,

$$E\left[h\{X_d(s,n,m,k)\} \mid X_d(l,n,m,k)=x\right]=g_{s|l}(x)=h(x)+\frac{1}{a}\sum_{j=l}^{s-1}\frac{1}{\gamma_{j+1}},$$
$$l=r,r+1,\quad a\neq 0 \text{ and } \gamma_{j+1}\neq 0 \qquad (2.9)$$

if and only if

$$F(x)=e^{-ah(x)}.$$

Proof:

Necessary part can be proved on lines of Theorem 2.1.

To prove the sufficiency part, let

$$g_{s|r}(x)=E\left[h\{X_d(s,n,m,k)\} \mid X_d(r,n,m,k)=x\right].$$

Therefore, we have

$$g_{s|r+1}(x)-g_{s|r}(x)=-\frac{1}{a}\frac{1}{\gamma_{r+1}}.$$

Now, in view of Khan *et al.* (2010), we get

$$\frac{f(x)}{F(x)}=-ah'(x),$$

implying that

$$F(x)=e^{-ah(x)}.$$

Hence the theorem.

Theorem 2.4

Under the condition as stated in Theorem 2.2 and for two consecutive values $s-1$ *and* s.

$$E\left[h\{X_d(r,n,m,k)\} \mid X_d(l,n,m,k)=y\right]=g_{r|l}(y)=h(y)+\frac{1}{a}\sum_{j=r}^{l-1}\frac{1}{j},$$
$$l=s-1,s,\quad a\neq 0 \text{ and } j\neq 0 \qquad (2.10)$$

if and only if

$$1-\left[F(x)\right]^{m+1}=e^{-ah(x)},\quad m\neq -1 \qquad (2.11)$$

and

$$F(x) = \exp\left[\exp\left(\frac{h(x)-h(p)}{\delta}\right)\right], \quad m=-1, \ \delta = \frac{1}{a} \neq 0. \tag{2.12}$$

Proof:

Theorem can be established on the lines of Theorem 2.2.

3. EXAMPLES

In this section several examples based on Theorem 2.1 and Theorem 2.3 are listed with proper choice of a and $h(x)$.

Table 4.1

Examples based on the *df* $F(x) = 1 - e^{-ah(x)}$

Distribution	$F(x)$	a	$h(x)$
Exponential	$1-e^{-\theta x}$, $0<x<\infty, \theta>0$	θ	x
Weibull	$1-e^{-\theta x^p}$, $0<x<\infty, \theta, p>0$	θ	x^p
Pareto	$1-\left(\frac{x}{\alpha}\right)^{-\theta}$, $\alpha<x<\infty$, $\theta>0$	θ	$\log\left(\frac{x}{\alpha}\right)$
Lomax	$1-\left[1+\left(\frac{x}{\alpha}\right)\right]^{-p}$, $0<x<\infty$, $\alpha, p>0$	p	$\log\left(1+\left(\frac{x}{\alpha}\right)\right)$
Gompertz	$1-\exp\left[-\frac{\lambda}{\mu}\left(e^{\mu x}-1\right)\right]$, $0<x<\infty$, $\lambda, \mu>0$	$\frac{\lambda}{\mu}$	$e^{\mu x}-1$
Beta of I	$1-(1-x)^{\theta}$, $0<x<1$, $\theta>0$	θ	$-\log(1-x)$
Beta of II	$1-(1+x)^{-1}$, $0<x<\infty$	1	$\log(1+x)$
Extreme value I	$1-\exp\left[-e^{x}\right]$, $0<x<\infty$	1	e^{x}
Log logistic	$1-\left(1+\theta x^p\right)^{-1}$, $0<x<\infty$, $p, \theta>0$	1	$\log\left(1+\theta x^p\right)$
Burr type IX	$1-\left[\frac{c\left\{\left(1+e^{x}\right)^{k}-1\right\}}{2}+1\right]^{-1}$, $-\infty<x<\infty$	1	$\log\left[\frac{c\left\{\left(1+e^{x}\right)^{k}-1\right\}}{2}+1\right]$
Burr type XII	$1-\left(1+\theta x^p\right)^{-m}$, $0<x<\infty$, $p, \theta, m>0$	m	$\log\left(1+\theta x^p\right)$

Table 4.2
Examples based on $F(x)=e^{-ah(x)}$

Distribution	$F(x)$	a	$h(x)$
Inverse Weibull	$e^{-\theta x^{-p}}$, $0 \le x < \infty$, $p,\theta > 0$	θ	x^{-p}
Burr Type II	$\left(1+e^{-x}\right)^{-k}$, $-\infty < x < \infty$, $k > 0$	k	$\log\left(1+e^{-x}\right)$
Burr Type III	$\left(1+x^{-c}\right)^{-k}$, $0 \le x < \infty$, $c,k > 0$	k	$\log\left(1+x^{-c}\right)$
Burr Type IV	$\left[1+\left(\frac{c-x}{x}\right)^{1/c}\right]^{-k}$, $0 \le k \le c$	k	$\log\left[1+\left(\frac{c-x}{x}\right)^{-1/c}\right]$
Burr Type V	$\left[1+ce^{-\tan x}\right]^{-k}$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$	k	$\log\left(1+ce^{-\tan x}\right)$
Burr Type VI	$\left[1+ce^{-k\sinh x}\right]^{-k}$, $-\infty < x < \infty$	k	$\log\left(1+ce^{-\sinh x}\right)$
Burr Type VII	$2^{-k}\left(1+\tanh x\right)^{k}$, $-\infty < x < \infty$	$-k$	$\log\left[\frac{1+\tanh x}{2}\right]$
Burr Type VIII	$\left(\frac{2}{\pi}\tan^{-1}e^{x}\right)$, $-\infty < x < \infty$	$-k$	$\log\left[\frac{2}{\pi}\tan^{-1}e^{x}\right]$
Burr Type X	$\left(1+e^{-x^2}\right)^{k}$, $0 \le x < \infty$	$-k$	$\log\left(1+e^{-x^2}\right)$
Burr Type XI	$\left(x-\frac{1}{2\pi}\sin 2\pi x\right)^{k}$, $0 \le x \le 1$	$-k$	$\log\left[x-\frac{1}{2\pi}\sin 2x\right]$
Cauchy	$\frac{1}{2}+\frac{1}{\pi}\tan^{-1}x$, $-\infty < x < \infty$	-1	$\log\left[\frac{1}{2}+\frac{1}{\pi}\tan^{-1}x\right]$

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