Pak. J. Statist.
2015 Vol. 31(1), 135-158

A TWO HETEROGENEOUS SERVERS PERISHABLE INVENTORY SYSTEM OF A FINITE POPULATION WITH ONE UNRELIABLE SERVER AND REPEATED ATTEMPTS

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ABSTRACT

In this paper, we consider a continuous review perishable inventory system with a service facility having two heterogeneous servers, server 1 and server 2, and the demands originate from a finite population of N sources. In such systems, the customers demand is satisfied only after some service is completed by any one of the two servers. Compared to most inventory models in which inventory is depleted at the demand rate, here it is depleted at the rate that the service is completed. It is assumed that server 1 is perfectly reliable and server 2 is subject to interruptions. The interrupted server is repaired at an exponential rate. The primary customer who finds either both servers are busy or there is no item (excluding those in service) in the stock, enters into an orbit with probability p and is lost forever with probability $1-p.$ A retrial customer in the orbit, finding the stock level (excluding those in service) is zero or both servers are busy, returns to the orbit with probability q and is lost forever with probability $1-q.$ The interval between two successive repeated attempts is exponentially distributed. The items of inventory have exponential life times. As and when the on-hand inventory level drops to a prefixed level $s(\geq 2),$ an order for $Q(=S-s>s)$ units is placed. The ordered items are received after a random time which is distributed as exponential. The joint probability distribution of the number of customers in the orbit and the inventory level is obtained in the steady state case. The measures of system performance in the steady state are derived and the total expected cost rate is also calculated. The results are illustrated numerically.

KEYWORDS

Control policy; Inventory with service time; Perishable; Service inerrupion; Markov process; Finite population; Heterogeneous servers.

1. INTRODUCTION

In most of the inventory models considered in the literature, the demanded items are directly issued from the stock, if available. The demands that occurred during stock out period are either not satisfied (lost sales case) or satisfied only after the receipt of the ordered items (backlog case). In the latter, it is assumed either all (full backlog case) or only a prefixed number of demands (partial backlogging) that occurred during stock out period are satisfied. For review of these works see Nahmias [19], Raafat [23], Kalpakam and Arivarignan ([11], [12]), Elango and Arivarignan [8], Liu and Yang [17], Cakanyildirim et al. [1], Goyal and Giri [10] and Dura et al. [7] and the references therein. But in the case of inventories maintained at service facilities, the demanded items are issued to the customers only after some service is performed on it. In this situation the items are issued not at the time of demand but after a random time of service. This forces the formation of queues in these models. This necessitates the study of both the inventory level and the queue length joint distributions. Study of such models is beneficial to organizations which

1. Provide service to customers by using items from a stock.
2. Maintain stock of items each of which needs service such as assembly or initialization or installation, etc.

Examples for the first type include firms that are engaged in servicing consumer products such as Television sets, Computers, etc., and for the second type include firms that supply bicycles which need assembly of its parts, that supply food items which need heating or garnishing and that computers which need installation of basic services.

Single server queueing systems with server interruptions have been studied by many researchers including Federgruen and Green [9], Li et al. [16], Tang [28], Nakdimon and Yechiali [20], Wang et al. [33], Wang et al. [32], Choudhury and Tadj [2], to mention a few. Two other papers where the single server queueing model with inventory and server interruptions is considered, are by Krishnamoorthy et al. ([14], [15]). The article [14] is considered for an inventory model with instantaneous replenishment and the service process is subject to interruptions. The discussion in [15] is an inventory model with positive lead time, server interruptions and an orbit of infinite capacity, where no waiting space is provided for customers, other than for the one whose service gets interrupted.

In all the above models the authors assumed that the service facility had a single server. But in many real life situations the service facility may provide more than one server so that more customers are handled at a time. In queueing systems with multi-server and server interruptions have been widely studied in different contexts in the literature. We refer the reader to Mitrany and Avi-Itzhak [18], Vinod [30], Neuts and Lucantoni [22], Wartenhosrt [34] and, Wang and Chang [31] for details on multi-server queueing systems with server interruptions and further references. The multi-server queueing models mentioned above all assumed the servers to be homogeneous, where the individual service rates were the same for all the servers in the system. This assumption may be valid only when the service process is highly mechanically or electronically controlled. In a queueing system with human servers, we can not expect work to be carried out at the same rate. We face situations of this kind in our everyday life, e.g., at checkout counters in department stores, in banks, in hospitals, etc.

A retrial finite source perishable inventory system with two heterogeneous servers is motivated by the service facility system with restricted customers for example military canteen providing service to soldiers or a company canteen serving the members of the specific working community in the company. Machine service problems within an industry is also a problem which motivated us to create the present stochastic model. The problem we consider is more relevant to the real life situation.

The rest of the paper is organized as follows. In the next section, we describe the mathematical model and the notations used in this paper are defined. Analysis of the model and the steady state solution of the model are proposed in section 3. Some key system performance measures are derived in section 4. In section 5, the results are illustrated with numerical examples. The last section is meant for conclusion.

2. MATHEMATICAL MODEL

Consider a finite-source retrial inventory system with two servers, say, server 1 and server 2, where the primary customers are generated by N, $2 \le N < \infty$, sources. The arrival process of customers is quasi random distribution with parameter λ. Each source can be in three stages: generating a primary customer (free), sending repeated customers (retry), and under service (in service) by one of the servers. The maximum capacity of the inventory is S. The demand is for single item per customer and the item is delivered to the demanding customer after performing service on the item. The two servers provide heterogeneous exponential service to customers with service rates μ_1 and μ_2 for server 1 and server 2, respectively. It is assumed that server 1 is perfectly reliable and server 2 is subject to interruptions. While the server 2 serves a customer, the service may get interrupted with the interruption process governed by a Poisson process of rate α_1. Also, we assumed that while the server 2 is under interruption, no further interruption can befall the server 2. On completion of an interruption the service restarts, with the duration of an interruption exponentially distributed with parameter α_2. No waiting space is provided for customers, other than for the one whose service gets interrupted.

When the inventory level is positive and both servers are idle then an arriving customer (primary/retry) is taken up for service by the server 1. While both servers are busy or there is no item (excluding those in service) in the stock, an arriving primary customer enters the orbit of unsatisfied customers with probability p and a retrying customer goes back to the orbit with probability q. With complementary probabilities the customer leaves the system in both cases. If a customer finds any one of the server is idle and at least one item is not in service, then he/she immediately accedes to the service. The probability of a repeated attempt during the interval $(t, t+dt)$, given that j customers are in the orbit at time t, is $j\theta + o(dt)$ and the interretrial times follow an exponential distribution.

The items are perishable in nature and the life time of each item has a negative exponential distribution with parameter $\gamma > 0$. We assumed that the servicing item cannot perish. As and when the on-hand inventory level drops to a prefixed level $s(\ge 2)$, an order for $Q(= S - s > s)$ units is placed. The lead time distribution is exponential with parameter $\beta(> 0)$. All the times involved in the model construction are assumed to be mutually independent of each other.

2.1 Notations:

e : a column vector of appropriate dimension containing all ones

0 : Zero matrix

I : an identity matrix

$[A]_{ij}$: entry at $(i,j)^{th}$ position of a matrix A

$H(x)$: $\begin{cases} 1 & if \quad x \geq 0 \\ 0 & otherwise \end{cases}$

$k \in V_i^j$: $k = i, i+1, \ldots j$

$\underset{i=j}{\overset{k}{\Omega}} c_i$: $\begin{cases} c_j c_{j-1} \cdots c_k & \text{if} \quad j \geq k \\ 1 & \text{if} \quad j < k \end{cases}$

δ_{ij} : $\begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$

$\bar{\delta}_{ij}$: $1 - \delta_{ij}$

$Y(t)$: $\begin{cases} 0, & \text{if both servers are idle at time t} \\ 1, & \text{if both servers are busy at time t} \\ 2, & \text{if server 1 is busy while server 2 is on interruption a time t} \\ 3, & \text{if server 1 is busy while server 2 is idle at time t} \\ 4, & \text{if server 2 is busy while server 1 is idle at time t} \\ 5, & \text{if server 2 is on interruption while server 1 is idle at time t} \end{cases}$

E_1 : $\left\{ (i_1, 0, i_3) \mid i_1 \in V_0^S,\, i_3 \in V_0^N \right\}$

E_2 : $\left\{ (i_1, i_2, i_3) \mid i_1 \in V_2^S,\, i_2 \in V_1^2, i_3 \in V_0^{N-2} \right\}$

E_3 : $\left\{ (i_1, i_2, i_3) \mid i_1 \in V_1^S,\, i_2 \in V_3^5,\, i_3 \in V_0^{N-1} \right\}$

E : $E_1 \cup E_2 \cup E_3$

3. ANALYSIS

Let $L(t)$, $X_1(t)$ and $X_2(t)$ respectively, denote the inventory level, the server status and the number of customers in the orbit at time t. From the assumptions made on the input and output processes, it can be shown that the stochastic process $\left\{ \left(L(t), X_1(t), X_2(t) \right), t \geq 0 \right\}$ is a continuous time Markov chain with state space given by E. To obtain the intensity of passage, $a\left((i_1, i_2, i_3), (i_1, i_2, i_3) \right)$ of state (i_1, i_2, i_3), we note that the sum of the entries in any row of the matrix is zero. Hence the diagonal entry in any row of the matrix is equal to the negative of the sum of the other entries in that row. More explicitly,

$$a\left((i_1, i_2, i_3), (i_1, i_2, i_3) \right) = - \sum_{\substack{j_1 \\ (j_1, j_2, j_3) \neq (i_1, i_2, i_3)}} \sum_{j_2} \sum_{j_3} a\left((i_1, i_2, i_3), (j_1, j_2, j_3) \right).$$

Hence, we have $a\left(\left(i_1,i_2,i_3\right),\left(j_1,j_2,j_3,\right)\right)$

$$
=\begin{cases}
\left(N-i_3\right)\lambda, & j_1=i_1,\quad j_2=2,\quad j_3=i_3, \\
& i_1\in V_1^S,\quad i_2=0,\quad i_3\in V_0^{N-1}, \\
& \text{or} \\
& j_1=i_1,\quad j_2=1,\quad j_3=i_3, \\
& i_1\in V_2^S,\quad i_2\in V_3^4,\quad i_3\in V_0^{N-2}, \\
& \text{or} \\
& j_1=i_1,\quad j_2=2,\quad j_3=i_3, \\
& i_1\in V_2^S,\quad i_2=5,\quad i_3\in V_0^{N-2}, \\
& \\
p\left(N-i_3\right)\lambda, & j_1=0,\quad j_2=0,\quad j_3=i_3+1, \\
& i_1=0,\quad i_2=0,\quad i_3\in V_0^{N-1}, \\
& \text{or} \\
& j_1=i_1,\quad j_2=i_2,\quad j_3=i_3+1, \\
& i_1=1,\quad i_2\in V_3^5,\quad i_3\in V_0^{N-2}, \\
& \text{or} \\
& j_1=i_1,\quad j_2=i_2,\quad j_3=i_3+1, \\
& i_1\in V_2^S,\quad i_2=1,\quad i_3\in V_0^{N-3}, \\
& \text{or} \\
& j_1=i_1,\quad j_2=i_2,\quad j_3=i_3+1, \\
& i_1\in V_2^S,\quad i_2=2,\quad i_3\in V_0^{N-3}, \\
& \\
(1-q)i_3\theta, & j_1=0,\quad j_2=0,\quad j_3=i_3-1, \\
& i_1=0,\quad i_2=0,\quad i_3\in V_1^N, \\
& \text{or} \\
& j_1=i_1,\quad j_2=i_2,\quad j_3=i_3-1, \\
& i_1=1,\quad i_2\in V_3^5,\quad i_3\in V_1^{N-1}, \\
& \text{or} \\
& j_1=i_1,\quad j_2=i_2,\quad j_3=i_3-1, \\
& i_1\in V_2^S,\quad i_2=1,\quad i_3\in V_1^{N-2}, \\
& \text{or} \\
& j_1=i_1,\quad j_2=i_2,\quad j_3=i_3-1,
\end{cases}
$$

$$
\begin{cases}
 & j_1 = i_1, \quad j_2 = 2, \quad j_3 = i_3 - 1, \\
 & i_1 \in V_2^S, \quad i_2 = 5, \quad i_3 \in V_1^{N-1}, \\
i_1\gamma, & j_1 = i_1 - 1, \quad j_2 = 0, \quad j_3 = i_3, \\
 & i_1 \in V_1^S, \quad i_2 = 0, \quad i_3 \in V_0^N, \\
(i_1 - 1)\gamma, & j_1 = i_1 - 1, \quad j_2 = i_2, \quad j_3 = i_3, \\
 & i_1 \in V_2^S, \quad i_2 \in V_3^5, \quad i_3 \in V_0^{N-1}, \\
(i_1 - 2)\gamma, & j_1 = i_1 - 1, \quad j_2 = i_2, \quad j_3 = i_3, \\
 & i_1 \in V_3^S, \quad i_2 \in V_1^2, \quad i_3 \in V_0^{N-2}, \\
\beta, & j_1 = i_1 + Q, \quad j_2 = i_2, \quad j_3 = i_3, \\
 & i_1 \in V_0^s, \quad i_2 = 0, \quad i_3 \in V_0^N, \\
 & \text{or} \\
 & j_1 = i_1 + Q, \quad j_2 = i_2, \quad j_3 = i_3, \\
 & i_1 \in V_2^s, \quad i_2 \in V_1^2, \quad i_3 \in V_0^{N-2}, \\
 & \text{or} \\
 & j_1 = i_1 + Q, \quad j_2 = i_2, \quad j_3 = i_3, \\
 & i_1 \in V_1^s, \quad i_2 \in V_3^5, \quad i_3 \in V_0^{N-1}, \\
\alpha_1, & j_1 = i_1, \quad j_2 = 5, \quad j_3 = i_3, \\
 & i_1 \in V_1^S, \quad i_2 = 4, \quad i_3 \in V_0^{N-1}, \\
 & \text{or} \\
 & j_1 = i_1, \quad j_2 = 2, \quad j_3 = i_3, \\
 & i_1 \in V_2^S, \quad i_2 = 1, \quad i_3 \in V_0^{N-2}, \\
\alpha_2, & j_1 = i_1, \quad j_2 = 4, \quad j_3 = i_3, \\
 & i_1 \in V_1^S, \quad i_2 = 5, \quad i_3 \in V_0^{N-1}, \\
 & \text{or} \\
 & j_1 = i_1, \quad j_2 = 1, \quad j_3 = i_3,
\end{cases}
$$

$$
\begin{cases}
\mu_1\delta_{i_2 3}+\mu_2\delta_{i_2 4}, & j_1=i_1-1,\quad j_2=0,\quad j_3=i_3, \\
 & i_1\in V_1^S,\quad i_2\in V_3^4,\quad i_3\in V_0^{N-1}, \\
-\big(p(N-i_3)\lambda+ & j_1=i_1,\quad j_2=i_2,\quad j_3=i_3, \\
(1-q)i_3\theta+\beta), & i_1=0,\quad i_2=0,\quad i_3\in V_0^N, \\
-\big(p(N-i_3)\lambda+(1-q)i_3\theta+ & j_1=i_1,\quad j_2=i_2,\quad j_3=i_3, \\
\mu_1+(i_1-2)\gamma+H(s-i_1)\beta+ & i_1\in V_2^S,\quad i_2\in V_1^2,\quad i_3\in V_0^{N-2}, \\
\delta_{i_2 2}\alpha_2+\delta_{i_2 1}(\alpha_1+\mu_2)\big), & \\
-\big(p(N-i_3)\lambda+(1-q)i_3\theta+ & j_1=i_1,\quad j_2=i_2,\quad j_3=i_3, \\
\beta+\delta_{i_2 3}\mu_1+\delta_{i_2 4}(\alpha_1+\mu_2)+ & i_1=1,\quad i_2\in V_3^5,\quad i_3\in V_0^{N-1}, \\
\delta_{i_2 5}\,\alpha_2\big), & \\
-\big((N-i_3)\lambda+i_3\theta+i_1\gamma+ & j_1=i_1,\quad j_2=i_2,\quad j_3=i_3, \\
H(s-i_1)\beta\big), & i_1\in V_1^S,\quad i_2=0,\quad i_3\in V_0^N, \\
-\big((N-i_3)\lambda+i_3\theta+\delta_{i_2 3}\mu_1+ & j_1=i_1,\quad j_2=i_2,\quad j_3=i_3, \\
H(s-i_1)\beta+(i_1-1)\gamma+ & i_1\in V_2^S,\quad i_2\in V_3^5,\quad i_3\in V_0^{N-1}, \\
\delta_{i_2 4}(\alpha_1+\mu_2)+\delta_{i_2 5}\,\alpha_2\big), & \\
0, & \text{otherwise.}
\end{cases}
$$

The infinitesimal generator of this process is

$$\Theta=\Big(a\big((i_1,i_2,i_3),(j_1,j_2,j_3)\big)\Big),\ (i_1,i_2,i_3),(j_1,j_2,j_3)\in E$$

The infinitesimal generator Θ can be written in terms of submatrices $\Theta_{i_1 j_1}$, namely,

$$\Theta=\big(\big(\Theta_{i_1 j_1}\big)\big),$$

where

$$\Theta_{i_1 j_1} = \begin{cases} A_{i_1} & j_1 = i_1,\ i_1 = 0,1,2,\ldots,S \\ B_{i_1} & j_1 = i_1 - 1,\ i_1 = 1,2,\ldots,S-1,S \\ C & j_1 = i_1 + Q,\ i_1 = 2,\ldots,s, \\ C_1 & j_1 = i_1 + Q,\ i_1 = 1, \\ C_2 & j_1 = i_1 + Q,\ i_1 = 0, \\ 0 & \textit{Otherwise.} \end{cases}$$

With

$$[C_2]_{i_2 j_2} = \begin{cases} D_1, & j_2 = 0, \quad i_2 = 0, \\ 0, & \textit{otherwise.} \end{cases}$$

$$[D_1]_{i_3 j_3} = \begin{cases} \beta, & j_3 = i_3, \quad i_3 \in V_0^N, \\ 0, & \textit{otherwise.} \end{cases}$$

$$[C_1]_{i_2 j_2} = \begin{cases} D_1, & j_2 = 0, \quad i_2 = 0, \\ D_3, & j_2 = i_2, \quad i_2 \in V_3^5, \\ 0, & \textit{otherwise.} \end{cases}$$

$$[D_3]_{i_3 j_3} = \begin{cases} \beta, & j_3 = i_3, \quad i_3 \in V_0^{N-1}, \\ 0, & \textit{otherwise.} \end{cases}$$

$$[C]_{i_2 j_2} = \begin{cases} D_1, & j_2 = 0, \quad i_2 = 0, \\ D_2, & j_2 = i_2, \quad i_2 \in V_1^2, \\ D_3, & j_2 = i_2, \quad i_2 \in V_3^5, \\ 0, & \textit{otherwise.} \end{cases}$$

$$[D_2]_{i_3 j_3} = \begin{cases} \beta, & j_3 = i_3, \quad i_3 \in V_0^{N-2}, \\ 0, & \textit{otherwise.} \end{cases}$$

$$[B_1]_{i_2 j_2} = \begin{cases} T_1 & j_2 = i_2, \quad i_2 = 0, \\ D_4 & j_2 = 0, \quad i_2 = 3, \\ D_5 & j_2 = 0, \quad i_2 = 4, \\ 0, & \textit{otherwise.} \end{cases}$$

$$[D_4]_{i_3 j_3} = \begin{cases} \mu_1, & j_3 = i_3, \quad i_3 \in V_0^{N-1}, \\ 0, & otherwise. \end{cases}$$

$$[D_5]_{i_3 j_3} = \begin{cases} \mu_2 & j_3 = i_3, \quad i_3 \in V_0^{N-1}, \\ 0, & otherwise. \end{cases}$$

$$[B_2]_{i_2 j_2} = \begin{cases} T_2, & j_2 = i_2, \quad i_2 = 0, \\ W_0, & j_2 = 3, \quad i_2 = 1, \\ W_1, & j_2 = 4, \quad i_2 = 1, \\ W_1, & j_2 = 5, \quad i_2 = 2, \\ W_2, & j_2 = 0, \quad i_2 = 3, \\ W_3, & j_2 = 0, \quad i_2 = 4, \\ W_4, & j_2 = i_2, \quad i_2 \in V_3^5, \\ 0, & otherwise. \end{cases}$$

$$[W_0]_{i_3 j_3} = \begin{cases} \mu_2, & j_3 = i_3, \quad i_3 \in V_0^{N-2}, \\ 0, & otherwise. \end{cases}$$

$$[W_1]_{i_3 j_3} = \begin{cases} \mu_1, & j_3 = i_3, \quad i_3 \in V_0^{N-2}, \\ 0, & otherwise. \end{cases}$$

$$[W_2]_{i_3 j_3} = \begin{cases} \mu_1, & j_3 = i_3, \quad i_3 \in V_0^{N-1}, \\ 0, & otherwise. \end{cases}$$

$$[W_3]_{i_3 j_3} = \begin{cases} \mu_2, & j_3 = i_3, \quad i_3 \in V_0^{N-1}, \\ 0, & otherwise. \end{cases}$$

$$[W_4]_{i_3 j_3} = \begin{cases} \gamma, & j_3 = i_3, \quad i_3 \in V_0^{N-1}, \\ 0, & otherwise. \end{cases}$$

For $i_1 = 3, 4, \dots, S;$

$$\left[B_{i_1}\right]_{i_2 j_2} = \begin{cases} T_{i_1}, & j_2 = i_2, \quad i_2 = 0, \\ M_{(i_1-2)}, & j_2 = i_2, \quad i_2 \in V_1^2, \\ U_{(i_1-1)}, & j_2 = i_2, \quad i_2 \in V_3^5, \\ W_0, & j_2 = 3, \quad i_2 = 1, \\ W_1, & j_2 = 4, \quad i_2 = 1, \\ W_2, & j_2 = 0, \quad i_2 = 3, \\ W_3, & j_2 = 0, \quad i_2 = 4, \\ 0, & otherwise. \end{cases}$$

For $i_1 = 1, 2, 3, \dots, S;$

$$\left[T_{i_1}\right]_{i_3 j_3} = \begin{cases} i_1\gamma, & j_3 = i_3, \quad i_3 \in V_0^N, \\ 0, & otherwise. \end{cases}$$

For $i_1 = 3, 4, \dots, S;$

$$\left[M_{(i_1-2)}\right]_{i_3 j_3} = \begin{cases} (i_1 - 2)\gamma, & j_3 = i_3, \quad i_3 \in V_0^{N-2}, \\ 0, & otherwise. \end{cases}$$

For $i_1 = 3, 4, \dots, S;$

$$\left[U_{(i_1-1)}\right]_{i_3 j_3} = \begin{cases} (i_1 - 1)\gamma, & j_3 = i_3, \quad i_3 \in V_0^N, \\ 0, & otherwise. \end{cases}$$

$$\left[A_0\right]_{i_2 j_2} = \begin{cases} D_0, & j_2 = 0, \quad i_2 = 0, \\ 0, & otherwise. \end{cases}$$

$$\left[D_0\right]_{i_3 j_3} = \begin{cases} p(N - i_3)\lambda, & j_3 = i_3 + 1, \quad i_3 \in V_0^{N-1}, \\ (1-q)i_3\theta, & j_3 = i_3 - 1, \quad i_3 \in V_1^N, \\ -\left(p(N - i_3)\lambda + (1-q)i_3\theta + \beta\right), & j_3 = i_3, \quad i_3 \in V_0^N, \\ 0, & otherwise. \end{cases}$$

$$[A_1]_{i_2 j_2} = \begin{cases} L_1, & j_2 = i_2, \quad i_2 = 0, \\ D, & j_2 = 3, \quad i_2 = 0, \\ D_6, & j_2 = i_2, \quad i_2 = 3, \\ D_7, & j_2 = i_2, \quad i_2 = 4, \\ D_8, & j_2 = i_2, \quad i_2 = 5, \\ R_0, & j_2 = 5, \quad i_2 = 4, \\ R_1, & j_2 = 4, \quad i_2 = 5, \\ 0, & otherwise. \end{cases}$$

$$[D]_{i_3 j_3} = \begin{cases} (N - i_3)\lambda, & j_3 = i_3, \quad i_3 \in V_0^{N-1}, \\ i_3\theta, & j_3 = i_3, \quad i_3 \in V_1^{N}, \\ 0, & otherwise. \end{cases}$$

$$[D_6]_{i_3 j_3} = \begin{cases} p(N - i_3)\lambda, & j_3 = i_3 + 1, \quad i_3 \in V_0^{N-2}, \\ (1-q)i_3\theta, & j_3 = i_3 - 1, \quad i_3 \in V_1^{N-1}, \\ -(\bar{\delta}_{i_3 N-1} p(N - i_3)\lambda + (1-q)i_3\theta + \mu_1 + \beta), & j_3 = i_3, \quad i_3 \in V_0^{N-1}, \\ 0, & otherwise. \end{cases}$$

$$[D_7]_{i_3 j_3} = \begin{cases} p(N - i_3)\lambda, & j_3 = i_3 + 1, \quad i_3 \in V_0^{N-2}, \\ (1-q)i_3\theta, & j_3 = i_3 - 1, \quad i_3 \in V_1^{N-1}, \\ -(\bar{\delta}_{i_3 N-1} p(N - i_3)\lambda + (1-q)i_3\theta + \mu_2 + \alpha_1 + \beta), & j_3 = i_3, \quad i_3 \in V_0^{N-1}, \\ 0, & otherwise. \end{cases}$$

$$[D_8]_{i_3 j_3} = \begin{cases} p(N - i_3)\lambda, & j_3 = i_3 + 1, \quad i_3 \in V_0^{N-2}, \\ (1-q)i_3\theta, & j_3 = i_3 - 1, \quad i_3 \in V_1^{N-1}, \\ -(p(N - i_3)\lambda + (1-q)i_3\theta + \alpha_2 + \beta), & j_3 = i_3, \quad i_3 \in V_0^{N-1}, \\ 0, & otherwise. \end{cases}$$

$$[R_0]_{i_3 j_3} = \begin{cases} \alpha_1, & j_3 = i_3, \quad i_3 \in V_0^{N-1}, \\ 0, & otherwise. \end{cases}$$

$$[R_1]_{i_3 j_3} = \begin{cases} \alpha_2, & j_3 = i_3, \quad i_3 \in V_0^{N-1}, \\ 0, & otherwise. \end{cases}$$

For $i_1 = 2,3,\ldots,S;$

$$\left[A_{i_1}\right]_{i_2 j_2} = \begin{cases} L_{i_1}, & j_2 = i_2, & i_2 = 0, \\ F_{i_1}, & j_2 = i_2, & i_2 = 1, \\ G_{i_1}, & j_2 = i_2, & i_2 = 3, \\ H_{i_1}, & j_2 = i_2, & i_2 = 4, \\ J_{i_1}, & j_2 = i_2, & i_2 = 5, \\ K_{i_1}, & j_2 = i_2, & i_2 = 2, \\ D, & j_2 = 3, & i_2 = 0, \\ R_2, & j_2 = 2, & i_2 = 1, \\ R_3, & j_2 = 1, & i_2 = 2, \\ R_4, & j_2 = 1, & i_2 \in V_3^4, \\ R_4, & j_2 = 2, & i_2 = 5, \\ R_0, & j_2 = 5, & i_2 = 4, \\ R_1, & j_2 = 4, & i_2 = 5, \\ 0, & \textit{otherwise.} \end{cases}$$

For $i_1 = 1,2,3,\ldots,S;$

$$\left[L_{i_1}\right]_{i_3 j_3} = \begin{cases} -(\bar{\delta}_{i_3 N}(N-i_3)\lambda + i_3\theta + H(s-i_1)\beta + i_1\gamma), & j_3 = i_3, \quad i_3 \in V_0^N, \\ 0, & \textit{otherwise.} \end{cases}$$

For $i_1 = 2,3,\ldots,S;$

$$\left[F_{i_1}\right]_{i_3 j_3} = \begin{cases} p(N-i_3)\lambda, & j_3 = i_3+1, \quad i_3 \in V_0^{N-3}, \\ (1-q)i_3\theta, & j_3 = i_3-1, \quad i_3 \in V_1^{N-2}, \\ -(p(N-i_3)\lambda + (1-q)i_3\theta + (i_1-2)\gamma & j_3 = i_3, \quad i_3 \in V_0^{N-2}, \\ +H(s-i_1)\beta + \mu_1 + \mu_2 + \alpha_1), & \\ 0, & \textit{otherwise.} \end{cases}$$

For $i_1 = 2,3,\ldots,S;$

$$\left[G_{i_1}\right]_{i_3 j_3} = \begin{cases} -(\bar{\delta}_{i_3 N-1}(N-i_3)\lambda + i_3\theta + (i_1-1)\gamma & j_3 = i_3, \quad i_3 \in V_0^{N-1}, \\ +H(s-i_1)\beta + \mu_1 + \beta), & \\ 0, & \textit{otherwise.} \end{cases}$$

For $i_1 = 2,3,\ldots,S;$

$$\left[H_{i_1}\right]_{i_3 j_3} = \begin{cases} -(\overline{\delta}_{i_3 N-1}(N-i_3)\lambda + i_3\theta + (i_1-1)\gamma & j_3 = i_3, \quad i_3 \in V_0^{N-1}, \\ +H(s-i_1)\beta + \mu_2 + \alpha_1), & \\ 0, & otherwise. \end{cases}$$

For $i_1 = 2,3,\ldots,S;$

$$\left[K_{i_1}\right]_{i_3 j_3} = \begin{cases} p(N-i_3)\lambda, & j_3 = i_3+1, \quad i_3 \in V_0^{N-3}, \\ (1-q)i_3\theta, & j_3 = i_3-1, \quad i_3 \in V_1^{N-2}, \\ -(p(N-i_3)\lambda + (1-q)i_3\theta + (i_1-2)\gamma & j_3 = i_3, \quad i_3 \in V_0^{N-2}, \\ +H(s-i_1)\beta + \mu_1 + \alpha_2), & \\ 0, & otherwise. \end{cases}$$

For $i_1 = 2,3,\ldots,S;$

$$\left[J_{i_1}\right]_{i_3 j_3} = \begin{cases} -((N-i_3)\lambda + i_3\theta + (i_1-1)\gamma & j_3 = i_3, \quad i_3 \in V_0^{N-1}, \\ +H(s-i_1)\beta + \alpha_2), & \\ 0, & otherwise. \end{cases}$$

$$\left[R_2\right]_{i_3 j_3} = \begin{cases} \alpha_1, & j_3 = i_3, \quad i_3 \in V_0^{N-2}, \\ 0, & otherwise. \end{cases}$$

$$\left[R_3\right]_{i_3 j_3} = \begin{cases} \alpha_2, & j_3 = i_3, \quad i_3 \in V_0^{N-2}, \\ 0, & otherwise. \end{cases}$$

$$\left[R_4\right]_{i_3 j_3} = \begin{cases} (N-i_3)\lambda, & j_3 = i_3, \quad i_3 \in V_0^{N-2}, \\ i_3\theta, & j_3 = i_3-1, \quad i_3 \in V_1^{N-1}, \\ 0, & otherwise. \end{cases}$$

It can be noted that the matrices C_2 is of size $(N+1)\times(6N-1)$, C_1 is of size $(4N+1)\times(6N-1)$, C, A_i and B_i, $i = 2,\cdots,S$ are of size $(6N-1)\times(6N-1)$, A_1 and B_1 are of size $(4N+1)\times(4N+1)$. A_0, T_i, $i = 1,2,3,\ldots,S$ and D_1 are square matrices of size $(N+1)\times(N+1)$. D_3, $U_{(i-1)}$, $i = 3,4,\cdots,S$ and W_4 are square matrices of size $N\times N$. D_4, D_5, W_2, and W_3 are matrices of size $N\times(N+1)$, D_2 and $V_{(i-2)}$, $i = 3,4,\cdots,S$, are square matrices of size $(N-1)\times(N-1)$, W_0 and W_1 are matrices of size $(N-1)\times N$.

3.1 Steady State Analysis

It can be seen from the structure of Θ that the homogeneous Markov process $\{(L(t), X_1(t), X_2(t)) : t \geq 0\}$ on the finite space E is irreducible, aperiodic and persistent non-null. Hence the limiting distribution

$$\varphi^{(i_1,i_2,i_3)} = \lim_{t\to\infty} Pr\left[L(t) = i_1, X_1(t) = i_2, X_2(t) = i_3 \mid L(0), X_1(0), X_2(0)\right],$$

exists. Let $\Phi = \left(\varphi^{(0)}, \varphi^{(1)}, \ldots, \varphi^{(S)}\right)$,

partitioning the vector, $\varphi^{(i_1)}$ as follows:

$$\varphi^{(0)} = \left(\varphi^{(0,0)}\right),$$
$$\varphi^{(1)} = \left(\varphi^{(1,0)}, \varphi^{(1,3)}, \varphi^{(1,4)}, \varphi^{(1,5)}\right),$$
$$\varphi^{(i_1)} = \left(\varphi^{(i_1,0)}, \varphi^{(i_1,1)}, \varphi^{(i_1,2)}, \varphi^{(i_1,3)}, \varphi^{(i_1,4)}, \varphi^{(i_1,5)}\right), \quad i_1 = 2,3,\ldots,S;$$

which is partitioned as follows:

$$\varphi^{(0,0)} = \left(\varphi^{(0,0,0)}, \varphi^{(0,0,1)}, \varphi^{(0,0,2)}, \ldots, \varphi^{(0,0,N)}\right),$$
$$\varphi^{(1,0)} = \left(\varphi^{(1,0,0)}, \varphi^{(1,0,1)}, \varphi^{(1,0,2)}, \ldots, \varphi^{(1,0,N)}\right),$$
$$\varphi^{(1,i_2)} = \left(\varphi^{(1,i_2,0)}, \varphi^{(1,i_2,1)}, \varphi^{(1,i_2,2)}, \ldots, \varphi^{(1,i_2,N-1)}\right), \quad i_2 = 3,4,5;$$
$$\varphi^{(i_1,0)} = \left(\varphi^{(i_1,0,0)}, \varphi^{(i_1,0,1)}, \varphi^{(i_1,0,2)}, \ldots, \varphi^{(i_1,0,N)}\right), \quad i_1 = 2,3,\ldots,S;$$
$$\varphi^{(i_1,i_2)} = \left(\varphi^{(i_1,i_2,0)}, \varphi^{(i_1,i_2,1)}, \varphi^{(i_1,i_2,2)}, \ldots, \varphi^{(i_1,i_2,N-2)}\right), i_1 = 2,3,\ldots,S;\ i_2 = 1,2;$$
$$\varphi^{(i_1,i_2)} = \left(\varphi^{(i_1,i_2,0)}, \varphi^{(i_1,i_2,1)}, \varphi^{(i_1,i_2,2)}, \ldots, \varphi^{(i_1,i_2,N-1)}\right), i_1 = 2,3,\ldots,S;\ i_2 = 3,4,5;$$

where $\varphi^{(i_1,i_2,i_3)}$ denotes the steady state probability for the state (i_1,i_2,i_3) of the process, exists and is given by

$$\Phi\Theta = 0 \quad \text{and} \quad \sum_{(i_1,i_2,i_3)}\sum\sum \varphi^{(i_1,i_2,i_3)} = 1. \tag{1}$$

The first equation of the above yields the following set of equations:

$$\varphi^{(i_1)} B_{i_1} + \varphi^{(i_1-1)} A_{i_1-1} = 0, \qquad i_1 = 1,2,\ldots,Q,$$
$$\varphi^{(i_1)} B_{i_1} + \varphi^{(i_1-1)} A_{i_1-1} + \varphi^{(i_1-1-Q)} C_2 = 0, \qquad i_1 = Q+1, \quad (*)$$
$$\varphi^{(i_1)} B_{i_1} + \varphi^{(i_1-1)} A_{i_1-1} + \varphi^{(i_1-1-Q)} C_1 = 0, \qquad i_1 = Q+2,$$
$$\varphi^{(i_1)} B_{i_1} + \varphi^{(i_1-1)} A_{i_1-1} + \varphi^{(i_1-1-Q)} C = 0, \qquad i_1 = Q+3, Q+4, \ldots, S,$$
$$\varphi^{(S)} A_S + \varphi^{(s)} C = 0.$$

After lengthy simplifications, the above equations, except $(*)$, yields

$$\varphi^{(i_1)} = (-1)^{(Q-i_1)}\varphi^{(Q)} \mathop{\Omega}_{j=Q}^{i_1+1} B_j A_{j-1}^{-1}, \qquad i_1 = Q-1, Q-2, \ldots, 0$$

$$= (-1)^{(Q)}\varphi^{(Q)} \sum_{j=0}^{s-1}\left[\left\{\left(\mathop{\Omega}_{k=Q}^{(s+1)-j} B_k A_{k-1}^{-1}\right) C A_{S-j}^{-1}\right\}\left\{\left(\mathop{\Omega}_{l=S-j}^{i_1+1} B_l A_{l-1}^{-1}\right) C_1 A_i^{-1}\right\}\right], \qquad i_1 = Q+1$$

$$= (-1)^{(2Q-i_1+1)}\varphi^{(Q)} \sum_{j=0}^{S-i_1}\left[\left(\mathop{\Omega}_{k=Q}^{s+1-j} B_k A_{k-1}^{-1}\right) C A_{S-j}^{-1}\left(\mathop{\Omega}_{l=S-j}^{i_1+1} B_l A_{l-1}^{-1}\right)\right], \qquad i_1 = S, S-1, \ldots, Q+2$$

where $\varphi^{(Q)}$ can be obtained by solving,

$$\varphi^{(Q+1)} B_{Q+1} + \varphi^{(Q)} A_Q + \varphi^{(0)} C_2 = 0 \quad \text{and} \quad \sum_{i_1=0}^{S} \varphi^{(i_1)} e = 1,$$

that is

$$\varphi^{(Q)}\left[(-1)^{(Q)}\varphi^{(Q)} \sum_{j=0}^{s-1}\left[\left\{\left(\mathop{\Omega}_{k=Q}^{(s+1)-j} B_k A_{k-1}^{-1}\right) C A_{S-j}^{-1}\right\}\left\{\left(\mathop{\Omega}_{l=S-j}^{i_1+1} B_l A_{l-1}^{-1}\right) C_1 A_i^{-1}\right\}\right] B_{Q+1} + A_Q + \left\{(-1)^{(Q)} \mathop{\Omega}_{j=Q}^{1} B_j A_{j-1}^{-1}\right\} C_2\right] = 0,$$

and

$$\varphi^{(Q)}\left[\sum_{i_1=0}^{Q-1}\left((-1)^{(Q-i_1)} \mathop{\Omega}_{j=Q}^{i_1+1} B_j A_{j-1}^{-1}\right) + I + (-1)^{(Q)} \sum_{j=0}^{s-1}\left[\left\{\left(\mathop{\Omega}_{k=Q}^{(s+1)-j} B_k A_{k-1}^{-1}\right) C A_{S-j}^{-1}\right\}\left\{\left(\mathop{\Omega}_{l=S-j}^{i_1+1} B_l A_{l-1}^{-1}\right) C_1 A_i^{-1}\right\}\right] + \sum_{i_1=Q+1}^{S}\left((-1)^{(2Q-i_1+1)} \sum_{j=0}^{S-i_1}\left[\left(\mathop{\Omega}_{k=Q}^{s+1-j} B_k A_{k-1}^{-1}\right) C A_{S-j}^{-1}\left(\mathop{\Omega}_{l=S-j}^{i_1+1} B_l A_{l-1}^{-1}\right)\right]\right)\right] e = 1.$$

4. SYSTEM PERFORMANCE MEASURES

In this section some performance measures of the system under consideration in the steady state are derived.

4.1 Expected Inventory Level

Let η_I denote the average inventory position in the steady state which is given by

$$\eta_I = \sum_{i_1=1}^{S}\sum_{i_3=0}^{N} i_1 \varphi^{(i_1,0,i_3)} + \sum_{i_1=1}^{S}\sum_{i_2=3}^{5}\sum_{i_3=0}^{N-1} i_1 \varphi^{(i_1,i_2,i_3)} + \sum_{i_1=2}^{S}\sum_{i_2=1}^{2}\sum_{i_3=0}^{N-2} i_1 \varphi^{(i_1,i_2,i_3)} \qquad (2)$$

4.2 Expected Reorder Rate

Let η_R denote the expected reorder rate in the steady state which is given by

$$\eta_R = \sum_{i_3=0}^{N}(s+1)\gamma\varphi^{(s+1,0,i_3)} + \sum_{i_2=3}^{5}\sum_{i_3=0}^{N-1}\left[\delta_{i_2 5}s\gamma + \delta_{i_2 3}\left(s\gamma+\mu_1\right)+\delta_{i_2 4}\left(s\gamma+\mu_2\right)\right]\varphi^{(s+1,i_2,i_3)}$$
$$+\sum_{i_2=1}^{2}\sum_{i_3=0}^{N-2}\left[\delta_{i_2 1}\left((s-1)\gamma+\mu_1+\mu_2\right)+\delta_{i_2 2}\left((s-1)\gamma+\mu_1\right)\right]\varphi^{(s+1,i_2,i_3)} \qquad (3)$$

4.3 Expected Perishable Rate

Let η_P denote the expected perishable rate in the steady state which is given by

$$\eta_P = \sum_{i_1=1}^{S}\sum_{i_3=0}^{N} i_1\gamma\varphi^{(i_1,0,i_3)} + \sum_{i_1=2}^{S}\sum_{i_2=1}^{2}\sum_{i_3=0}^{N-2}(i_1-2)\gamma\varphi^{(i_1,i_2,i_3)} + \sum_{i_1=2}^{S}\sum_{i_2=3}^{5}\sum_{i_3=0}^{N-1}(i_1-1)\gamma\varphi^{(i_1,i_2,i_3)} \qquad (4)$$

4.4 Expected Number of Customers in the Orbit

Let η_O denote the expected number of customers in the orbit which is given by

$$\eta_O = \sum_{i_1=0}^{S}\sum_{i_3=1}^{N} i_3\varphi^{(i_1,0,i_3)} + \sum_{i_1=2}^{S}\sum_{i_2=1}^{2}\sum_{i_3=1}^{N-2} i_3\varphi^{(i_1,i_2,i_3)} + \sum_{i_1=1}^{S}\sum_{i_2=3}^{5}\sum_{i_3=1}^{N-1} i_3\varphi^{(i_1,i_2,i_3)} \qquad (5)$$

4.5 Expected Number of Customers Lost Before Entering the Orbit

Let η_{L1} denote the expected number of customers lost before entering the orbit which is given by

$$\eta_{L1} = \sum_{i_3=0}^{N-1}(1-p)(N-i_3)\lambda\varphi^{(0,0,i_3)} + \sum_{i_2=3}^{5}\sum_{i_3=0}^{N-2}(1-p)(N-i_3)\lambda\varphi^{(1,i_2,i_3)}$$
$$+\sum_{i_1=2}^{S}\sum_{i_2=1}^{2}\sum_{i_3=0}^{N-3}(1-p)(N-i_3)\lambda\varphi^{(i_1,i_2,i_3)} \qquad (6)$$

4.6 Expected Number of Customers Lost after Retrials

Let η_{L2} denote the expected number of customers lost after retrials which is given by

$$\eta_{L2} = \sum_{i_3=1}^{N}(1-q)i_3\theta\varphi^{(0,0,i_3)} + \sum_{i_2=3}^{5}\sum_{i_3=1}^{N-1}(1-q)i_3\theta\varphi^{(1,i_2,i_3)} + \sum_{i_1=2}^{S}\sum_{i_2=1}^{2}\sum_{i_3=1}^{N-2}(1-q)i_3\theta\varphi^{(i_1,i_2,i_3)} \qquad (7)$$

4.7 Effective Interruption Rate

Let η_{INTR} denote the effective interruption rate which is given by

$$\eta_{INTR} = \sum_{i_1=2}^{S}\sum_{i_3=0}^{N-2}\alpha_1\varphi^{(i_1,1,i_3)} + \sum_{i_1=1}^{S}\sum_{i_3=0}^{N-1}\alpha_1\varphi^{(i_1,4,i_3)} \qquad (8)$$

4.8 Effective Repair Rate

Let η_{REP} denote the effective repair rate which is given by

$$\eta_{REP} = \sum_{i_1=2}^{S}\sum_{i_3=0}^{N-2}\alpha_2\varphi^{(i_1,2,i_3)} + \sum_{i_1=1}^{S}\sum_{i_3=0}^{N-1}\alpha_2\varphi^{(i_1,5,i_3)} \tag{9}$$

4.9 The Overall Retrial Rate

Let η_{OR} denote the overall rate of retrials in the steady state which is given by

$$\eta_{OR} = \sum_{i_1=0}^{S}\sum_{i_3=1}^{N} i_3\theta\varphi^{(i_1,0,i_3)} + \sum_{i_1=2}^{S}\sum_{i_2=1}^{2}\sum_{i_3=1}^{N-2} i_3\theta\varphi^{(i_1,i_2,i_3)} + \sum_{i_1=1}^{S}\sum_{i_2=3}^{5}\sum_{i_3=1}^{N-1} i_3\theta\varphi^{(i_1,i_2,i_3)} \tag{10}$$

4.10 The Successful Retrial Rate

Let η_{SR} denote the successful retrial rate in the steady state which is given by

$$\eta_{SR} = \sum_{i_1=1}^{S}\sum_{il=1}^{N} i_3\theta\varphi^{(i_1,0,i_3)} + \sum_{i_1=2}^{S}\sum_{i_2=3}^{5}\sum_{i_3=1}^{N-1} i_3\theta\varphi^{(i_1,i_2,i_3)} \tag{11}$$

4.11 The Fraction of Successful Rate of Retrial

Let η_{FR} denote the successful retrial rate in the steady state which is given by

$$\eta_{FR} = \frac{\eta_{SR}}{\eta_{OR}} \tag{12}$$

4.12 Total Expected Cost Rate

To compute the total expected cost per unit time, we consider the following costs.

c_h : The inventory carrying cost per unit item per unit time.

c_s : Setup cost per order.

c_p : Failure cost per unit item per unit time.

c_w : Waiting time cost of a orbiting customer per unit time.

c_l : Cost due to loss of customers per unit per unit time.

c_i : Cost per interruption per unit time.

c_r : Cost per repair per unit time

The long run total expected cost rate is given by

$$TC(S,s,M) = c_h\eta_I + c_s\eta_R + c_p\eta_P + c_w\eta_O + c_l\left(\eta_L = \eta_{L1} + \eta_{L2}\right) + c_i\eta_{INTR} + c_r\eta_{REP}. \tag{13}$$

where $\eta's$ are as given in $(2)-(9)$.

Since the computation of the φ's are recursive, it is quite difficult to study the qualitative behaviour of model analytically. However, we present the following numerical examples to demonstrate the computability of the results derived in our work and to illustrate the effect of the costs on the main performance characteristics.

5. NUMERICAL ILLUSTRATIONS

In this section, we discuss some numerical examples that reveal the possible convexity of the total expected cost rate. First, we explore the behavior of the cost function by considering it as function of any two variables by fixing the other one at a constant level.

Table 1
Total Expected Cost Rate as a Function of S and s

$\lambda = 11, \beta = 1.2, \gamma = 3, \alpha_1 = 5, \alpha_2 = 9, \mu_1 = 1.2, \mu_2 = 0.09, \theta = 0.006, p = 0.5,$
$q = 0.5, c_h = 0.1, c_s = 1.3, c_p = 0.007,$
$c_w = 0.002, c_l = 0.009, c_i = 0.005, c_r = 0.009, N = 10.$

S \ s	7	8	9	10	11
43	11.724783	11.714067	**11.713352**	11.719872	11.732426
44	11.724085	11.713254	**11.711574**	11.717067	11.728524
45	11.723767	11.713303	**11.710742**	11.715299	11.725758
46	11.724088	11.714165	**11.710803**	11.714510	11.724060
47	11.724709	11.715798	**11.711709**	11.714644	11.725371

Table 2
Total Expected Cost Rate as a Function of S and N

$\lambda = 0.11, \beta = 12, \gamma = 3, \alpha_1 = 5, \alpha_2 = 0.9, \mu_1 = 0.002, \mu_2 = 9, \theta = 0.6, p = 0.5,$
$q = 0.5, c_h = 0.1, c_s = 1.3, c_p = 0.007, c_w = 1.2, c_l = 0.009, c_i = 0.005, c_r = 0.09, s = 9.$

S \ N	16	17	18	19	20
22	3.080566	**3.077946**	3.078023	3.080460	3.084950
23	3.068105	3.065511	**3.065449**	3.067596	3.071661
24	3.062852	3.060376	**3.060285**	3.062272	3.066056
25	3.063412	3.061126	**3.061096**	3.063027	3.066649
26	3.068720	**3.066682**	3.066787	3.068748	3.072304

Tables 1-3, give the total expected cost rate as a function of (S,s), (S,N) and (s,N) by fixing respectively N, s and S each at a constant level. All the costs and other parameters are assigned fixed values which are indicated in each Table.

The value that is shown **bold** is the least among the values in that row and the value that is shown underlined is the least in that column. It may be observed that, these values in each table exhibit a (possibly) local minimum of the function of the two variables. We also note that the total expected cost rate is more sensitive to s than to S and N.

Example 1

In this example, we study the impact of the costs c_s, c_h, c_p, c_w, c_l, c_i and c_r on the optimal values $\left(s^*, S^*\right)$ and the corresponding total expected cost rate TC^*. Towards this end, we first fix the parameter values as $\lambda = 11$, $\beta = 1.2$, $\gamma = 3$, $\alpha_1 = 5$, $\alpha_2 = 9$, $\mu_1 = 1.2$, $\mu_2 = 0.09$, $\theta = 0.006$, $p = 0.5$ and $q = 0.05$. We observe the following from Table 4 to Table 8:

Table 3
Total Expected Cost Rate as a Function of s and N
$\lambda = 11$, $\beta = 1.2$, $\gamma = 3$, $\alpha_1 = 5$, $\alpha_2 = 9$, $\mu_1 = 1.2$, $\mu_2 = 0.09$, $\theta = 0.006$, $p = 0.5$,
$q = 0.05$, $c_h = 0.1$, $c_s = 1.3$, $c_p = 0.007$,
$c_w = 1.2$, $c_l = 0.009$, $c_i = 0.005$, $c_r = 0.09$, $S = 45$.

s \ N	17	18	19	20	21
3	2.597325	2.595521	2.581732	**2.573957**	2.586196
4	2.525087	2.521657	**2.518703**	2.519175	2.524028
5	2.491462	2.491142	**2.491051**	2.491166	2.491462
6	2.504236	**2.503449**	2.504761	2.506163	2.507645
7	2.537725	2.536626	**2.535568**	2.538547	2.540558

1. When each of c_s, c_h, c_p, c_w, c_l, c_i and c_r increases, the long run total expected cost rate $TC(s,S)$ also increases.

2. As is to be expected, S^* increases with c_s and c_w. This is because if setup cost increases, then we have to maintain high inventory to avoid frequent ordering. Similarly the waiting cost increases we have to maintain high inventory to reduce the number of waiting customers. Also, we note that c_h, c_l, c_i and c_r increase when S^* decreases.

3. We cannot predict the behaviour of s^* when each of c_l, c_i and c_r increases.

Table 4

Sensitivity of c_h and c_s on the Optimal Values

$c_p = 0.007,\ c_w = 1.2,\ c_l = 0.009,\ c_i = 0.005,\ c_r = 0.09,\ N = 10.$

c_s / c_h	1.0		1.3		1.6		1.9		2.2	
0.1	44	10	45	9	46	8	48	8	49	7
	11.468530		11.710742		11.945108		12.173855		12.395647	
0.115	40	9	41	8	43	8	44	7	45	7
	11.604646		11.850068		12.087985		12.318465		12.545149	
0.130	37	9	38	8	39	7	40	7	41	6
	11.729517		11.977051		12.216978		12.451999		12.679924	
0.145	34	8	35	7	36	7	37	6	38	6
	11.842550		12.093232		12.336618		12.572686		12.804368	
0.160	31	7	33	7	33	6	34	6	34	5
	11.947932		12.200619		12.445494		12.684988		12.920425	

Table 5

Variation in Optimal Values for Different Values of c_s and c_w

$c_h = 0.1,\ c_p = 0.007,\ c_l = 0.009,\ c_i = 0.005,\ c_r = 0.09,\ N = 10.$

c_s / c_w	1.0		1.3		1.6		1.9		2.2	
1.0	27	6	27	5	29	5	30	5	30	4
	10.292182		10.545433		10.790461		11.031254		11.267123	
1.2	31	7	33	7	33	6	34	6	34	5
	11.947932		12.200619		12.445494		12.684988		12.920425	
1.4	36	9	36	8	37	7	38	7	39	6
	13.570887		13.824408		14.070792		14.309684		14.545621	
1.6	40	10	40	9	42	9	42	8	44	8
	15.169402		15.423611		15.670769		15.910007		16.146450	
1.8	43	11	44	10	45	10	46	9	46	8
	16.748020		17.002679		17.250183		17.490262		17.726801	

Table 6
Variation in Optimal Values for Different Values of c_s and c_p

$c_h = 0.1,\ c_w = 1.2,\ c_l = 0.009,\ c_i = 0.005,\ c_r = 0.09,\ N = 10.$

c_s / c_p	1.0		1.3		1.6		1.9		2.2	
0.005	33	8	33	7	34	6	35	6	37	6
	11.908610		12.160056		12.404951		12.642520		12.877113	
0.007	31	7	33	7	33	6	34	6	34	5
	11.947932		12.200619		12.445494		12.684988		12.920425	
0.009	30	7	32	7	32	6	34	6	34	6
	11.985455		12.239957		12.485130		12.726510		12.961464	
0.011	30	7	30	6	31	6	33	6	33	5
	12.022278		12.277350		12.523968		12.766876		13.001492	
0.013	29	7	29	6	31	6	31	5	32	5
	12.057869		12.313469		12.561893		12.804539		13.040698	

Table 7
Variation in Optimal Values for Different Values of c_w and c_l

$c_h = 0.1,\ c_s = 1.3,\ c_p = 0.007,\ c_i = 0.005,\ c_r = 0.09,\ N = 10.$

c_w / c_l	1.0		1.3		1.6		1.9		2.2	
0.007	27	5	33	7	37	8	40	9	44	10
	10.541381		12.196313		13.819853		15.418988		16.997864	
0.009	27	5	33	7	36	8	40	9	44	10
	10.545433		12.200619		13.824406		15.423611		17.0026679	
0.011	27	5	32	7	36	8	40	9	44	10
	10.549485		12.204776		13.828808		15.428235		17.007494	
0.013	26	5	32	7	35	8	40	9	44	10
	10.553537		12.208917		13.833208		15.432858		17.012309	
0.015	26	5	31	7	35	8	39	9	43	10
	10.557589		12213058		13.837608		15.437482		17.017124	

Table 8
Variation in Optimal Values for Different Values of c_r and c_i
$c_h = 0.1,\ c_s = 1.3,\ c_p = 0.007,\ c_w = 1.2,\ c_l = 0.009,\ N = 10.$

c_r / c_i	0.06		0.09		0.12		0.15		0.18	
0.005	33	7	33	7	32	7	32	7	32	7
	12.189126		12.200619		12.211944		12.223254		12.234563	
0.010	33	7	32	7	32	7	32	7	32	7
	12.191265		12.202735		12.214044		12.225354		12.236663	
0.015	32	7	32	7	32	7	32	7	32	7
	12.193405		12.204835		12.216145		12.227454		12.238763	
0.020	32	7	31	7	31	7	31	7	31	7
	12.195544		12.206936		12.218245		12.229554		12.240864	
0.025	32	7	31	7	31	7	30	7	30	7
	12.197683		12.209036		12.220345		12.231654		12.242964	

6. CONCLUSIONS

The stochastic model discussed here is useful in studying a two heterogeneous servers perishable inventory system of a finite population with one unreliable server and repeated attempts. The joint probability distribution of the number of customers in the orbit and the inventory level is derived in the steady state. Various system performance measures are derived and the long-run total expected cost rate is calculated. By assuming a suitable cost structure on the inventory system, we have presented extensive numerical illustrations to show the effect of change of values for constants on the total expected cost rate.

ACKNOWLEDGEMENT

V.S.S. Yadavalli is thankful to National Research Foundation for their support. N. Anabzhagan's research was supported by the National Board for Higher Mathematics (DAE), Government of India through research project 2/48(11)/2011/R&D II/1141. The authors are grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

REFERENCES

1. Cakanyildirim, M., Bookbinder, J.H. and Gerchak, Y. (2000). Continuous review inventory models where random lead time depends on lot size and reserved capacity. *International Journal of Production Economics*, 68(3), 217 - 228.
2. Choudhury. G. and Tadj, L. (2009). An M/G/1 queue with two phases of service subject to the server breakdown and delayed repair. *Applied Mathematical Modelling*, 33, 2699-2709.

3. Daniel, J.K. and Ramanarayanan, R. (1987). An inventory system with two servers and rest periods. *Cahiers du C.E.R.O, Universite Libre De Bruxelles*, 29, 95-100.
4. Daniel, J.K. and Ramanarayanan, R. (1988). An (s,S) inventory system with rest periods to the server. *Naval Research Logistics*, John Wiley & Sons, 35, 119-123.
5. Doshi, B.T. (1986). Queueing systems with vacations: A survey. *Queueing Systems*, 1, 29-66.
6. Doshi, B.T. (1990). Single server queues with vacations, in H. Takagi (Ed.), *Stochastic Analysis of Computer and Communication Systems*, Elsevier Science Publishers B.V. (North-Holland), Amsterdam, 217-265.
7. Dura, N.A., Gutierrez, G. and Zequeira, R.I. (2004). A continuous review inventory model with order expediting. *International Journal of Production Economics*, 87(2), 157-169.
8. Elango, C. and Arivarignan, G. (2003). A continuous review perishable inventory systems with poisson demand and partial backlogging, In: Balakrishnan, N., Kannan, N., & Srinivasan, M.R. (Eds), *Satistical Methods and Practice: Recent Advances*. Narosa Publishing House, New Delhi.
9. Federgruen, A. and Green, L. (1986). Queueing systems with service interruptions, *Operations Research*, 34, 752-768.
10. Goyal, S.K. and Giri, B.C. (2001). Recent trends in modeling of deteriorating inventory. *European Journal of Operational Research*, 34(1), 1-16.
11. Kalpakam, S. and Arivarignan, G. (1990). Inventory system with random supply quantity. *OR Spektrum*, 12, 139-145.
12. Kalpakam, S. and Arivarignan, G. (1993). A coordinated multicommodity (s,S) inventory system. *Mathematical and Computer Modelling*, 18, 69-73.
13. Krishnamoorthy, A. and Anbazhagan, N. (2008). Perishable inventory system at service facility with N policy. *Stochastic Analysis and Applications*, 26, 1-17.
14. Krishnamoorthy, A., Nair, S.S. and Narayanan, V.C. (2010). An inventory model with server interruptions. *Opsearch*, 1-17.
15. Krishnamoorthy, A., Nair, S.S. and Narayanan, V.C. (2012). An inventory model with server interruptions and retrials. *Operations Research International Journal*, 12(2), 151-171.
16. Li, W., Shi, D. and Chao, X. (1997). Reliability analysis of M/G/1 queueing systems with server breakdowns and vacations. *Journal of Applied Probability*, 34, 546-555.
17. Liu, L. and Yang, T. (1999). An (s,S) random lifetime inventory model with a positive lead time. *European Journal of Operational Research*, 113, 52-63.
18. Mitrany, I.L. and Avi-Itzhak, B. (1968). A many-server queue with service interruptions. *Operations Research*, 16, 628-638.
19. Nahmias, S. (1982). Perishable inventory theory: A review. *Operations Research*, 30, 680-708.
20. Nakdimon, O. and Yechiali, U. (2003). Polling systems with breakdowns and repairs. *European Journal of Operational Research*, 149, 588-613.
21. Narayanan, V.C., Deepak, T.G., Krishnamoorthy, A. and Krishnakumar, B. (2008). On an (s,S) inventory policy with service time, vacation to server and correlated lead time. *Qualitative Technology & Quantitative Management*, 5(2), 129-143.

22. Neuts M.F. and Lucantoni, D.M. (1979). A Markovian queue with N servers subject to breakdowns and repairs. *Management Science*, 25, 849-861.
23. Raafat, F. (1991). A survey of literature on continuously deteriorating inventory models. *Journal of Operational Research Society*, 42, 27-37.
24. Ross, S.M. (2000). *Introduction to Probability Models*, Harcourt Asia PTE Ltd, Singapore.
25. Sivakumar B. (2011). An inventory system with retrial demands and multiple server vacation. *Quality Technology & Quantitative Management*, 8(2), 125-146.
26. Takagi, H. (1991). *Queueing Analysis - A Foundation of Performance Evaluation*, Vol. I., Elsevier Science Publishers B.V.
27. Takagi, H. (1993). *Queueing Analysis - A Foundation of Performance Evaluation*, Vol. III. Elsevier Science Publishers B.V.
28. Tang, Y. (1997). Single-server M/G/1 queueing system subject to breakdowns - some reliability and queueing problems, *Microelectronics Reliability*, 37, 315-321.
29. Tian, N. and Zhang, Z.G. (2006). *Vacation Queueing Models - Theory and Applications*, International Series in Operations Research & Management Science, Springer-Verlag, New York, Inc. Secaucus, NJ, USA.
30. Vinod, B. (1985). Unreliable queueing systems. *Computers and Operations Research*, 12, 322-340.
31. Wang K.H. and Chang, Y.C. (2002). Cost analysis of a finite M/M/R queueing system with balking, reneging and server breakdowns. *Mathematical Methods of Operations Research*, 56, 169-180.
32. Wang, J., Liu, B. and Li, J. (2008). Transient analysis of an M/G/1 retrial queue subject to disasters and server failures. *European Journal of Operational Research*, 189, 1118-1132.
33. Wang, K., Wang, T. and Pearn, W. (2007). Optimal control of the N policy M/G/1 queueing system with server breakdowns and general startup times. *Applied Mathematical Modelling*, 31, 2199-2212.
34. Wartenhosrt, P. (1995). N parallel queueing systems with server breakdown and repair. *European Jopurnal of Operational Research*, 82, 302-322.