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AN ASYMMETRIC GENERALIZED FGM COPULA AND ITS PROPERTIES

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ABSTRACT

In this paper, we introduce a new family of Farlie–Gumbel–Morgenstern (FGM) copulas. This family is an asymmetric generalization family of Farlie–Gumbel–Morgenstern (FGM) copulas and includes some of its recent extensions. Some general formulas for well-known association measures of these copulas are obtained and various properties of the proposed model are studied.

KEYWORDS

FGM Copula; Associated Measures; Admissible Range; Concepts of Dependence.

1. INTRODUCTION

A copula is a function that links univariate marginal distributions into a joint multivariate one, specially, into a bivariate one. As Sklar (1959) stated, these functions are bivariate distribution functions by support $[0,1]^2$, whose margins are uniform in $[0,1]$. He showed that if H is a bivariate distribution function with marginal distributions $F(X)$ and $G(Y)$, then there must exist a copula C such that $H_\theta(X,Y)=C\left(F(X),G(Y);\theta\right)$, where θ is introduced as dependence parameter. This strategy provides an approach for modeling a bivariate distribution function. However, if univariate marginal distribution functions and an appropriate copula are determined, it is easy to compute the joint distribution, function (More details can be found in Joe, 1997 and Nelson, 2006). One of the most popular parametric families of copulas, which were studied by Farlie (1960), Gumbel (1960) and Morgenstern (1956), is the Farlie-Gumbel-Morgenstern *(FGM)* copula. Because of their simple analytical form, *FGM* copula is widely used in modeling and studying the efficiency of nonparametric procedures. However, this copula has been shown to be somewhat limited. In detail for copula dependence parameter $\theta\in[-1,1]$, the Spearman's Rho and Kendall's Tau are $\rho_S=\theta/3\in[-1/3,1/3]$ and $\tau_k=2\theta/9\in[-2/9,2/9]$, respectively. Since the correlation domain of *FGM* copula is limited, more general copulas have been introduced with the aim of improving the correlation range. Huang and Kotz (1999) developed Polynomial-type single-parameter extensions of *FGM* copula. They showed that ρ_S can be increased up to approximately 0.375 while the lower bound remains -0.33. Bairamov and Kotz (2002) further extended the family given by Huang and Kotz (1999) to the associated Spearman's $\rho_S\in[-0.48,0.5016]$. Lai and Xie (2000)

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gave conditions for positive quadrant dependence and studied a class of bivariate uniform distribution with positive quadrant dependence property by generalizing the uniform representation of a well-known *FGM* copula. By a simple transformation, they also obtained families of bivariate distributions with pre-specified marginals. An alternative approach to generalize the *FGM* family of the symmetric semi-parametric is defined by Fischer and Klein (2004). It was extensively studied in Amblard and Girard (2002). Cuadras and Diaz (2012) introduced an extended *FGM* family in two dimensions and studied how to approximate any distribution to this family. Bekrizadeh, et al. (2012) proposed a new class of generalized *FGM* copula and showed that their generalization can improve the correlation domain of *FGM* copula.

In this regard, this paper propose another generalization of *FGM* copula, which includes some of extended copulas introduced in recent years and can improve the correlation range i.e. the proposed family covers some of the introduced family in the literature and its correlation range is more efficient. From another perspective, this presented family, is an asymmetric extension of the generalized *FGM* copula discussed in Bekrizadeh, et al. (2012).

The main contribution of this paper includes the followings: first, an extension of *FGM* copula and some fine properties are presented. Second, asymmetry properties and the general formulas for association measures of this family are studied. The main feature of this family is capability for modeling a wider range of dependence. This permits us to extend the range of potential applications of the family in various branches of sciences.

The rest of the present paper is as follows: the new asymmetric extension and their basic characteristics are described in Section 2. The associated Spearman's Rho and Kendall's tau are studied in Section 3. Section 4 is dedicated to the dependence structure properties of this new family of copulas. Finally, the results of this research are stated in Section 5.

2. A NEW CLASS OF FGM COPULA

The copula is mostly defined as a function $C:[0,1]^2 \to [0,1]$ that satisfies the boundary conditions

A1. $C(u,0)=C(0,u)=0$ and $C(u,1)=C(1,u)=u$, $\forall u \in [0,1]$,

A2. $\forall \left(u_1,u_2,v_1,v_2\right) \in [0,1]^4$, such that $u_1 \le u_2$ and $v_1 \le v_2$,
$C\left(u_2,v_2\right)-C\left(u_2,v_1\right)-C\left(u_1,v_2\right)+C\left(u_1,v_1\right) \ge 0$.

Eventually, for twice differentiable, 2-increasing property (A2) can be replaced by the condition

$$c(u,v)=\frac{\partial^2}{\partial u \partial v} C(u,v) \ge 0 , \qquad (2.1)$$

where $c(u,v)$ is the so-called copula density. A copula C is *symmetric* if $C(u,v)=C(v,u)$, for every $(u,v) \in [0,1]^2$, otherwise C is *asymmetric*.

Many different copulas can be found in the literature, see for instance Nelson (2006). The *FGM* copula is one of these copulas which are widely used in application. The *FGM* copula is defined by

$$C^{FGM}(u,v) = uv\left[1+\theta(1-u)(1-v)\right], \theta \in [-1,1].$$

These copulas are the only ones whose functional form is quadratic in both u and v. Regarding limitation of correlation range in *FGM* copula, along with other generalizations, Bekrizadeh, et al. (2012) proposed a new class of symmetric generalized *FGM* family whose dependence is as follows:

$$C(u,v) = uv\left[1+\theta\left(1-u^{\alpha}\right)\left(1-v^{\alpha}\right)\right]^{p}, \ \alpha > 0 \text{ and } p = 0,1,2,\ldots.$$

In the following definition, an asymmetric extension of the above copula family is introduced in order to extend the *FGM* copula.

Definition 2.1

Suppose that the continuous functions $A_1, A_2 : [0,1] \to [0,1]$ are differentiable on $(0,1)$. An *asymmetric* function $C_{\theta,p}^{A_1,A_2} : [0,1]\times[0,1] \to [0,1]$ is defined as

$$C_{\theta,p}^{A_1,A_2}(u,v) = uv\left[1+\theta A_1(u)A_2(v)\right]^{p}, \ p \in [0,\infty), \ \forall (u,v) \in [0,1]^2, \qquad (2.2)$$

where the parameter $\theta \in \Theta = [-1,1]$ is called the associated parameter.

We assume that A_1 and A_2 do not change their sign on $[0,1]$ in order to obtain unique determined dependence structure. Note that the copula is limited to the range of $[0,1]$ and therefore, $\left[1+\theta A_1(u)A_2(v)\right]^{p}$ should be bounded on $[0,1]$.

The following theorem gives sufficient and necessary conditions on A_1 and A_2 to ensure that $C_{\theta,p}^{A_1,A_2}$ is a bivariate copula.

Theorem 2.1

Let $A_1, A_2 : [0,1] \to [0,1]$ be continuously differentiable functions on $(0,1)$. The function $C_{\theta,p}^{A_1,A_2}$ is a bivariate copula if and only if A_1 and A_2 satisfy the following conditions:

B1. $A_i(1) = 0$, for $i = 1,2$,

B2. $\left|xA_i'(x)\right| \le 1$ and $\left|A_i(x) + pxA_i'(x)\right| \le 1$, for every $x \in [0,1]$, and for $i = 1,2$, where $A_i'(x) = \partial A_i(x)/\partial x$.

Proof:

The proof involves two steps:

First, it is clear that $C_{\theta,p}^{A_1,A_2}(x,1) = C_{\theta,p}^{A_1,A_2}(1,x) = x, \ \forall x \in [0,1] \Leftrightarrow$ (B1).

Second, since A_1 and A_2 are continuously differentiable functions and $[1+\theta A_1(u)A_2(v)]^p$ is bounded on $[0,1]$ and 2-increasing function, by (2.1) the condition $c_{\theta,p}^{A_1,A_2}(u,v)\geq 0$ hold, if and only if $|xA_i'(x)|\leq 1$ and $|A_i(x)+pxA_i'(x)|\leq 1$, for every $x\in[0,1]$, and $i=1,2$, where $c_{\theta,p}^{A_1,A_2}(u,v)$ is as follows:

$$
\begin{aligned}
c_{\theta,p}^{A_1,A_2}(u,v) &= \partial^2 C_{\theta,p}^{A_1,A_2}(u,v)/\partial u\partial v \\
&= [1+\theta A_1(u)A_2(v)]^{p-2} \\
&\quad \times\Big\{\Big(1+\theta A_2(v)\big[A_1(u)+puA_1'(u)\big]\Big)\Big(1+\theta A_1(u)\big[A_2(v)+pvA_2'(v)\big]\Big) \\
&\quad + p\theta\, uA_1'(u)vA_2'(v)\Big\}. \blacksquare \qquad (2.3)
\end{aligned}
$$

Note that θ is a parameter that shows dependence structure of the family $C_{\theta,p}^{A_1,A_2}$ and $\theta=0$ or $p=0$, leads to the independence of u and v. By theorem 2.1, the concrete amount of the parameter space θ is dependent on the properties of the function of A_1 and A_2 that has been investigated via (2.3) for every amount of u and v in $[0,1]$. If $A_1(x)=A_2(x)$, $\forall x\in[0,1]$, then the family $C_{\theta,p}^{A_1,A_2}$ is *symmetric*.

Remark 2.1

The family $C_{\theta,p}^{A_1,A_2}$ includes some known family of *FGM* copulas introduced by researchers in recent years, which are as follows:

i) $A_i(x)=1-x$, $\forall x\in[0,1]$, for $i=1,2$, and $p=1$, the family $C_{\theta,p}^{A_1,A_2}$ leads to the symmetric *FGM* copula discussed by Farlie (1960), Gumbel (1960) and Morgenstern (1956).

ii) if $A_i(x)=1-x^{\alpha}$, $\forall x\in[0,1]$, for $i=1,2$, $\alpha\geq 0$, and $p=1$, the family $C_{\theta,p}^{A_1,A_2}$ leads to the symmetric extended *FGM* copula introduced by Huang and Kotz (1999)

iii) if $A_i(x)=x^q(1-x)^q$, $\forall x\in[0,1]$, for $i=1,2$, $q\geq 1$ and $p=1$, the family $C_{\theta,p}^{A_1,A_2}$ leads to the symmetric extended *FGM* copula introduced by Lai and Xie (2000).

iv) if $A_i(x)=(1-x^{\gamma})^{\lambda}$, $\forall x\in[0,1]$, for $i=1,2$, $\gamma\geq 0$, $\lambda\geq 1$ and $p=1$, the family $C_{\theta,p}^{A_1,A_2}$ leads to the symmetric extended *FGM* copula introduced by Bairamov-Kotz (2002)

v) if $A_i(x)=A(x)$, $\forall x\in[0,1]$, for $i=1,2$, and $p=1$, the family $C_{\theta,p}^{A_1,A_2}$ leads to the symmetric copula introduced by Rodriguez-Lallena and Ubeda-Flores (2004).

Moreover, via the family $C_{\theta,p}^{A_1,A_2}$, some new generalizations can be defined by introducing additional parameters p for the families (i)-(v), and we can generate some copulas of the family $C_{\theta,p}^{A_1,A_2}$ of type (2.2) through functions A_1 and A_2.

Proposition 2.1

Two limited properties of the family $C_{\theta,p}^{A_1,A_2}$ are as follows:

i. $\lim_{p\to 0} C_{\theta,p}^{A_1,A_2}(u,v) = \lim_{p\to 0} uv\left[1+\theta A_1(u)A_2(v)\right]^p = uv = \Pi(u,v)$, where $\Pi(u,v)$ is the independent copula.

ii. Let $\theta = \dfrac{\gamma}{p}$, where $\gamma \le p$, then

$$\lim_{p\to\infty} C_{\theta,p}^{A_1,A_2}(u,v) = \lim_{p\to\infty} uv\left[1+\frac{\gamma}{p}A_1(u)A_2(v)\right]^p$$
$$= uv\exp\left[\gamma A_1(u)A_2(v)\right] = C_\gamma(u,v). \blacksquare \quad (2.4)$$

The new family of copula in (2.4) can be a new asymmetric generalization of Cuadras copula (2009).

3. MEASURES OF DEPENDENCE

Measures of dependence are common instruments to summarize a complicated dependence structure in the bivariate case. For a historical review of measures of dependence, see Joe (1997) and Nelsen (2006). In this section, we compute the measures of dependence for the family $C_{\theta,p}^{A_1,A_2}$. Since we cannot give formulas for the properties of dependence in terms of elementary functions, it is replaced by its expansion series on

$$\Omega = \left\{\left(\theta, A_1, A_2\right):\ \left|\theta A_1(u)A_2(v)\right| < 1\right\}.$$

Based on Ω, the family $C_{\theta,p}^{A_1,A_2}$ in (2.2) for every $p \in [0,\infty)$ may also be written by polynomial expansion with respect to A_1 and A_2 as

$$C_{\theta,p}^{A_1,A_2}(u,v) = uv + \sum_{k=1}^{g}\binom{p}{k}\theta^k uA_1^{\,k}(u)vA_2^{\,k}(v). \quad (3.1)$$

Note that, in (3.1), we have $g = p$ when p is integer, otherwise, g equals to $+\infty$.

3.1 Spearman's rho

Let X and Y be continuous random variables whose copula is C. Then the population version of Spearman's rho for X and Y is given by

$$\rho_S = 12\int_0^1\int_0^1 C(u,v)\,du\,dv - 3.$$

Note that, ρ_S coincides with correlation coefficient ρ between the uniform marginal distributions.

Proposition 3.1

Let (X,Y) be a pair of random variables with the family $C_{\theta,p}^{A_1,A_2}$. The Spearman's rho (ρ_S) for the family $C_{\theta,p}^{A_1,A_2}$ is given by

$$\rho_S = 12\sum_{k=1}^{g}\binom{p}{k}\theta^k B_1(k)B_2(k) \ , \tag{3.2}$$

where $B_i(k) = \int_0^1 x\,A_i^{\,k}(x)dx$, for $i = 1,2$.

Proof:

By using (3.1), the Spearman's ρ_S can be expanded as

$$\begin{aligned}
\rho_S &= 12\int_0^1\int_0^1 C_{\theta,p}^{A_1,A_2}(u,v)dudv - 3 \\
&= 12\int_0^1\int_0^1 uv[1+\theta A_1(u)A_2(v)]^p\, dudv - 3 \\
&= 12\int_0^1\int_0^1 uv\sum_{k=0}^{g}\binom{p}{k}\theta^k A_1^{\,k}(u)A_2^{\,k}(v)dudv - 3 \\
&= 12\int_0^1\int_0^1 \left\{uv + \sum_{k=1}^{g}\binom{p}{k}\theta^k u\, A_1^{\,k}(u)\, v\, A_2^{\,k}(v)\right\} dudv - 3 \\
&= 12\left\{\int_0^1\int_0^1 uvdudv + \int_0^1\int_0^1 \sum_{k=1}^{g}\binom{p}{k}\theta^k uA_1^{\,k}(u)vA_2^{\,k}(v)dudv\right\} - 3 \\
&= 12\left\{\sum_{k=1}^{g}\binom{p}{k}\theta^k \int_0^1\int_0^1 uA_1^{\,k}(u)vA_2^{\,k}(v)dudv\right\} \\
&= 12\left\{\sum_{k=1}^{g}\binom{p}{k}\theta^k \left(\int_0^1 uA_1^{\,k}(u)du\right)\left(\int_0^1 vA_2^{\,k}(v)dv\right)\right\} \\
&= 12\sum_{k=1}^{g}\binom{p}{k}\theta^k B_1(k)B_2(k)\ . \ \blacksquare
\end{aligned}$$

3.2 Kendall's tau

In terms of copula, Kendall's tau τ_k is defined as (see Nelsen, 2006)

$$\tau_k = 4\int_0^1\int_0^1 c(u,v)C(u,v)dudv - 1.$$

Proposition 3.2

Let (X,Y) be a pair of random variables with the family $C_{\theta,p}^{A_1,A_2}$ and the family density $c_{\theta,p}^{A_1,A_2}$. The Kendall's tau (τ_k) can be expanded as

$$\tau_k = 4\sum_{k=0}^{g}\binom{2p-2}{k}\theta^k\left\{B_1(k)B_2(k)+2\theta\left(1-\frac{2p}{(k+1)^2}\right)B_1(k+1)B_2(k+1)\right.$$
$$\left.+\theta^2\left(\frac{k+2-2p}{k+2}\right)^2 B_1(k+2)B_2(k+2)\right\}-1\,. \qquad (3.3)$$

Proof:

For the sake textual unity the proposition's proof deferred to the Appendix. ■

Remark 3.1

For $A_i(x)=1-x$, $i=1,2$, $\forall x\in[0,1]$, and $p=1$ in (2.2), we have for the classical *FGM* copula as discussed in Farlie (1960), Gumbel (1960) and Morgenstern (1956) that $\rho_S=\theta/3$ and $\tau_k=2\theta/9$. Hence, we have $-0.33\le\rho_S\le 0.33$ and $-0.22\le\tau_k\le 0.22$ (as $-1\le\theta\le 1$).

As the remark (3.1) shows, the domain of correlation of *FGM* copula is limited and therefore it is not allowed for modeling of strong dependence. One of the advantages of the family $C_{\theta,p}^{A_1,A_2}$ is capability to improve the domain of correlation by introducing additional parameter p in *FGM* copula and some generalized *FGM* families presented in recent years.

Example 3.1

In the family $C_{\theta,p}^{A_1,A_2}$, let $A_i(x)=1-x$, for $i=1,2$, and $\forall x\in[0,1]$. Then the family $C_{\theta,p}^{A_1,A_2}$ leads to a new symmetric generalized *FGM* copula with $-\left(\max\{1,p\}\right)^{-1}\le\theta\le p^{-1}$.

Since $B_i(k)=\int_0^1 xA_i^k(x)dx=\frac{1}{(k+1)(k+2)}$, we have by using (3.2) that

$$\rho_S = 12\sum_{k=1}^{g}\binom{p}{k}\theta^k\left[\frac{1}{(k+1)(k+2)}\right]^2,$$

where the upper bound of above Spearman's ρ_S can be increased up to approximately 0.3805 as $p\to\infty$, while the lower bound -0.3333 remains unchanged. Therefore, the admissible range of Spearman's ρ_S in the new symmetric generalized *FGM* family is $[-0.3333, 0.3805]$.

Example 3.2

In the family $C_{\theta,p}^{A_1,A_2}$, let $A_i(x)=1-x^{\alpha}$, for $i=1,2$, $\forall x\in[0,1]$, and $\alpha\succ 0$. Then the family $C_{\theta,p}^{A_1,A_2}$ leads to a new symmetric generalized Hung-Kotz family with $-\left(\max\left\{1,p\alpha^2\right\}\right)^{-1}\le\theta\le(p\alpha)^{-1}$.

Since $B_i(k)=\int_0^1 xA_i^k(x)dx=\dfrac{\Gamma(k+1)\Gamma(2/\alpha)}{\alpha\Gamma(k+1+2/\alpha)}$, for $i=1,2$, we have

$$\rho_S=12\sum_{k=1}^{g}\binom{p}{k}\theta^k\left(\frac{\Gamma(k+1)\Gamma(2/\alpha)}{\alpha\Gamma(k+1+2/\alpha)}\right)^2 . \tag{3.4}$$

Taking $\alpha\cong 1.85$ and $p>500$ in (3.4), we have $\rho_{S,\max}=0.43$. Similarly, taking $\alpha\cong 0.1$ and $p>500$, we obtain $\rho_{S,\min}\cong -0.50$. Therefore, the admissible range of Spearman's ρ_S in the generalized Hung-Kotz family is $[-0.50,0.43]$. So, the generalized Hung-Kotz copula improves the amplitude ρ_S of Hung-Kotz family.

Remark 3.2

For $p=1$, (3.4) reduces to $\rho_S=3\theta\left(\dfrac{\alpha}{\alpha+2}\right)^2$ whose range is

$$-3\left(\max\left\{1,\alpha^2\right\}\right)^{-1}\left(\frac{\alpha}{\alpha+2}\right)^2\le\rho_S\le 3\frac{\alpha}{(\alpha+2)^2} ,$$

which is the same as the one discussed by Huang and Kotz (1999).

4. SOME CONCEPTS OF DEPENDENCE

In this section, we focus on some concepts of dependence for our presented generalized FGM copula. Several concepts of positive (negative) dependence and dependence stochastic orders have been introduced in the literature (Nelsen, 2006). In the following definition, we recall some of these concepts and then, study dependence structure of the family $C_{\theta,p}^{A_1,A_2}$ given in (2.2).

Definition 4.1

The random variables X and Y are

i) Positively Quadrant Dependent (PQD) if

$$P(X\le x,Y\le y)\ge P(X\le x)P(Y\le y) ,$$

for all $(x,y)\in\Re^2$ or equivalently

$$C(u,v)\ge uv ,\ \forall(u,v)\in[0,1]^2 . \tag{4.1}$$

ii) Left Tail Decreasing (LTD) if $P(Y \le y | X \le x)$ is non-increasing in x for all y, or equivalently,

$$u \to \frac{C(u,v)}{u} \tag{4.2}$$

is non-increasing for all $v \in \mathrm{I}$.

iii) Left Corner Set Decreasing $(LCSD)$ if $P(X \le x, Y \le y | X \le x_1, Y \le y_1)$ is non-increasing in x_1 and y_1 for all x and y, or equivalently, C is a Totally Positive function of order $2(TP_2)$, for all $(u_1, u_2, v_1, v_2) \in \mathrm{I}^4$ with $u_1 \le u_2$ and $v_1 \le v_2$ if

$$\Lambda = C(u_1, v_1)C(u_2, v_2) - C(u_1, v_2)C(u_2, v_1) \ge 0. \tag{4.3}$$

Proposition 4.1

Let (X, Y) be a pair of random variables with the family $C_{\theta,p}^{A_1,A_2}$. The random variables X and Y are

a. PQD if and only if $\theta \ge 0$.
b. LTD if and only if $\theta A_1'(u) \ge 0$.
c. $LCSD$ if and only if both A_1 and A_2 be decreasing or increasing.

Proof.

a. Using (4.1), the proof is straightforward.

b. Based on (4.2), $\frac{C_{\theta,p}^{A_1,A_2}(u,v)}{u} = \frac{uv[1+\theta A_1(u)A_2(v)]^p}{u}$ is non-increasing with respect to u if and only if,

$$\frac{\partial}{\partial u} v[1+\theta A_1(u)A_2(v)]^p = p\theta A_1'(u)vA_2(v)[1+\theta A_1(u)A_2(v)]^{p-1} \le 0,$$

on necessary and sufficient condition that $\theta A_1'(u) \ge 0$.

c. Suppose $u_1 \le u_2$ and $v_1 \le v_2$. By (4.3), we have

$$\Lambda = u_1v_1u_2v_2 \Big\{ \Big(\big[1+\theta A_1(u_1)A_2(v_1)\big]\big[1+\theta A_1(u_2)A_2(v_2)\big] \Big)^p - \Big(\big[1+\theta A_1(u_2)A_2(v_1)\big]\big[1+\theta A_1(u_1)A_2(v_2)\big] \Big)^p \Big\} \ge 0.$$

By simple computations, we have $\Lambda \ge 0$ if and only if

$$\big(A_1(u_1) - A_1(u_2)\big)\big(A_2(v_1) - A_2(v_2)\big) \ge 0. \tag{4.4}$$

The relation (4.4) holds, if and only if both A_1 and A_2 be decreasing or increasing. ■

As an example, for $A_i(x)=1-x$, $i=1,2$ and $\forall x\in[0,1]$, we have that the classical FGM copula are PQD, if $0\le\theta\le1$, LTD if $-1\le\theta\le0$ and $LCSD$ for all $-1\le\theta\le1$.

Recently, Bairamov and Bayramoglu (2011) showed that if in the Baker's model, one uses the dependent random variables (X,Y) with positive quadrant dependent (PQD) joint distribution function $F(x,y)$, instead of independent random variables, then the correlation increases, and in contrast, for negative quadrant dependent (NQD) $F(x,y)$, it decreases.

Lai and Xie (2000) stated conditions for having PQD and studied a class of bivariate uniform distributions having PQD property by generalizing the uniform representation of a well-known FGM copula. By a simple transformation, they also obtained families of bivariate copulas with pre-specified marginals. We showed that the association parameter in this class can improve the PQD property for a wider range. Thus, we believe that the family $C_{\theta,p}^{A_1,A_2}$ is more applicable in a greater variety of situations.

5. CONCLUSION

In this paper, we introduced a generalization of an asymmetric family of Farlie–Gumbel–Morgenstern (FGM) copula; including some of their new extensions. Also, the general formulas for measures of association of these copulas were studied. Moreover, we studied necessary and sufficient conditions for some dependence concepts in this family. Since the domain of correlation of *FGM* copula is limited, it is necessary to extend this domain. By using the generalized *FGM* family, the correlation coefficient of *FGM* copula improved. The main feature of this family is capability for modeling a wider range of dependence. This permits us to extend the range of potential applications of the family in various branches of sciences.

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APPENDIX

The proof of Proposition 2.1:

By using (2.2) and (2.3), we have

$$c_{\theta,p}^{A_1,A_2}(u,v)C_{\theta,p}^{A_1,A_2}(u,v) = uv\left[1+\theta A_1(u)A_2(v)\right]^{2p-2}$$
$$\times\Big\{\Big(1+\theta A_2(v)\left[A_1(u)+puA_1'(u)\right]\Big)\Big(1+\theta A_1(u)\left[A_2(v)+pvA_2'(v)\right]\Big)$$
$$+p\theta u\, A_1'(u)vA_2'(v)\Big\}$$

The relation above may be written through polynomial sections with respect to A_1 and A_2 as:

$$c_{\theta,p}^{A_1,A_2}(u,v)C_{\theta,p}^{A_1,A_2}(u,v) = \sum_{k=0}^{g}\binom{2p-2}{k}\theta^k uA_1^k(u)vA_2^k(v)$$
$$\times\Big\{\big(1+\theta A_2(v)[A_1(u)+puA_1'(u)]\big)\big(1+\theta A_1(u)[A_2(v)+pvA_2'(v)]\big)$$
$$+p\theta u\, A_1'(u)vA_2'(v)\Big\}$$
$$= \sum_{k=0}^{g}\binom{2p-2}{k}\theta^k$$
$$\times\Big\{uA_1^k(u)vA_2^k(v)+2\theta\, uA_1^{k+1}(u)vA_2^{k+1}(v)+p\theta\, uA_1^{k+1}(u)v^2A_2'(v)A_2^k(v)$$
$$+\theta^2uA_1^{k+2}(u)vA_2^{k+2}(v)+p\theta^2u\, A_1^{k+2}(u)v^2A_2'(v)A_2^{k+1}(v)$$
$$+p\theta u^2A_1'(u)A_1^k(u)vA_2^{k+1}(v)+p\theta^2u^2A_1'(u)A_1^{k+1}(u)vA_2^{k+2}(v)$$
$$+p^2\theta^2u^2A_1'(u)A_1^{k+1}(u)v^2A_2'(v)A_2^{k+1}(v)+p\theta u^2A_1'(u)A_1^k(u)v^2A_2'(v)A_2^k(v)\Big\}$$

So, the Kendall's tau (τ_k) is

$$\tau_k = 4\int_0^1\int_0^1 c_{\theta,p}^{A_1,A_2}(u,v)C_{\theta,p}^{A_1,A_2}(u,v)dudv - 1$$

$$= 4\sum_{k=0}^{g}\binom{2p-2}{k}\theta^k \left\{\left(\int_0^1 uA_1^k(u)du\right)\left(\int_0^1 vA_2^k(v)dv\right)\right.$$
$$+2\theta\left(\int_0^1 uA_1^{k+1}(u)du\right)\left(\int_0^1 vA_2^{k+1}(v)dv\right)$$
$$+p\theta\left(\int_0^1 uA_1^{k+1}(u)du\right)\left(\int_0^1 v^2A_2'(v)A_2^k(v)dv\right)$$
$$+\theta^2\left(\int_0^1 uA_1^{k+2}(u)du\right)\left(\int_0^1 vA_2^{k+2}(v)dv\right)$$
$$+p\theta^2\left(\int_0^1 uA_1^{k+2}(u)du\right)\left(\int_0^1 v^2A_2'(v)A_2^{k+1}(v)dv\right)$$
$$+p\theta\left(\int_0^1 u^2A_1'(u)A_1^k(u)du\right)\left(\int_0^1 vA_2^{k+1}(v)dv\right)$$
$$+p\theta^2\left(\int_0^1 u^2A_1'(u)A_1^{k+1}(u)du\right)\left(\int_0^1 vA_2^{k+2}(v)dv\right)$$
$$+p^2\theta^2\left(\int_0^1 u^2A_1'(u)A_1^{k+1}(u)du\right)\left(\int_0^1 v^2A_2'(v)A_2^{k+1}(v)dv\right)$$
$$\left.+p\theta\left(\int_0^1 u^2A_1'(u)A_1^k(u)du\right)\left(\int_0^1 v^2A_2'(v)A_2^k(v)dv\right)\right\} - 1$$

$$= 4\sum_{k=0}^{g}\binom{2p-2}{k}\theta^k \left\{B_1(k)B_2(k) + 2\theta B_1(k+1)B_2(k+1) - \frac{2p\theta}{k+1}B_1(k+1)B_2(k+1)\right.$$
$$+\theta^2 B_1(k+2)B_2(k+2) - \frac{2p\theta^2}{k+2}B_1(k+2)B_2(k+2) - \frac{2p\theta}{k+1}B_1(k+1)B_2(k+1)$$
$$-\frac{2p\theta^2}{k+2}B_1(k+2)B_2(k+2) + 4p^2\theta^2\left(\frac{1}{k+2}\right)^2 B_1(k+2)B_2(k+2)$$
$$\left.+4p\theta\left(\frac{1}{k+1}\right)^2 B_1(k+1)B_2(k+1)\right\} - 1$$

$$= 4\sum_{k=0}^{g}\binom{2p-2}{k}\theta^k \left\{B_1(k)B_2(k) + 2\theta\left(1 - \frac{2p}{(k+1)^2}\right)B_1(k+1)B_2(k+1)\right.$$
$$\left.+\theta^2\left(\frac{k+2-2p}{k+2}\right)^2 B_1(k+2)B_2(k+2)\right\} - 1 .$$

■