**Pak. J. Statist.**
**2015 Vol. 31(1), 21-32**

# A CONFLUENT HYPERGEOMETRIC GENERALIZED INVERSE GAUSSIAN DISTRIBUTION

**Abdus Saboor**[1], **Serge B. Provost**[2] and **Munir Ahmad**[3]
[1] Department of Mathematics, Kohat University of Science and Technology, Kohat, Pakistan. Email: saboorhangu@gmail.com
[2] Department of Statistical and Actuarial Sciences, The University of Western Ontario, London, Ontario, Canada. Email: sp@uwo.ca
[3] National College of Business Administration & Economics Lahore, Pakistan. Email: drmunir@ncbae.edu.pk

## ABSTRACT

A new generalized extended inverse Gaussian density function involving a confluent hypergeometric function is defined. Closed form representations of a generalized extended inverse Gaussian density function and the corresponding moment generating function are provided in terms of Meijer's G-function. Some other statistical functions which are associated with the proposed distribution are also derived. Many distributions can be obtained as limiting cases of the proposed density function. This density is utilized to model an actual data set. The new distribution provides a better fit than related distributions as measured by the Anderson-Darling and Cramér-von Mises statistics. The proposed distribution could, for instance, find applications in the physical and biological sciences and in particular be helpful to researchers working in special functions as well as reliability analysis, survival models and diffraction theory.

## KEYWORDS

Confluent hypergeometric function; Meijer's G-function; Generalized inverse Gaussian distribution; Lifetime data; Goodness-of-fit statistics.

## 1. INTRODUCTION

Some exponential distributions such as the gamma, Weibull and inverse Gaussian distributions have a wide range of applications in research areas such as reliability, survival analysis, queuing theory, the physical and biological sciences, hydrology, medicine, meteorology and engineering. In fact, the power function along with the exponential part which are both present in the gamma, Weibull and inverse Gaussian densities enable one to model a wide variety of data sets. To extend the scope of these distributions, many generalizations involving additional parameters have been considered, see for instance, Good (1953), Pham and Almhana (1995), Agarwal and Kalla (1996), Lee and Wang (2003), Chou and Huang (2004), Silva *et al.* (2006), Nadarajah and Kotz (2006), Nadarajah (2008), Provost *et al.* (2011), Provost and Mabrouk (2011), Lemonte and Cordeiro (2011), Kumar and Vellaisamy (2012), Saboor and Ahmad (2012), Saboor *et al.* (2012), Badmus *et al.* (2013), Cordeiro *et al.* (2013b), Cordeiro *et al.* (2013a), Cordeiro *et al.* (2013c), Cordeiro *et al.* (2013d), Cordeiro and Lemonte (2013), Cordeiro *et al.* (2014a), Gómez-

 

Déniz *et al.* (2013), Favaro and Lijoi (2013), Nagatsukaa and Balakrishnan (2013), Saboor and Ahmad (2013), Cordeiro *et al.* (2014b), Peng and Yan (2014), Saboor *et al.* (2014a), Saboor *et al.* (2014b), Saboor and Pogány (2014) and Tojeiro *et al.* (2014).

A generalized extended inverse Gaussian density function involving a confluent hypergeometric function is defined in Section 2. Graphical representations of the effects of the parameters are also included in Section 2. Explicit representations of certain statistical functions are provided in the Section 3. The parameter estimation technique described in Section 3.1 is utilized in connection with the modeling of a data set. Section 4 includes some concluding remarks.

## 2. A CONFLUENT HYPERGEOMETRIC GENERALIZED INVERSE GAUSSIAN DISTRIBUTION

This paper proposes a confluent hypergeometric generalized inverse Gaussian distribution (CHGIG) whose associated density function is defined as follows:

$$f\left(x\right)=C\, x^{m-1}\, exp\left(-p_1 x-p_2 x^{-1}\right)\, {}_1F_1(\lambda;b;-\alpha x),\quad x>0\,, \tag{1}$$

where $f(\mathrm{x})>0$ whenever $m>0, p_1>0, p_2>0,\ \alpha\ge 0,\ \lambda, b\ne\ 0,-1,-2,...$ and $f(\mathrm{x})=0$ elsewhere, ${}_1F_1(\lambda;b;z)=\sum_{k=0}^{\infty}\frac{(\lambda)_k\ z^k}{(\mathrm{b})_k\ k!}, (\theta)_k\ =\ \Gamma(\theta+k)\,/\,\Gamma(\theta)$ denoting the Pocchammer function and $\Re\left(b_1\right)>0,\Re\left(b_2\right)>0,\Re(b)>\Re(\lambda),\Re(b)\ne 0,-1,-2,...,$ see (Rainville, 1960) and

$$\begin{aligned} C^{-1} &=\Gamma^{(\Theta)}\left(m\right)=\int_0^{\infty} t^{m-1}\ exp\left(-p_1 t-p_2 t^{-1}\right){}_1F_1(\lambda;b;-\alpha t)\ \mathrm{d}t,\\ &=\frac{\Gamma(\mathrm{b})}{\Gamma(\lambda)}\sum_{n=0}^{\infty}\frac{\left(p_1 p_2\right)^n}{\mathrm{n}!}\left(p_1{}^{-m}\ G_{3,3}^{2,2}\left(\alpha/\mathrm{p}1\ \middle|\begin{matrix}1-m,1-\lambda,-m+n+1\\ 0,1-m,1-b\end{matrix}\right)\right.\\ &\qquad\left.-p_2{}^{m}G_{2,4}^{2,2}\left(\mathrm{p}_2\ \alpha\ \middle|\begin{matrix}1-m,1-\lambda\\ 0,1-m,1-b,-m-n\end{matrix}\right)\right), \end{aligned} \tag{2}$$

where $\Theta=\left\{\lambda,b,p_1,p_2,\alpha\right\},\Re\left(\mathrm{b}_1\right)>0,\Re\left(\mathrm{b}_2\right)>0,-\infty<\Re\left(\alpha\right)<\infty,\ \Gamma^{(\Theta)}\left(m\right)$ is a generalized extended gamma function (Saboor *et al.*, 2013), $\Re(\mathrm{b}),\mathrm{Re}(\lambda),\Re(\mathrm{m})\ne 0,-1,-2,...,$ and the symbol $G_{\mathrm{p,q}}^{\mathrm{m,n}}(.|.)$ denotes Meijer's G-function (Meijer, 1946) defined in terms of Mellin-Barnes integral as

$$G_{p,q}^{m,n}\left(z\middle|\begin{matrix}a_1,...,a_p\\ b_1,...,b_p\end{matrix}\right)=\frac{1}{2\pi i}\int_c\frac{\prod_{j=1}^{m}\Gamma\left(b_j-s\right)\prod_{j=1}^{n}\Gamma\left(1-a_j+s\right)}{\prod_{j=m+1}^{q}\Gamma\left(1-b_j-s\right)\prod_{j=n+1}^{p}\Gamma\left(a_j-s\right)}z^j\mathrm{d}s\,,$$

where $0\le m\le q, 0\le n\le p$ and $a_j, b_j$ are such the poles of $\Gamma\left(b_j-s\right).j=\overline{1.m}$ do not coincide with those of $\Gamma\left(1-a_j-s\right).j=\overline{1.n}$ i.e. $a_k-b_j\notin\mathbb{N}$ while $z\ne 0$. $c$ is a suitable integration contour which start at $c^{-}i\infty$ and goes to $c^{+}i\infty$ and separate the poles $\Gamma\left(b_j - s\right).j = \overline{1.m}$ which lie to the right of the contour, from all poles of $\Gamma\left(1-a_j - s\right).j = \overline{1.n}$ which lie to the left of $C$.

The integral converges if $\delta = m+n-\frac{1}{2}\left(p+q\right) > 0$ and $\left|\arg(z)\right| < \delta\pi$, see (Luke, 1969 p. 143) and Meijer (1946).

The Mathematica code for the G-function reads

$$Meijer\,G\left[\left\{\left\{a_1,...,a_n\right\}\left\{a_{n+1},...,a_p\right\}\left\{b_1,...,b_m\right\}\left\{b_{m+1},...,b_q\right\}\right\},z\right].$$

The structure of the density function (1) can be motivated as follow:

- The integral involving the function ${}_1F_1$ may be evaluated as a Laplace transform by making use of inverse Mellin transform technique. The proposed distribution may provide a more realistic model in connection with certain physical problems, as its density function includes a confluent hypergeometric function.
- The density function (1) provides more flexible distributions than the generalized inverse Gaussian and many other exponentiated type distributions.

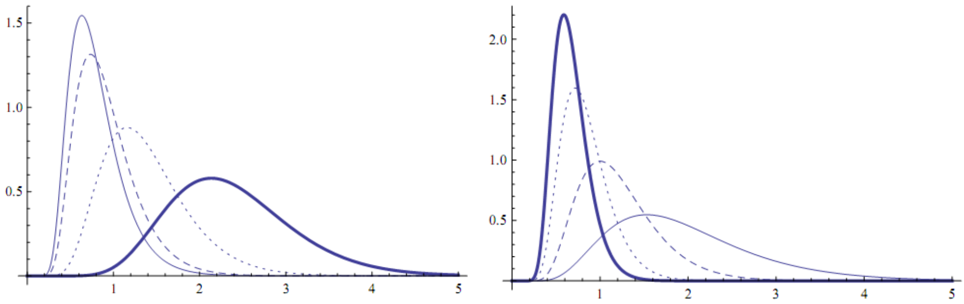


Fig. 1: The confluent hypergeometric generalized inverse Gaussian pdf.
Left panel: $\lambda = 1.9$, $b = 2.3$, $\alpha = 10.1$, $p_1 = 2$, $p_2 = 3$ and $m = 1.1$ (solid line), $m = 2.1$ (dashed line), $m = 5.1$ (dotted line), $m = 10.1$ (thick line).
Right panel: $\lambda = 1.9$, $b = 2.3$, $\alpha = 10.1$, $p_2 = 3$, $m = 4.1$ and $p_1 = 2$ (solid line), $p_1 = 4$ (dashed line), $p_1 = 7$ (dotted line), $p_1 = 10$ (thick line).

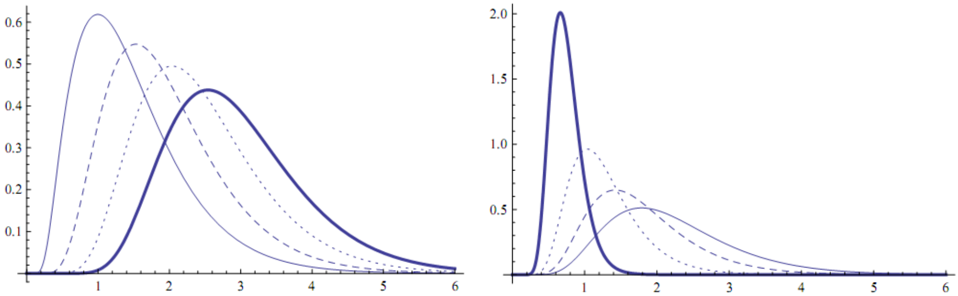


Fig. 2: The confluent hypergeometric generalized inverse Gaussian pdf.
Left panel: $\lambda = 1.9$, $b = 2.3$, $\alpha = 10.1$, $p_1 = 2$, m = 4.1 and $p_2 = 1$ (solid line), $p_2 = 3$ (dashed line), $p_2 = 6$ (dotted line), $p_2 = 10$ (thick line).
Right panel: $b = 12.3$, $\alpha = 10.1$, $p_1 = 2$, $p_2 = 3$, $m = 4.1$ and $\lambda = 1.9$ (solid line), $\lambda = 3.9$ (dashed line), $\lambda = 6.9$ (dotted line), $\lambda = 11.9$ (thick line).

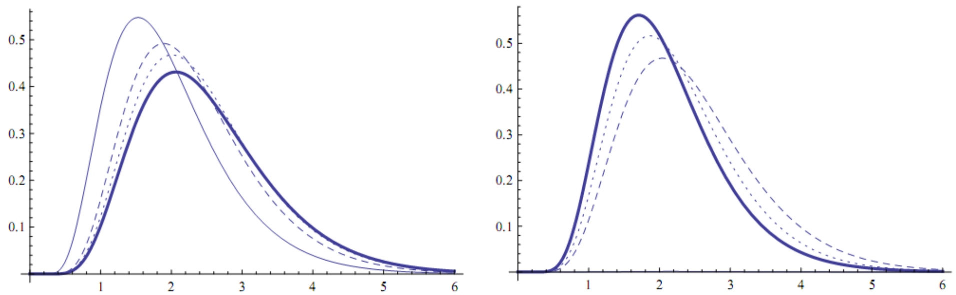


Fig. 3: The confluent hypergeometric generalized inverse Gaussian pdf.
Left panel: $\lambda = 1.9$, $\alpha = 10.1$, $p_1 = 2$, $m = 4.1$, $p_2 = 3$ and $b = 2.3$ (solid line), $b = 22.3$ (dashed line), $b = 47.3$ (dotted line), $b = 72.3$ (thick line).
Right panel: $\lambda = 1.9$, $b = 2.3$, $p_1 = 2$, $p_2 = 3$, $m = 4.1$ and $\alpha = 0.1$ (solid line), $\alpha = 0.3$ (dashed line), $\alpha = 0.7$ (dotted line), $\alpha = 1.2$ (thick line).

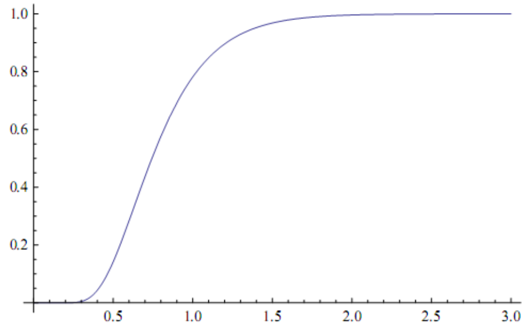


Fig. 4: The confluent hypergeometric generalized inverse Gaussian cdf for $\lambda = 1.9$, $b = 2.3$, $p_1 = 4$, $m = 1.1$, $p_2 = 3$ and $\alpha = 10.1$.

The corresponding cumulative distribution function (c.d.f), $F\left(x\right)$ and survival function, $S\left(x\right)$, are given by

$$F(x)=C\int_0^x t^{m-1}\ \exp\ \left(-p_1 t-p_2 t^{-1}\right)\ {}_1F_1\left(\lambda;b;-\alpha t\right)\mathrm{d}t\ =\frac{\Gamma^{(\Theta)}\begin{pmatrix}x\\0\end{pmatrix}(m)}{\Gamma^{(\Theta)}\,(m)} \tag{3}$$

and

$$S(x)=\frac{\Gamma^{(\Theta)}\begin{pmatrix}x\\0\end{pmatrix}(m)}{\Gamma^{(\Theta)}\,(m)}\,. \tag{4}$$

Since $f(x)>0, \lim\limits_{x\to 0} f(x)=\lim\limits_{x\to\infty} f(x)=0$ and $\int_0^\infty f(\mathrm{x})\,\mathrm{d}\,x=1$, $f(\mathrm{x})$ is a bone fide probability function.

The left and right panels of Figure 1, 2 and 3 illustrate how the parameters $m$, $p_1$, $p_2$, $\lambda, b$ and $\alpha$ affect the CHGIG distribution. The curves tend to flatten as $m$ increases or as $p_1$ and $\lambda$ decreases when keeping the other parameters fixed. The left panel of Figure 2 illustrates that the parameter $p_2$ acts somewhat as a location parameter. Figure 4 shows the cdf of the CHGIG distribution for a given set of parameters.

### 2.1 *Statistical Functions*

Closed form representations of the moment generating function of a CHGIG random variable which will be denoted by $X$, as well as the associated moments, hazard rate, mean residual life function are provided in this section. The resulting expressions can be evaluated exactly or numerically with symbolic computational packages such as *Mathematica*, MATLAB or Maple. In numerical applications, infinite sums can be truncated whenever convergence is observed.

- The moment generating function associated with the density function (1), that is

$$\begin{aligned} M_X\left(t\right)&=C\int_0^\infty e^{tx} t^{m-1}\ exp\left(-p_1 t-p_2 t^{-1}\right)\ {}_1F_1(\lambda;b;-\alpha t)\ \mathrm{d}t,\\ &=C\int_0^\infty t^{m-1}\ exp\left(-\left(p_1-t\right)\mathrm{x}-p_2 t^{-1}\right)\ {}_1F_1(\lambda;b;-\alpha t)\ \mathrm{d}t, \end{aligned} \tag{5}$$

  where $\left(p_1-t\right)>0, p_2>0$ and $b\neq 0,-1,-2,\ldots,$ can be derived as follows.

  Let

$$\int_0^\infty t^{m-1}\ exp\left(-\left(p_1-t\right)\mathrm{x}-p_2 t^{-1}\right)\ {}_1F_1(\lambda;b;-\alpha t)\ dt \tag{6}$$

$$=\frac{\Gamma(\mathrm{b})}{\Gamma(\lambda)}\int_0^\infty t^{m-1}\ exp\left(-\left(p_1-t\right)\mathrm{x}-p_2 t^{-1}\right)\ \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\ \frac{(\alpha\,\mathrm{t})^{\mathrm{S}}\,\Gamma(-\mathrm{s})\Gamma(\lambda+s)}{\Gamma(\mathrm{b+s})}\,\mathrm{d}s\,\mathrm{d}t$$

$$= \frac{\Gamma(\text{b})}{\Gamma(\lambda)} 2\left(\frac{p_2}{(p_1-t)}\right)^{m/2} \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \left(\frac{\sqrt{p_2}}{\sqrt{p_1-t}\alpha}\right)^{-s} \frac{\Gamma(-\text{s})\Gamma(\lambda+s)}{\Gamma(\text{b+s})}$$

$$\times k_{m+s}\left(2\sqrt{(p_1-t)\,p_2}\right)\text{d}s\,, \tag{7}$$

where $k_v$ is the Macdonald function (Gradshteyn and Ryzhik, 1994), defined by

$$k_v = \pi/2\,\frac{I_{-v}(\text{z})I_v(\text{z})}{\sin v\pi}\,.$$

$I_v(\text{z})$ denoting the modified Bessel function of the first kind (Abramowitz and Stegun, 1972), given by

$$I_v(z) = \sum_{n=0}^{\infty} \frac{(z/2)^{v+2n}}{n!\Gamma(v+n+1)}\,.$$

Making use of the extended gamma function (Chaudhry and Zubair, 2001), the modified Bessel function of first kind kind (Abramowitz and Stegun, 1972) and the residue theorem (Ahlfors, 1966) and (7), one has

$$M_X(t) = \frac{\Gamma(b)}{\Gamma^{(\Theta)}(m)\Gamma(\lambda)} \sum_{n=1}^{\infty} \frac{((p_1-t)\,p_2)^n}{n!} \left( (p_1-t)^{-m} G_{3,3}^{2,2}\left(\alpha/p_1 \middle|\begin{matrix} 1,m,b \\ \lambda,m,m-n\end{matrix}\right)\right.$$

$$\left. -p_2{}^m G_{2,4}^{2,2}\left(1/\text{p}_2\,\alpha \middle|\begin{matrix} 1,m,b,m+n+1 \\ \lambda,m\end{matrix}\right)\right), \tag{8}$$

which can be expressed as

$$M_X(t) = \frac{\Gamma(b)}{\Gamma^{(\Theta)}(m)\Gamma(\lambda)} \sum_{n=1}^{\infty} \frac{((p_1-t)\,p_2)^n}{n!} \left( (p_1-t)^{-m}\right.$$

$$\times G_{3,3}^{2,2}\left(\alpha/(p_1-t) \middle|\begin{matrix} 1-m,1-\lambda,-m+n+1 \\ 0,1-m,1-b,-m-n\end{matrix}\right)$$

$$\left. -p_2{}^m G_{2,4}^{2,2}\left(\text{p}_2\,\alpha \middle|\begin{matrix} 1-m,1-\lambda \\ 0,1-m,1-b,-m-n\end{matrix}\right)\right),$$

where $(p_1-t)>0,\ p_2>0$ and $b,\lambda,m \neq 0,-1,-2,\ldots$

- The mean and variance of the distribution of $X$ are given by

$$E(X) = \frac{\Gamma^{(\Theta)}(m+1)}{\Gamma^{(\Theta)}(m)} \tag{9}$$

and

$$V(X) = \frac{1}{\Gamma^{(\Theta)}(m)}\left(\Gamma^{(\Theta)}(m+2) - \frac{\left(\Gamma^{(\Theta)}(m+1)\right)^2}{\Gamma^{(\Theta)}(m)}\right). \tag{10}$$

The information function is simply the inverse of the variance.

- The factorial moments for the probability density function defined in (1) are as follows

$$E\left(X(X-1)(X-2)....(X-\gamma+1)\right) \square \sum_{j=1}^{\gamma-1} \varphi_j (-1)^j E\left(X^{\gamma-j}\right)$$

$$= \sum_{j=1}^{\gamma-1} \varphi_j (-1)^{\mathrm{j}} \frac{\Gamma^{(\Theta)}(\mathrm{m}+\gamma+1)}{\Gamma^{(\Theta)}(\mathrm{m})}, \tag{11}$$

where the $\varphi_j$ 's are non-null real numbers that satisfy the first identity.

- The $i^{th}$ order negative moment is

$$E\left(\mathrm{x}^{-\mathrm{i}}\right) = \sum_{j=1}^{\gamma-1} \frac{\Gamma^{(\Theta)}(\mathrm{m}-\mathrm{i})}{\Gamma^{(\Theta)}(\mathrm{m})} E, \quad \Re(m-1) > 0. \tag{12}$$

- The hazard rate function (failure rate) which is defined as

$$Z(x) = \frac{f(x)}{S(x)},$$

where $f(x)$ and $S(x)$ are given in (1) and (4), has its origin in reliability theory. In this case,

$$Z(x) = \frac{x^{m-1}\ e^{-p_1 \mathrm{x} - p_2 x^{-1}}\ {}_1F_1(\lambda; b; -\alpha x)}{\Gamma^{(\Theta)}\begin{pmatrix}\infty \\ x\end{pmatrix}(m)}. \tag{13}$$

- The mean residue life function is defined as

$$K(x) = \frac{1}{S(x)}\int_x^\infty (y-x) f(y)dy = \frac{1}{S(x)}\int_x^\infty y\, f(y)dy - x = \frac{1}{S(x)}\left[E(Y) - \int_0^x y f(y)dy\right] - x,$$

where $f(y)$, $S(x)$ and $E(Y)$ is given in (1), (4) and (9) respectively. Accordingly,

$$K(x) = \frac{1}{S(x)}\left[E(Y) - \frac{\Gamma^{(\Theta)}(m+1)}{\Gamma^{(\Theta)}(m)} dy\right] - x. \tag{14}$$

## 3. APPLICATION

In this section, we will use the two parameter gamma, ESGamma (Nadarajah and Kotz, 2006), generalized gamma (GG) (Agarwal and Kalla, 1996; Nadarajah, 2008), the generalized inverse Gaussian (GIG) and the confluent hypergeometric generalized inverse Gaussian (CHGIG) distributions to model the well-known ‘Ball bearings’ data set. The maximum likelihood approach will be utilized to estimate the parameters. Additionally, two goodness-of-fit measures are defined and employed to compare the density estimates.

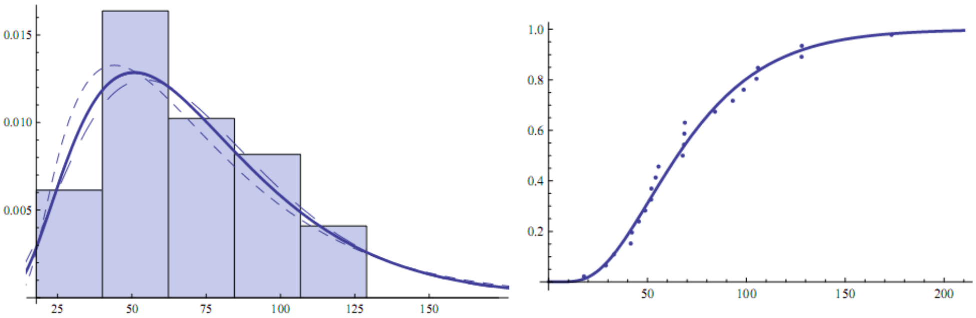


Fig. 5: Left panel: The graphs of gamma (dotted line), ESGamma (small dashes), GG (long dashes), GIG (dolid line) and CHGIG (thick line) density estimates superimposed on the histogram for ball bearings data.
Right panel: The CHGIG cdf estimates and empirical cdf.

### *3.1 Maximum Likelihood Estimation*

In order to estimate the parameters of the proposed confluent hypergeometric generalized inverse Gaussian distribution as specified by the density function appearing in Equation (1), the loglikelihood of the sample is maximized with respect to the parameters by making use of the *NMaximize* command in the symbolic computational package *Mathematica*. Given the data $x_i, i=1,2,\ldots,n$ the loglikelihood function is given by

$$l(\Theta)=-n\log\left(\Gamma^{\Theta}(m)\right)+(m-1)\sum_{i=0}^{n}\log\left(x_i\right)-p_1\sum_{i=0}^{n}x_i-p_2\sum_{i=0}^{n}x_i^{-1}$$
$$+\sum_{i=0}^{n}\log\left({}_1F_1\left(\lambda;b,-\alpha x_i\right)\right), \qquad (15)$$

where $f\left(x\right)$ is as given in (1).

### *3.2 Goodness-of-Fit Statistics*

The Anderson-Darling and the Cramḗr-von Mises statistics are widely utilized to determine how closely a specific distribution whose associated cumulative distribution function denoted by cdf(·) fits the empirical distribution associated with a given data set. The smaller these statistics are, the better the fit. Upper tail percentiles of the asymptotic distributions of these goodness–of–fit statistics were tabulated in Stephens (1976).

**Table 1**
**Estimates of Parameters and Goodness-of-Fit Statistics for the Ball Bearings Data under Various Distributions**

| Estimates | Gamma | ESGamma | GG | GIG | GHGIG |
|---|---|---|---|---|---|
| $\hat{m}$ | 4.02802 | 22.131 | 4.03031 | 2.51277 | 6.45952 |
| $\hat{p}_1$ | 0.0557669 | -- | 0.0557399 | 0.0441474 | 0.0441129 |
| $\hat{p}_2$ | -- | -- | -- | 37.2274 | 3.11946 |
| $\hat{\beta}$ | -- | 0.0335813 | -- | -- | -- |
| $\hat{\lambda}$ | -- | -- | 0.0040165 | -- | 3.88653 |
| $\hat{b}$ | -- | -- | -- | -- | 6.49732 |
| $\hat{z}$ | -- | -- | 0.00100415 | -- | -- |
| $\hat{\alpha}$ | -- | -- | -- | -- | 0.260609 |
| $A_0^2$ | 0.215968 | 0.227772 | 0.215878 | 0.190673 | 0.189267 |
| $W_0^2$ | 0.0392492 | 0.0348883 | 0.03924 | 0.0325841 | 0.0324115 |

The confluent hypergeometric generalized inverse Gaussian model was applied to a data set published in (Lawless, 1982, p. 228), which consists of the number of million revolutions before failure for each of 23 ball bearings in a life testing experiment. The pdf and cdf estimates are plotted in Figure 5 for CHGIG distribution. The estimates of the parameters and the values of the Anderson-Darling and Cramėr-von Mises goodness-of-fit statistics are given in Table 1. It is seen that the CHGIG model provides the best fit.

## 4. CONCLUDING REMARKS

A confluent hypergeometric generalized inverse Gaussian density function has been defined and some of its properties were discussed. Certain related statistical functions have also been derived. An application of the new distribution to an actual data set was presented; it was shown that it can provide a better fit than other exponential–type models. The presented extension could be utilized to model data arising in numerous areas of scientific investigation.

## 5. ACKNOWLEDGEMENTS

The research of Abdus Saboor has been supported in part by the Higher Education Commission of Pakistan under NRPU project. The financial support of NSERC is gratefully acknowledged by the second author. The authors are indebted to referees and Associate Editor for improving the original version of the paper.

## REFERENCES

1. Abramowitz, M. and Stegun, I.A. (1972). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. New York.
2. Agarwal, S.K. and Kalla, S.K. (1996). A generalized gamma distribution and its application in reliability, *Commun. Statist. Theo. Meth.*, 25, 201-210.
3. Ahlfors, L.V. (1966). *Complex Analysis*. McGraw-Hill, New York.
4. Badmus, N.I., Ikegwu and Emmanuel, M. (2013). The beta-weighted Weibull distribution: some properties and application to bladder cancer data, *J. Appl. Computat. Math.*, 2(145) [cited 2013 Nov 25]. Available from: DOI: 10.4172/2168-9679.1000145.
5. Chaudhry, M.A. and Zubair, S.M. (2001). Extended gamma and digamma functions, *Frac. Calc. Appl. Anal.*, 4, 303-326.
6. Chou, C.W. and Huang, W.J. (2004). On characterizations of the gamma and generalized inverse Gaussian distributions. *Statist. Probabil. Lett.*, 69, 381-388.
7. Cordeiro, G.M. and Lemonte, A.J. (2013). On the Marshall-Olkin extended Weibull distribution. *Stat. Papers*, 54 (2013) 333-353.
8. Cordeiro, G.M., Edwin, M.M. Ortega and Cunha, D.C.C. (2013). The exponentiated generalized class of distributions. *J. Data Sci.*, 11, 1-27.
9. Cordeiro, G.M., Edwin, M.M. Ortega and Lemonte, A.J. (2013). The exponential-Weibull lifetime distribution. *J. Statist. Comput. Simul.*, 2013 [cited 2013 May 14]. Available from: DOI: 10.1080/00949655.2013.797982.
10. Cordeiro, G.M., Edwin, M.M. Ortega and Silva, G.O. (2014). The Kumaraswamy modified Weibull distribution: theory and applications. *J. Statist. Comput. Simul.*, 84, 1387-1411.
11. Cordeiro, G.M., Gomes, A.E., da-Silva, C.Q. and Edwin, E.M.M. Ortega. (2013). The beta exponentiated Weibull distribution. *J. Statist. Compu. Simul.*, 83(1), 114-138.
12. Cordeiro, G.M., Hashimoto, E.M. and Edwin, M.M. Ortega. (2014). The McDonald Weibull model. *Statistics: A Journal of Theoretical and Applied Statistics*, 48, 256-278.
13. Cordeiro, G.M., Silva, G.O. and Edwin, M.M. Ortega. (2013). The beta-Weibull geometric distribution. *Statistics*, 47, 817-834.
14. Favaro, S., Lijoi, A. and Prnster, I. (2012). On the stick-breaking representation of normalized inverse Gaussian priors. *Biometrika*, 99(3), 663-674.
15. Gómez-Déniz, E., Calder´ın-Ojeda, E. and Sarabia, J.M. (2013). Gamma-generalized inverse Gaussian class of distributions with applications. *Commun. Statist. : Theo. Meth*., 42(6), 919-933.
16. Ghitany, M.A., Al-Awadi, S. and Kalla, S.L. (2002). On hypergeometric generalized negative binomial distribution. *Int. J. Math. Sci.*, 29, 727-736.
17. Good, I.J. (1953). The population frequencies of species and the estimation of population parameters. *Biometrika*, 40, 237-260.
18. Gradshteyn, I. and Ryzhik, I. (1994). *Tables of Integrals, Series, and Products* (English Transl. Ed. by Alan Jeffrey). Academic Press, New York.
19. Hoem, I.N. (1976). The statistical theory of demographic rates, *Scand. J. Statist.*, 3, 169-185.

20. Jórgensen, B. (1982). *Statistical Properties of Generalized Inverse Gaussian Distributions*. Lecture Notes in Statistics vol. 9. Springer. New York.
21. Kumar, A. and Vellaisamy, P. (2012). Fractional normal inverse Gaussian process. *Methodol. Comput. Appl.*, 14(2), 263-283.
22. Lawless, J. (1982). *Statistical Models and Methods for Lifetime Data*. John Wiley & Sons. New York.
23. Lee, E. and Wang, J. (2003). *Statistical Methods for Survival Data Analysis*. 3rd Edition. Wiley & Sons. New York.
24. Lemonte, A.J. and Cordeiro, G.M. (2011). The exponentiated generalized inverse Gaussian distribution, *Stat. Probabil. Lett.*, 81, 506-517.
25. Luke, Y.L. (1969). *The Special Functions and Their Approximation*. Vol. I, Academic Press, New York and London.
26. Meijer, C.S. (1946). On the *G*-function I-VIII, Proc. Kon. Ned. Akad. Wet. 49, 227-237, 344-356, 457-469, 632-641, 765-772, 936-943, 1063-1072, 1165-1175.
27. Nadarajah, S. (2008). Some gamma distributions, S*tatistics: A Journal of Theoretical and Applied Statistics.*, 42(1), 77-94.
28. Nadarajah, S. and Kotz, S. (2006). The exponentiated type distributions, *Acta Applicandae Mathematica.*, 92, 97-111.
29. Nagatsukaa, H. and Balakrishnan, N. (2013). A consistent method of estimation for the parameters of the three-parameter inverse Gaussian distribution, *J. Statist. Comput. Simul.*, 83, 1915-1931.
30. Peng, X. and Yan, Z. (2014). Estimation and application for a new extended Weibull distribution. *Reliab. Eng. Syst. Safe.*, 121, 34-42.
31. Pham, T. and Almhana, J. (1995). The generalized gamma distribution: its hazard rate and strength model, *IEEE Transaction on Reliability*, 44, 392-397.
32. Provost, S.B. and Mabrouk, I. (2011). An extended inverse Gaussian model, *Int. J. Stat. Sci.*, 11, 135-144.
33. Provost, S.B., Saboor, A. and Ahmad M. (2011). The gamma-Weibull distribution, *Pak. J. Statist.*, 27, 111-131.
34. Rainville, E.D. (1960). *Special Function*. The Macmillan Company. New York.
35. Saboor, A. and Ahmad, M. (2012). Bivariate gamma-type probability function using a hypergeometric function of two variables, *Pak. J. Statist.*, 28(1), 81-91.
36. Saboor, A. and Ahmad, M. (2013). Further properties of a bivariate gamma-type function and its probability distribution, *Pak. J. Statist.*, 29(1), 49-57.
37. Saboor, A., Khan, M.N., Kamal, M. and Mubeen, S. (2014a). The modified beta exponential-Weibull distribution. *Bothalia Journal*, 44(8), 36-50.
38. Saboor, A., Khan, A. and Mubeen, S. (2014b). A $\Psi_2$ hypergeometric generalized inverse Gaussian distribution. *Bothalia Journal*, 44 (8), 18-25.
39. Saboor, A., Mubeen, S. and Khan, A. (2013). Further generalization of gamma and beta functions, *World Appl. Sci. J.*, 22(9), 1272-1278.
40. Saboor, A and Pogány, T. (2014). Marshall-Olkin Gamma-Weibull Distribution with Application. *Commu. Statist. Theo. Meth*. To Appear.
41. Saboor, A., Provost, S.B. and Ahmad, M. (2012). The moment generating function of a bivariate gamma-type distribution, *Appl. Math. Comput.*, 218(24), 11911-11921.

42. Silva, R.S., Lopes, H.F. and Migon, H.S. (2006). The extended generalized inverse Gaussian distribution for log-linear and stochastic volatility models, *Braz. J. Prob. Stat.*, 67-91.
43. Stephens, M. (1976). Asymptotic results for goodness-of-fit statistics with unknown parameters. *Annal. Statist.*, 4, 357-359.
44. Tojeiro, C., Louzada, F., Roman, M. and Borges, P. (2014). The complementary Weibull geometric distribution. *J. Statist. Comput. Simul.*, 84, 1345-1362.